# Math 442

## Exercise set 6, 2008 — sketch solutions

1. For $n = 0$, we understand $\dfrac{d^0}{dx^0}$ as the "zeroth power of the differentiation" i.e. as the identity operator: $P_0(x) = 1$ (doing nothing to the constant polynomial 1). For $n = 1$,

$$P_1(x) = \frac{1}{2.1!}\frac{d}{dx}(x^2 - 1) = x\,.$$

For $n = 2$,

$$P_2(x) = \frac{1}{4.2!}\frac{d^2}{dx^2}(x^2 - 1)^2 = \frac{1}{8}\frac{d}{dx}\{4x(x^2 - 1)\} = \tfrac{1}{2}(3x^2 - 1)\,.$$

It is clear that the calculations rapidly become complicated.

2. Set $m = n + k$, where $k \in \mathbb{N}$. For the appropriate $\alpha > 0$

$$\begin{aligned}
\alpha\int_{-1}^{1} P_{n+k}(x)\, P_n(x)\, dx &= \int_{-1}^{1} \frac{d^{n+k}}{dx^{n+k}}\{(x^2-1)^{n+k}\}\, \frac{d^n}{dx^n}\{(x^2-1)^n\}\, dx \\
&= \left[\frac{d^{n+k-1}}{dx^{n+k-1}}\{(x^2-1)^{n+k}\}\, \frac{d^n}{dx^n}\{(x^2-1)^n\}\right]_{-1}^{1} \\
&\qquad - \int_{-1}^{1} \frac{d^{n+k-1}}{dx^{n+k-1}}\{(x^2-1)^{n+k}\}\, \frac{d^{n+1}}{dx^{n+1}}\{(x^2-1)^n\}\, dx\,.
\end{aligned}$$

In the first expression (the evaluation bracket) on the right, apply Leibniz's product formula for higher derivatives of a product. Indeed, for any $l$ such that $0 \le l < n + k$,

$$\begin{aligned}
&\frac{d^l}{dx^l}\{(x-1)^{n+k}(x+1)^{n+k}\} \\
&\quad = \sum\nolimits_{p=0}^{l} \binom{l}{p} \frac{d^p}{dx^p}(x-1)^{n+k}\, \frac{d^{l-p}}{dx^{l-p}}(x+1)^{n+k} \\
&\quad = \sum\nolimits_{p=0}^{l} \binom{l}{p} \frac{(n+k)!}{(n+k-p)!}(x-1)^{n+k-p}\, \frac{(n+k)!}{(n+k-l+p-1)!}(x+1)^{n+k-l+p}\,, \qquad (1)
\end{aligned}$$

so $(x-1)(x+1)$ still divides the derivative. Hence the evaluation bracket vanishes, and

$$\begin{aligned}
\alpha\int_{-1}^{1} P_{n+k}(x)\, P_n(x)\, dx &= -\int_{-1}^{1} \frac{d^{n+k-1}}{dx^{n+k-1}}\{(x^2-1)^{n+k}\}\, \frac{d^{n+1}}{dx^{n+1}}\{(x^2-1)^n\}\, dx \\
&= \int_{-1}^{1} \frac{d^{n+k-2}}{dx^{n+k-2}}\{(x^2-1)^{n+k}\}\, \frac{d^{n+2}}{dx^{n+2}}\{(x^2-1)^n\}\, dx
\end{aligned}$$

$$= (-1)^{n+1} \int_{-1}^{1} \frac{d^{n+k-n-1}}{dx^{n+k-n-1}} \{(x^2-1)^{n+k}\} \, \frac{d^{2n+1}}{dx^{2n+1}} \{(x^2-1)^n\} \, dx \,,$$

which vanishes. On the other hand, if $k = 0$ the same argument (minus the very last step) still works, and (again letting $\beta$ be the appropriate constant)

$$\beta \int_{-1}^{1} P_n(x)^2 \, dx = (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}} \{(x^2-1)^n\} \, dx$$
$$= (2n)! \int_{-1}^{1} (1-x^2)^n \, dx \,.$$

It is not difficult to find a reduction formula for this integral, but we may exploit a known formula instead by substituting $x = \sin\theta$, $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\theta$. Then we get

$$(2n)! \int_{-1}^{1} (1-x^2)^n \, dx = (2n)! \int_{-\pi/2}^{\pi/2} \cos^{2n+1}\theta \, d\theta = \frac{(2n)!\, 2n}{2n+1} \int_{-\pi/2}^{\pi/2} \cos^{2n-1}\theta \, d\theta$$
$$= \frac{(2n)!\, 2n.(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{(2^n n!)^2}{2n+1}.2 \,.$$

As $\beta = (2^n n!)^2$ from the original definition of $P_n$, we deduce the stated result that

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1} \,.$$

[Incidentally, if one takes $l = n$ and $k = 0$ in (1), one finds that

$$\frac{d^n}{dx^n} \{(x^2-1)^n\} \Big|_{x=1} = n!(1+1)^n \,, \quad \frac{d^n}{dx^n} \{(x^2-1)^n\} \Big|_{x=-1} = n!(-1-1)^n \,,$$

from which $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.] (2)

3. Let $E$ be the vector subspace of $H := L^2([-1,1];\lambda)$ spanned by the Legendre polynomials. It is clear that $E$ is also the subspace spanned by all polynomials; indeed, if a polynomial of degree $n$ is a linear combination of $P_0, P_1, \ldots, P_n$, then, as $P_{n+1}$ is of degree exactly $n+1$, $x^{n+1}$ is a combination of $P_{n+1}$ and a polynomial of degree $n$, and therefore is a combination of $P_0, P_1, \ldots, P_{n+1}$. (Indeed, the *normalized* Legendre polynomials $\sqrt{n+\frac{1}{2}}\, P_n(x)$ result from applying the Gram-Schmidt procedure in $H$ to the sequence $1, x, x^2, \ldots$ ). But, by Weierstrass's theorem, the polynomials are dense in $C([-1,1])$. Thus $E$ is dense in $C([-1,1])$ in the supremum norm. The assumption in 17.16 was that $C([-1,1])$ is dense in $L^2([-1,1])$ in $L^2$-norm; hence, linear combinations of Legendre polynomials are also dense in $H$. However, this is precisely what we need to ensure that the Legendre polynomials form an orthonormal basis, i.e. a complete orthonormal set.

4. *(a)* If $0 \le m < n-1$, then, by the remarks in question 3, $xP_m(x)$, being a polynomial of degree $m+1$, is a linear combination of $P_0, P_1, \ldots, P_{m+1}$. Each of these is orthogonal to $P_n$, by question 2. So $\int_{-1}^{1} xP_n(x)\, P_m(x) \, dx = 0$.

If $m = n-1$, there are a coefficient $\gamma$ and a polynomial $Q$ of degree $n-1$ or less with

$$xP_{n-1}(x) = \gamma P_n(x) + Q(x)\,.$$

$Q$ is orthogonal to $P_n$, so that

$$\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \int_{-1}^{1} \gamma P_n(x)^2\,dx = \frac{2\gamma}{2n+1}\,. \tag{3}$$

However, $\gamma$ may be determined by looking at the coefficients of $x^n$ in $P_n$ and in $xP_{n-1}$:

$$\frac{1}{2^{n-1}(n-1)!}\,\frac{(2(n-1))!}{(n-1)!} = \gamma\frac{1}{2^n n!}\,\frac{(2n)!}{n!}\,, \qquad \gamma = \frac{n}{2n-1}\,. \tag{4}$$

The desired result follows from (3) and (4).

*(b)* Similarly, if $n \le m$, $P_n'$ is of degree $n-1 < m$, and the integral is $0$. If $n > m$, then

$$\begin{aligned}\int_{-1}^{1} P_m(x)\,P_n'(x)\,dx &= \Big[P_n(x)\,P_m(x)\Big]_{-1}^{1} - \int_{-1}^{1} P_m'(x)\,P_n(x)\,dx \\ &= 1-(-1)^{m+n}-0 \quad \text{by (2) and the last remark.}\end{aligned}$$

*(c)* If $n < m$, then $xP_n'(x)$ is of degree $n$ and the integral is $0$. If $n > m$, then

$$\begin{aligned}\int_{-1}^{1} xP_m(x)\,P_n'(x)\,dx &= \Big[xP_m(x)P_n(x)\Big]_{-1}^{1} - \int_{-1}^{1} (xP_m'(x)+P_m(x))\,P_n(x)\,dx \\ &= 1+(-1)^{m+n} - \int_{-1}^{1} xP_m'(x)\,P_n(x)\,dx = 1+(-1)^{m+n}\,.\end{aligned}$$

But, if $m = n$, then

$$\begin{aligned}\int_{-1}^{1} xP_n(x)\,P_n'(x)\,dx &= \Big[\tfrac{1}{2}x(P_n(x))^2\Big]_{-1}^{1} - \int_{-1}^{1} \tfrac{1}{2}(P_n(x))^2\,dx \\ &= 1-\frac{1}{2n+1} = \frac{2n}{2n+1} \quad \text{from question 2.}\end{aligned}$$

5. *(a)* Both sides are of degree $n+1$. If $k < n-1$, questions 2 and 4 show that both the left-hand and the right-hand sides are orthogonal to $P_k$. If $k = n-1$, then

$$\begin{aligned}&\int_{-1}^{1} \{(2n+1)xP_n(x) - nP_{n-1}(x)\}P_{n-1}(x)\,dx \\ &\quad = \frac{2n}{2n-1} - n.\frac{2}{2(n-1)+1} = 0 = \int_{-1}^{1} (n+1)P_{n+1}(x)\,P_{n-1}(x)\,dx\,.\end{aligned}$$

If $k = n$, notice that, directly from Rodrigues' formula, $P_n(x)$ is an even function when $n$ is even and an odd function when $n$ is odd. Thus $\int_{-1}^{1} xP_n(x)^2\,dx = 0$ for all $n$, and

$$\int_{-1}^{1} \{(2n+1)xP_n(x) - nP_{n-1}(x)\} P_n(x)\,dx = 0\,.$$

Finally, $\int_{-1}^{1} \{(n+1)P_{n+1}(x)\} P_{n+1}(x)\,dx = \dfrac{2(n+1)}{2n+3}$ by question 2, whilst by question 4

$$\int_{-1}^{1} \{(2n+1)xP_n(x) - nP_{n-1}(x)\} P_{n+1}(x)\,dx$$
$$= (2n+1)\frac{2(n+1)}{(2n+3)(2n+1)} - n.0 = \frac{2(n+1)}{2n+3}\,.$$

The two sides have the same inner product with every member of an orthonormal basis of the space of polynomials of degree $n+1$. They must, therefore, be the same.

*(b)* Again, the two sides are both of degree $n$. If $m \le n-1$, question 2 shows that

$$\int_{-1}^{1} nP_n(x)\,P_m(x) = 0\,,$$

whilst, by 4*(c)* and 4*(b)*, if $m < n-1$,

$$\int_{-1}^{1} xP_n'(x)\,P_m(x)\,dx = 1 + (-1)^{m+n}\,, \quad \int_{-1}^{1} P_{n-1}'(x)\,P_m(x)\,dx = 1 - (-1)^{m+n-1}\,,$$

which implies that $\int_{-1}^{1} (xP_n'(x) - P_{n-1}'(x))\,P_m(x)\,dx = 0$. If $m = n-1$, then

$$\int_{-1}^{1} P_{n-1}'(x)\,P_{n-1}(x)\,dx = 0 \qquad \text{as } P_{n-1}' \text{ is of degree } n-2\,,$$
$$\int_{-1}^{1} xP_{n-1}(x)\,P_n'(x)\,dx = 1 + (-1)^{2n-1} = 0 \quad \text{by 4}(c).$$

Lastly, if $m = n$, question 2 tells us that $\int_{-1}^{1} nP_n(x)\,P_n(x)\,dx = \dfrac{2n}{2n+1}$. But by 4*(c)*

$$\int_{-1}^{1} xP_n'(x)\,P_n(x)\,dx = \frac{2n}{2n+1}\,,$$

whilst $\int_{-1}^{1} P_{n-1}'(x)\,P_n(x)\,dx = 0$ as the degree of $P_{n-1}'$ is $n-2$.

Hence, as in *(a)*, the two polynomials of degree $n$ on the two sides of the supposed equality must indeed be the same.

*(c)* This equality may be proved in the same way, but it is easier to argue as follows. Take $n = 1$; then, as asserted,

$$nP_{n-1}(x) = P_0(x) = 1 = 1 - xP_0'(x) = P_n'(x) - xP_{n-1}'(x)\,.$$

For $n \ge 1$, we know $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$, and we may differentiate *(a)*:

$$(n+1)P_{n+1}'(x) = (2n+1)P_n(x) + (2n+1)xP_n'(x) - nP_{n-1}'(x)\,; \tag{5}$$

from these facts, we deduce

$$\begin{aligned}(n+1)(P'_{n+1} - xP'_n) &= nxP'_n + (2n+1)P_n - nP'_{n-1} \\ &= n(nP_n + P'_{n-1}) + (2n+1)P_n - nP'_{n-1} \\ &= (n^2 + 2n + 1)P_n = (n+1)^2 P_n\,,\end{aligned}$$

and so $P'_{n+1}(x) - xP'_n(x) = (n+1)P_n$. This proves that *(c)* holds for the index $n+1$.

[Incidentally, the algebra reverses, so that "*(c)*$_{n+1}$" and "*(b)*$_n$" imply "*(a)*$'_n$".] (6)

*(d)* Eliminate $P'_{n-1}(x)$ between *(b)* and *(c)*.

6. Differentiate 5*(d)*, and then use 5*(b)* to eliminate $P'_{n-1}$:

$$\begin{aligned}(x^2-1)P''_n(x) + 2xP'_n(x) &= nxP'_n(x) + nP_n(x) - nP'_{n-1}(x) \\ &= nxP'_n(x) + nP_n(x) + n^2P_n(x) - nxP'_n(x) \\ &= n(n+1)P_n(x)\,.\end{aligned}$$

Thus, $P_n(x)$ does satisfy the stated differential equation. [This equation, with $\cos\theta$ as the variable $x$, arises from separation of variables for Laplace's equation in spherical polars.]

7. For small $h$ (how small, will depend on $x$; for instance, if $|2xh| + |h|^2 < 1$), $(1 + (-2xh + h^2))^{-1/2}$ may be expanded as a series in powers of $(-2xh + h^2)$ by the binomial theorem (for non-integer and negative exponent! — it is not a trivial result), and the powers of $(-2xh + h^2)$ in turn may be expanded by the "elementary" binomial theorem. The result will be a series of polynomials in the two variables $x$ and $h$. If $h$ is small, it may be rearranged by collecting together powers of $h$:

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} Q_n(x)h^n\,. \tag{7}$$

[When I say the series may be rearranged, I mean that the expression on the right is a convergent power series in $h$ for, let us say, $-2 \le x \le 2$ and for sufficiently small values of $h$, and the equality is then a a genuine equality. It is also true that it may be differentiated term-by-term with respect to $x$ or with respect to $h$. All these facts will be taken for granted below; they were *roughly* covered by one of the MATH 312 handouts.]

Differentiate (7) with respect to $h$:

$$-\tfrac{1}{2}(2h - 2x)(1 - 2xh + h^2)^{-3/2} = \sum_{n=0}^{\infty} nQ_n(x)h^{n-1}\,, \quad \text{and from (7)}$$
$$(x-h)\sum_{n=0}^{\infty} Q_n(x)h^n = (1 - 2xh + h^2)\sum_{n=0}^{\infty} nQ_n(x)h^{n-1}\,.$$

Equating coefficients of $h^n$ (for $n \ge 1$) one finds that

$$xQ_n(x) - Q_{n-1}(x) = (n+1)Q_{n+1}(x) - 2nxQ_n(x) + (n-1)Q_{n-1}(x)\,, \tag{8}$$

which is exactly the relation 5*(a)* for the polynomials $Q_n$ instead of $P_n$. However,

$$(1 - 2xh + h^2)^{-1/2} = 1 - \tfrac{1}{2}(-2x + h)h + \cdots,$$

all the omitted terms being of degree at least 2 in $h$, and therefore from question 1,

$$Q_0(x) = 1 = P_0(x)\,, \quad Q_1(x) = x = P_1(x)\,.$$

The relations 5*(a)* and (8) now prove inductively that $P_n(x) = Q_n(x)$ for all $n$.

[This at first sight strange fact becomes more transparent in view of the Laplace equation.]

8. *(a)* Notice, first of all, that $q_0, q_1, q_2, \ldots, q_n$ (being linearly independent) span the space of real polynomials of degree not exceeding $n$.

Being of degree $n$, $q_n$ has $n$ zeros in all (including non-real zeros, and counting multiplicities).

Suppose that a real polynomial $q(x)$ has (real) roots $c_1, c_2, \ldots, c_r$ in $(a, b)$, where the multiplicity of $c_k$ is $m(k)$, and then the number of zeros of $q$ in $(a, b)$, "counted according to multiplicity", is $M(k) := \sum_{k=1}^{r} m(k)$.

Let $\epsilon(k)$ be the *parity* of $m(k)$; that is, $\epsilon(k) = 1$ if $m(k)$ is odd, and $\epsilon(k) = 0$ if $m(k)$ is even. (Thus $\epsilon(k) = m(k)$ only if $m(k) = 1$, and otherwise $\epsilon(k) \le m(k) - 2$.) The real polynomial

$$h(x) := \prod_{k=1}^{r} (x - c_k)^{\epsilon(k)} \tag{9}$$

is of degree not exceeding $r$, and strictly less than $M(k) - 1$ unless $\epsilon(k) = m(k)$ for each $k$, i.e. each zero of $q$ in $(a, b)$ is of multiplicity 1, or "simple".

Apply this construction to $q := q_n$. If $M(k) < n$, or, alternatively, if $M(k) = n$ but one or more of the zeros $c_1, c_2, \ldots c_r$ is of multiplicity greater than 1, the degree of $h(x)$ will be strictly less than $n$. $h(x)$ is therefore a linear combination of $q_0, q_1, \ldots, q_r$, and, consequently,

$$\int_a^b q_n(x)\, h(x)\, w(x)\, dx = 0\,. \tag{10}$$

On the other hand, at any zero $c_k$ of $q_n$, $h(x)$ either does not have a root (if $m(k)$ is even) or has a root of multiplicity 1 (if $m(k)$ is odd). In either case, $q_n(x)\, h(x)$ does not change sign at $c_k$ (which is a zero of *even* multiplicity of $q_n(x)\, h(x)$). Hence, it has the same sign throughout $(a, b)$, and

$$\int_a^b q_n(x)\, h(x)\, w(x)\, dx \ne 0\,.$$

In view of (10), this is absurd. So $q_n$ must have $n$ (real) zeros in $(a, b)$, and they must all be of multiplicity 1, so that there are $n$ *distinct* real zeros in $(a, b)$. This proves *(a)*.

*(b)* For any polynomial $h(x)$ of degree less than $n$, as at (10)

$$\int_a^b (q_{n+1}(x) - \alpha q_n(x))\, h(x)\, w(x)\, dx = 0\,. \tag{11}$$

As in *(a)*, if $q_{n+1}(x) - \alpha q_n(x)$ has fewer than $n$ (real) zeros (counting multiplicities) in $(a, b)$, or has exactly $n$ zeros there not all of multiplicity 1, there is a real polynomial $h(x)$ of degree less than $n$ such that $(q_{n+1}(x) - \alpha q_n(x))h(x)$ preserves the same sign throughout $(a, b)$; and this contradicts (11). So there are *at least* $n$ zeros in $(a, b)$ (counting multiplicities), and if there are *no more* than $n$, they are all of multiplicity 1. In this case there must be one more (complex) zero of $q_{n+1}(x) - \alpha q_n(x)$; it must be real (as non-real roots of a real polynomial must occur in conjugate complex pairs) and outside $(a, b)$, so that there are in all $n + 1$ distinct real roots.

What still remains to be shown is that, if $q_{n+1}(x) - \alpha q_n(x)$ has $n+1$ real zeros (counting multiplicity) in $(a,b)$, they must all be distinct, i.e. of multiplicity 1. Suppose not. Apply the construction (9) to the polynomial $q := q_{n+1} - \alpha q_n$. $h(x)$ is of degree at least 2 less than $n+1$, and so of degree less than $n$; but, as before, the integrand in (11) preserves the same sign throughout $(a,b)$, which is impossible. So the proof of *(b)* is complete.

*(c)* Suppose that $\beta\, q_{n+1} - \alpha q_n$ and $\beta' q_{n+1} - a' q_n$ have a common root $\xi$, where $\alpha\beta' \neq \alpha'\beta$; then $\xi$ is a common root of $q_{n+1}$ and $q_n$. Since it is a simple root of (both, but particularly of) $q_n$, $q_n'(\xi) \neq 0$, and $\xi$ is a repeated root of $q_{n+1} - (q_{n+1}'(\xi)/q_n'(\xi))q_n$. This contradicts *(b)*. Therefore, the sets of roots of $\beta q_{n+1} - \alpha q_n$ for linearly independent vectors $(\alpha, \beta)$ are always disjoint from each other.

Hence, if $q_{n+1}(\xi) - \alpha q_n(\xi) = 0$, for any $\alpha$, then $q_n(\xi) \neq 0$, and

$$\frac{\partial}{\partial \alpha}(q_{n+1}(x) - \alpha q_n(x))\Big|_{(\alpha,\xi)} \neq 0\,.$$

By the Implicit Function Theorem, there is a small neighbourhood $W$ of $\alpha$ and there is a C$^\infty$ function $\xi(\tau)$ on $W$ such that, for $\tau \in W$, $\xi(\tau)$ is the only root of $q_{n+1}(x) - \tau q_n(x)$ in a sufficiently small neighbourhood of $\xi$. ****

9. Notice that

$$\exp(-(t-x)^2) = \sum\nolimits_{n=0}^{\infty} \frac{\exp(-x^2)H_n(x)}{n!} t^n\,. \tag{12}$$

Using general theorems on convergent power series, we find that

$$\exp(-x^2)H_n(x) = \frac{\partial^n}{\partial t^n}\exp(-(t-x)^2)\Big|_{t=0}\,.$$

However, if we write $u := t - x$, then by the chain rule $\dfrac{\partial}{\partial u} = \dfrac{\partial}{\partial t} = -\dfrac{\partial}{\partial x}$. Thus, in fact,

$$\exp(-x^2)H_n(x) = (-1)^n \frac{\partial^n}{\partial x^n}\exp(-(t-x)^2)\Big|_{t=0} = (-1)^n \frac{d^n}{dx^n}\exp(-x^2)\,, \tag{13}$$

since the differentiation does not involve $t$. Notice now that $H_n$ is of degree $n$.

We can now proceed very much as in question 2. Suppose that $m \geq n$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} H_m(x)\, H_n(x) \exp(-x^2)\, dx &= (-1)^m \int_{-\infty}^{\infty} \frac{d^m}{dx^m}(\exp(-x^2))\, H_n(x)\, dx \\ &= (-1)^m \left[\frac{d^{m-1}}{dx^{m-1}}(\exp(-x^2))\, H_n(x)\right]_{-\infty}^{\infty} - (-1)^m \int_{-\infty}^{\infty} \frac{d^{m-1}}{dx^{m-1}}(\exp(-x^2))\, H_n'(x)\, dx \\ &= (-1)^{m+1} \int_{-\infty}^{\infty} \frac{d^{m-1}}{dx^{m-1}}(\exp(-x^2))\, H_n'(x)\, dx\,.\end{aligned}$$

These integrals are Cauchy-Riemann integrals. Strictly speaking, they are limits of integrals over bounded intervals. As each integrand is the product of a polynomial and $\exp(-x^2)$, all the integrals are absolutely convergent. Similarly, each of the evaluation brackets is really a limit; but the evaluand, being again a polynomial times $\exp(-x^2)$, tends to zero both as the upper limit of the evaluation tends to $+\infty$ and as the lower limit tends to $-\infty$. The integration by parts may be repeated in the same way, until we reach

$$(-1)^{m+n}\int_{-\infty}^{\infty}\frac{d^{m-n}}{dx^{m-n}}(\exp(-x^2))\,H_n^{(n)}(x)\,dx\,. \tag{14}$$

If $m > n$, we can go one step further, and, as $H_n$ is of degree $n$, at this stage the integrand will be zero. This proves the orthogonality relation: if $m > n$,

$$\int_{-\infty}^{\infty} H_m(x)\,H_n(x)\exp(-x^2)\,dx = 0\,.$$

If, on the other hand, $m = n$, (14) tells us that

$$\int_{-\infty}^{\infty}(H_n(x))^2\exp(-x^2)\,dx = \int_{-\infty}^{\infty} H_n^{(n)}(x)\exp(-x^2)\,dx = c_n\int_{-\infty}^{\infty}\exp(-x^2)\,dx\,,$$

where $c_n$ is the constant value of $H_n^{(n)}$ and (as is well-known) the integral is $\sqrt{\pi}$. Now $c_n$ will be $n!$ times the highest coefficient in $H_n(x)$. Let $\dfrac{d^k}{dx^k}(\exp(-x^2)) = p_k(x)\exp(-x^2)$; then $p_k(x)$ is a polynomial, and it is clear that the highest power of $x$ in $p_{k+1}(x)$ will arise from multiplying the highest power in $p_k(x)$ by $-2x$, the other term in the product rule yielding lower powers. Hence $c_n = (-1)^n(-2)^n n! = 2^n n!$, and the desired result follows.

Differentiate (12) with respect to $t$:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\exp(-x^2)H_n(x)}{n!}nt^{n-1} &= -2(t-x)\exp(-(t-x)^2)\\ &= -2(t-x)\sum_{n=0}^{\infty}\frac{\exp(-x^2)H_n(x)}{n!}t^n\,.\end{aligned}$$

Cancel $\exp(-x^2)$ (which could have been omitted anyway) and equate coefficients of $t^n$:

$$\frac{H_{n+1}(x)}{n!} = 2x\frac{H_n(x)}{n!} - 2\frac{H_{n-1}(x)}{(n-1)!} \quad \text{for } n \geq 1\,.$$

This is the first relation demanded. (If $n = 0$, the last term does not appear. Indeed, $H_0(x) = 1$, $H_1(x) = 2x$.)

Differentiate (12) with respect to $x$:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\exp(-x^2)(H_n'(x) - 2xH_n(x))}{n!}t^n &= 2(t-x)\exp(-(t-x)^2)\\ &= 2(t-x)\sum_{n=0}^{\infty}\frac{\exp(-x^2)H_n(x)}{n!}t^n\,,\end{aligned}$$

and, again equating coefficients, for $n \geq 1$

$$\frac{H_n'(x) - 2xH_n(x)}{n!} = \frac{2H_{n-1}(x)}{(n-1)!} - \frac{2xH_n(x)}{n!}\,,$$

from which the second stated relation follows trivially.

10. The normalization that is needed is manifestly to take instead of $H_n$

$$\widehat{H}_n(x) = \frac{H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \,,$$

and then $\{\widehat{H}_n : n \in \mathbb{N}\}$ is an orthonormal set in the Hilbert space.