MATH 442

Test 2

2008

1. Let $(E, \|\|)$ be a Banach space, and suppose that $[\,,]$ is an inner product on E which is bounded in the sense that, for some positive constant K,

$$(\forall x, y \in E) \quad |[x, y]| \leq K\|x\|\|y\|\,. \tag{1}$$

(The norm $\|\!|\ \|\!|$ induced by $[\,,]$ is not supposed to be the same as $\|\|$). Prove that, if E is a Hilbert space with respect to $[\,,]$, then there exists a positive constant L such that

$$(\forall x \in E) \quad L\|x\| \leq \|\!|x\|\!| \leq \sqrt{K}\,\|x\|\,.$$

Thus $\|\!|\ \|\!|$ and $\|\|$ define the same open sets and the same Cauchy sequences. [Hint: this follows from a Big Theorem, and is more or less impossible otherwise.]

2. In the space $C([0,1])$ with supremum norm $\|\|$, there is an inner product

$$[f, g] := \int_0^1 f(t)\,\overline{g(t)}\,dt\,,$$

where λ denotes Lebesgue measure in $\mathbb{R}$. Show that (1) holds in this case, and give an example of a sequence in $C([0,1])$ which is Cauchy with respect to the norm $\|\!|\ \|\!|$ induced by $[\,,]$ but not with respect to $\|\|$.

3. Let ℓ^∞ denote the Banach space of bounded *real-valued* sequences $(\xi_1, \xi_2, \dots)$ with the supremum norm. For any finite sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ in $\mathbb{N}$, define for $(\xi_n) \in \ell^\infty$

$$M((\xi_n); \alpha_1, \alpha_2, \dots, \alpha_k) := \limsup_{n\to\infty} \frac{1}{k} \sum\nolimits_{i=1}^{k} \xi_{n+\alpha_i}\,.$$

Why does this definition make sense? Next, let

$$p((\xi_n)) := \inf M((\xi_n); \alpha_1, \alpha_2, \dots, \alpha_k)\,,$$

the infimum being taken over all choices of $k \in \mathbb{N}$ and all sequences $\alpha_1, \alpha_2, \dots, \alpha_k$. Again, why does this definition make sense?

Prove that p is a sublinear functional on ℓ^∞.

4. Suppose that $\phi : \ell^\infty \longrightarrow \mathbb{R}$ is a linear functional dominated by the sublinear functional p of the last question. Show that ϕ must have the following properties.

(a) If $\xi_n \geq 0$ for all n, then $\phi((\xi_n)) \geq 0$ [hint: prove that $\phi(-(\xi_n)) \leq 0$, using p].

(b) If $\eta_n := \xi_{n+1}$ for all $n \in \mathbb{N}$, $\phi((\eta_n)) = \phi((\xi_n))$ [consider $(\eta_n) - (\xi_n)$; use *(a)*].

(c) If $\xi_n = 1$ for all n, then $\phi((\xi_n)) = 1$.

5. A metric (or, indeed, topological) space Ω is *separable* if there is a countable subset of Ω that is dense in Ω.

(a) Prove that the (real) sequence spaces ℓ^1 and ℓ^2 are separable as normed spaces.

(b) Show that the sequence space ℓ^∞ is not separable. [Find an uncountable subset of ℓ^∞ such that any two distinct elements are at a distance 1 or greater from each other.]

(c) Let E be a separable normed space, with a countable dense subset $X = \{x_k : k \in \mathbb{N}\}$. Show that the formula

$$d(\phi, \psi) := \sum_{k=1}^{\infty} \frac{2^{-k}|\phi(x_k) - \psi(x_k)|}{1 + |\phi(x_k) - \psi(x_k)|}$$

defines a metric on the closed unit ball $\widehat{B}$ in the dual space E'. (Here ϕ, ψ are linear functionals in $\widehat{B}$.)

(d) Prove that $\widehat{B}$ is sequentially compact with respect to the metric defined by d. [In fact, the d-topology on $\widehat{B}$ is the subspace topology induced by the weak* topology on E', so it does not depend on the choice of the dense countable subset X; it follows that this is a weaker form of the Bourbaki-Alaoglu theorem. Hint: use a diagonal construction — subsequences of subsequences, etc.]

6. Suppose that E and F are normed spaces; let $\phi \in E'$ and $f \in F$. Define a mapping $T : E \longrightarrow F$ by

$$(\forall x \in E) \quad Tx := \phi(x)f\,.$$

Prove that T is a bounded linear mapping $E \longrightarrow F$, and calculate its operator-norm in terms of the given norms in E and in F. Show that T is of rank 1; and prove that, if $A : E \longrightarrow F$ is any bounded linear operator of rank 1, there must exist $g \in F$ and $\psi \in E'$ such that

$$(\forall x \in E) \quad Ax := \psi(x)g\,.$$

To what extent are ψ and g completely determined once A is given?

7. Let S be an orthonormal set in the Hilbert space H. Show that S is an orthonormal basis if and only if, for any $x, y \in H$, the unordered sum $\sum_{a\in S} \langle x, a\rangle\langle a, y\rangle$ is defined and

$$\langle x, y\rangle = \sum_{a\in S} \langle x, a\rangle\langle a, y\rangle\,.$$

(This is sometimes called *Parseval's identity*.)

8. *(a)* Let H be a *complex* Hilbert space, and let A be a bounded linear operator in H. Show that A is self-adjoint if and only if $\langle Ax, x\rangle$ is real for all $x \in H$, and that A is skew-adjoint if and only if $\langle Ax, x\rangle$ is pure imaginary for all $x \in H$.

(b) Show also that, if H is a *real* Hilbert space and A a bounded operator therein, then $\langle Ax, x\rangle = 0$ for all $x \in H$ if and only if A is skew-adjoint.

9. Let $\phi : [a, b] \longrightarrow \mathbb{C}$ be a continuous function. In the Hilbert space $L^2([a, b])$, define an operator T_ϕ by

$$T_\phi(f) := \phi f$$

(pointwise multiplication). This operator is clearly well-defined. Show that it is bounded; find its adjoint; prove it is normal. When is it self-adjoint?

10. A bounded self-adjoint operator A in a Hilbert space H is called *positive* if $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

(a) Show that, if H is *complex*, a bounded operator such that $\langle Ax, x\rangle \geq 0$ for all $x \in H$ must be self-adjoint. Is this still true if H is real?

(b) If A is bounded self-adjoint and positive, then $\langle Ax, y\rangle$ defines a "semi-inner-product" in H. Define this concept, and show that the Cauchy-Schwarz inequality still holds for it. Prove that, for any bounded semi-inner-product $[\,,]$ in H (in the sense of (1)), there is a bounded self-adjoint positive operator A such that $[x, y] = \langle Ax, y\rangle$ for all $x, y \in H$.

(c) Deduce from the Cauchy-Schwarz inequality that, for any bounded self-adjoint positive operator A, $\|A\| = \sup\{\langle Ax, x\rangle : x \in H \ \&\ \|x\| \leq 1\}$.

(d) Prove that, for any bounded self-adjoint operator A, $\sup\{\langle Ax, x\rangle : \|x\| \leq 1\}$ and $\inf\{\langle Ax, x\rangle : \|x\| \leq 1\}$ both belong to the spectrum of A.