MATH 442

Test 1

2008

1. Suppose that (Ω, d) is a complete uncountable metric space, and $f : \Omega \longrightarrow \Omega$ a continuous map. For each $x \in \Omega$, define $f^1(x) := f(x)$, and set inductively $f^{n+1}(x) := f(f^n(x))$ for each $n \in \mathbb{N}$. Suppose that $(f^n(x))_{n=1}^{\infty}$ is convergent in Ω for every $x \in \Omega$.
   (a) Show that f must have at least one fixed point.
   (b) Give an example to show that f need not be a contraction mapping.
   (c) Show that f may have countably infinitely many fixed points. [3,3,4]

2. Let (Ω, d) be a metric space, and A a nonnull subset of Ω. For $x \in \Omega$, define

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

(This is the usual notation. In principle it is deplorable; in practice, it is rarely ambiguous, since subsets of Ω are customarily denoted differently from points.) Show that A is closed if and only if $d(x, A) = 0$ only when $x \in A$. [6]

3. Let $\mathfrak{C}$ denote the class of all nonnull closed bounded subsets of the metric space Ω. Define, for $A_1, A_2 \in \mathfrak{C}$,

$$D(A_1, A_2) := \max\{\sup\{d(x, A_2) : x \in A_1\}, \sup\{d(y, A_1) : y \in A_2\}\}.$$

Prove that D is a metric in $\mathfrak{C}$. (This is the *Hausdorff metric* in $\mathfrak{C}$.) [10]

4. Let Ω be the space whose points are closed circular disks in $\mathbb{R}^2$, of the form

$$C(a; r) := \{x \in \mathbb{R}^2 : \|x - a\| \leq r\}$$

for $r > 0$ and $a \in \mathbb{R}^2$. If D_1, D_2 are two such disks, define the distance $\Delta(D_1, D_2)$ between them to be the area of their symmetric difference $(D_1 \setminus D_2) \cup (D_2 \setminus D_1)$.

Show that Δ is a metric in Ω [you may assume obvious properties of area]. Is (Ω, Δ) a complete metric space? [8,4]

5. Let Ω be as in exercise 4, and let K be the subset of Ω consisting of all the disks of radius exactly 1 that are included in $C(0; 100)$. Show that K is a compact subset of Ω. [10]

6. Prove that any continuous function $f : [0, \infty) \longrightarrow \mathbb{C}$ which vanishes at infinity may be uniformly approximated by linear combinations of the functions $\exp((-m + in)x^2)$, in which m is to be a positive integer and n any integer. [10]

7. Let Ω, Ψ be compact Hausdorff topological spaces. Prove that any continuous function $f : \Omega \times \Psi \longrightarrow \mathbb{R}$ may be uniformly approximated by linear combinations of functions of the form $g(x)h(y)$, where $g : \Omega \longrightarrow \mathbb{R}$ and $h : \Psi \longrightarrow \mathbb{R}$ are continuous.

$\longrightarrow$ ***over***

[You may assume that $\Omega \times \Psi$ is itself compact in the product topology, that continuous functions separate the points of Ω and of Ψ, and (what is almost obvious) that functions such as $g(x)h(y)$ are continuous on $\Omega \times \Psi$. This result can be used to justify the rules about double integrals and repeated integrals, at least for continuous functions.] [10]

8. Let l^1 be the sequence space mentioned in 8.6*(ii)*: its elements are sequences $(\xi_1, \xi_2, \xi_3, \dots)$ in $\mathbb{C}$ such that $\sum_{k=1}^{\infty} |\xi_k|$ converges, and its norm is given by

$$\|(\xi_1, \xi_2, \xi_3, \dots)\| := \sum_{k=1}^{\infty} |\xi_k| .$$

Prove that, for every bounded linear functional $\phi : l^1 \longrightarrow \mathbb{C}$, there is a *bounded* sequence in $\mathbb{C}$, $(\eta_1, \eta_2, \eta_3, \dots)$, such that, for every $x = (\xi_1, \xi_2, \xi_3, \dots) \in l^1$,

$$\phi(x) = \sum_{k=1}^{\infty} \eta_k \xi_k . \qquad [12]$$

9. Let Ω be a compact Hausdorff topological space, and give $C := C(\Omega; \mathbb{R})$ the supremum norm. If $f \in C$, say that f is non-negative, or $f \geq 0$, if $(\forall x \in \Omega)\ f(x) \geq 0$.

A linear functional $\phi : C \longrightarrow \mathbb{R}$ is *positive* ["non-negative" would be better] if $\phi(f) \geq 0$ for every non-negative f. Prove that

(a) a positive linear functional is necessarily continuous, [6]

(b) a continuous linear functional is the difference of two positive linear functionals. [HINT: given a continuous linear functional ϕ, define, for a nonnegative function f,

$$\phi^+(f) := \sup\{\phi(g) : 0 \leq g \leq f\} .$$

Show that ϕ^+ extends to a positive linear functional on C, and that $\phi^+ - \phi$ is also a positive linear functional. Assume there are "plenty" of continuous maps $\Omega \longrightarrow \mathbb{R}$.] [14]