Math 442

Exercise set 2

1. *(a)* Prove Lemma 2.12: if $f : \Omega_1 \longrightarrow \Omega_2$ is a continuous map between topological spaces, then, for any $A \subseteq \Omega_1$, $f(\mathrm{cl}\, A) \subseteq \mathrm{cl}(f(A))$.

(b) Prove that, if $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \Omega_3$ are continuous maps between topological spaces, then $g \circ f : \Omega_1 \longrightarrow \Omega_3$ is also continuous.

2. Let $(\Omega, \mathcal{T})$ be a topological space. A class $\mathcal{B}$ of subsets of Ω is called a *base for the topology* of Ω if, firstly, $\mathcal{B} \subseteq \mathcal{T}$, and, secondly, every member of $\mathcal{T}$ is a union of [possibly very infinitely many] members of $\mathcal{B}$. Prove that, if $\mathcal{B}$ is a base for the topology of Ω and $A \subseteq \Omega$, and any cover of A by members of $\mathcal{B}$ has a finite subcover, then A is compact in Ω.

3. Show that, if $\mathcal{B}$ is a base for the topology of Ω, a function $f : \Psi \longrightarrow \Omega$ is continuous (where Ω is a second topological space) if and only if $f^{-1}(B)$ is open in Ψ for every $B \in \mathcal{B}$.

4. If (Ω, d) is a metric space, prove that the mapping $d : \Omega \times \Omega \longrightarrow \mathbb{R}$ is itself continuous when $\Omega \times \Omega$ is given the product topology.

5. The *Cantor set* is the subset E of $[0,1]$ defined as $\bigcap_{n=0}^{\infty} E_n$, where the subsets E_n of $[0,1]$ are defined inductively. E_0 is to be $[0,1]$. If E_n has been defined, and is a union of finitely many disjoint nondegenerate closed intervals, $E_n := \bigcup_{k=1}^{p(n)} [a_{kn}, b_{kn}]$ (where

$$a_{kn} < b_{kn} \quad \text{for } 1 \le k \le p(n), \qquad b_{kn} < a_{k+1,n} \quad \text{for } 1 \le k < p(n)),$$

then $E_{n+1} := \bigcup_{k=1}^{p(n)} \left([a_{kn}, \tfrac{2}{3}a_{kn} + \tfrac{1}{3}b_{kn}] \cup [\tfrac{1}{3}a_{kn} + \tfrac{2}{3}b_{kn}, b_{kn}]\right)$. That is, we remove the (open) "middle third" of each constituent subinterval of E_n to obtain E_{n+1}. [E is sometimes called the Cantor "middle third" set, because one could, in principle, also change the fraction removed from the subintervals at each stage.]

(a) Show that E is compact in $\mathbb{R}$.

(b) Show that E is nowhere dense in $\mathbb{R}$.

(c) Show that E is *dense in itself*, by which I mean that every point of E is the limit of a sequence in E without repeats, i.e. is an accumulation point of E.

(d) Show that E is uncountable. [Thus nowhere dense sets can easily be "large" in a purely set-theoretic sense.]

6. Show that the *diagonal mapping* $\Omega \longrightarrow \Omega \times \Omega : x \mapsto (x, x)$ is continuous if Ω is a topological space and $\Omega \times \Omega$ is given the product topology.

7. Give counterexamples to show that Dini's theorem fails if the assumption that the sequence be pointwise monotonic is omitted, if the functions of the sequence are not required to be continuous, or if the limit function is not required to be continuous.

8. Prove, along the same general lines as the argument of 3.2, that there is a sequence of polynomials $p_n(x)$ which converges monotonically (and therefore uniformly) on the interval $[0,1]$ to the function $\sqrt{x}$. (HINT: there are certainly many possible proofs, but the one that

is usually quoted proceeds by studying the function $\sqrt{1-x}$ instead. For $0 \leq u \leq 1$, let $\phi(u) := \frac{1}{2}(x+u^2)$, and show that, if $1-x < (1-u)^2$, then $1-x < (1-\phi(u))^2$ and $\phi(u) > u$.)

9. Prove that, for any continuous function $f : [0,1] \longrightarrow \mathbb{C}$, any point $c \in [0,1]$, and any $\epsilon > 0$, there is a polynomial function $P(x)$ such that $P'(c) = P''(c) = 0$ and

$$\sup\{|f(x) - P(x)| : x \in [0,1]\} < \epsilon .$$

10. Prove that, for any continuous function $f : [1,2] \longrightarrow \mathbb{C}$ and any $\epsilon > 0$, there is a polynomial function P consisting entirely of terms of even degree greater than 5 such that

$$\sup\{|f(x) - P(x)| : x \in [1,2]\} < \epsilon .$$

11. Let C be the class of *uniformly* continuous bounded functions $\mathbb{R} \longrightarrow \mathbb{R}$. Show that C is a Banach space with respect to pointwise linear operations and the norm $\|f\| := \sup\{|f(t)| : t \in \mathbb{R}\}$, and an algebra with respect to pointwise multiplication.

12. Let C be as in the previous question. Suppose that A is a subalgebra of C that contains the constant functions and separates the points of $\mathbb{R}$. Does the analogue of Stone's theorem hold in this case — that is, can any function in C be uniformly approximated by elements of A? Justify your answer.