Math 442

Exercise set 1

1. If $(\Omega_1, d_1), (\Omega_2, d_2)$ are metric spaces, $x \in \Omega_1$, and $f : \Omega_1 \longrightarrow \Omega_2$ a map, prove that f is continuous at $x \in \Omega_1$ in the sense of 1.5 if and only if, for any metrically open set U containing $f(x)$, there is a metrically open set V containing x and such that $f(V) \subseteq U$.

Deduce that f is metrically continuous at every point of Ω_1 if and only if $f^{-1}(W)$ is metrically open in Ω_1 for every metrically open set W of Ω_2.

2. *(a)* Let $f : \Omega \longrightarrow \Psi$ be a continuous map between metric spaces, and suppose that K is a sequentially compact subset of Ω. Prove that f is uniformly continuous on A. [Hint: if not, then there is a sequence in A for which a convergent subsequence yields a contradiction.]

(b) Prove similarly that, if $\mathcal{F}$ is a family of maps $\Omega \longrightarrow \Psi$ that is equicontinuous at each point of Ω, then $\mathcal{F}$ is uniformly equicontinuous on K.

3. Prove Theorem 1.15.

4. Let (f_n) be a sequence of functions between metric spaces, $f_n : \Omega \longrightarrow \Psi$. Say that it converges *subuniformly* to $f : \Omega \longrightarrow \Psi$ if

$$(\forall \epsilon > 0)(\forall x \in \Omega)(\exists \delta > 0)(\exists N \in \mathbb{N}) \quad d_\Omega(x, y) < \delta \ \ \& \ \ n \geq N \Longrightarrow d_\Psi(f_n(y), f(y)) < \epsilon \, .$$

[Notice that δ and N depend, in principle, both on x and on ϵ.] Prove that in that case, if each f_n is continuous at $a \in \Omega$, then f is continuous at a.

5. Show that the differential equation $\dfrac{dy}{dx} = y^{2/3}$ with initial condition $y(0) = 0$ has many solutions. (Hint: they are defined by different formulæ on different intervals of $\mathbb{R}$. But there is more than one way of seeing this.) How is this fact compatible with Theorem 2.7?

[This is not entirely a piece of formal nonsense. If you assume that a spherical raindrop accretes water at a rate proportional to its surface area, then its volume should satisfy the equation $\dfrac{dv}{dt} = kv^{2/3}$ for some constant k, which may be solved as in the question. It would be silly to say that this constitutes an “explanation” of raindrop formation, but at least it shows that a very simple-minded mathematical model is not entirely unrealistic.]

6. Let Ω be a topological space, and $f : \Omega \longrightarrow \mathbb{R}$. We say that f is *upper semicontinuous at* $a \in \Omega$ if, for any $\epsilon > 0$, the set $\{x \in \Omega : f(x) < f(a) + \epsilon\}$ is a neighbourhood of a in Ω. If f is upper semicontinuous at each point of the subset A of Ω, we say f is *upper semicontinuous on* A.

Suppose that A is compact and non-empty, and that f is upper semicontinuous on A. Prove that $f(A)$ is bounded above, and there exists $a \in A$ such that $f(a) = \sup A$. [This result is true in the general form stated; but I should be satisfied with a proof when Ω is a metric space and A is sequentially compact in Ω.]

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7. Suppose that $\mathcal{F}$ is a non-empty family of continuous functions $\Omega \longrightarrow [0, \infty)$, where Ω is a topological space. Prove that $g : \Omega \longrightarrow \mathbb{R} : x \mapsto \inf\{f(x) : f \in \mathcal{F}\}$ is upper semi-continuous on Ω.

8. Let (Ω, d) be a metric space. Prove that $d : \Omega \times \Omega \longrightarrow \mathbb{R}$ is uniformly continuous (when $\Omega \times \Omega$ is given the product metric defined in 3.18).

9. *(a)* Give an example of a mapping $f : \Omega \longrightarrow \Omega$ such that Ω is a compact metric space (with metric d) and

$$(\forall x, y \in \Omega) \quad d(f(x), f(y)) \leq d(x, y) ,$$

but f has no fixed point.

(b) Similarly, give an example of a mapping g of a compact metric space Ω into itself such that, whenever $x, y \in \Omega$ and $x \neq y$, then $d(g(x), g(y)) < d(x, y)$ (that is, g is "distance-decreasing"), but g is *not* a contraction mapping.

(c) Prove that, nevertheless, a distance-decreasing mapping of a compact metric space into itself has a unique fixed point. (Hint for one possible proof: consider the function function $x \mapsto d(x, f(x)) : \Omega \longrightarrow \mathbb{R}$. But there are other possibilities.)