Math 442

Exercise set 4 — sketch solutions

1. Take a sequence $(x^{(n)})_n := ((\xi_i^{(n)})_i)_n$, where each $x^{(n)} \in c_0$ and the sequence is Cauchy in c_0. This means that, for given $\epsilon > 0$, there is some $N(\epsilon)$ such that $m, n \geq N(\epsilon) \Longrightarrow \sup\left\{\left|\xi_i^{(m)} - \xi_i^{(n)}\right| : i \in \mathbb{N}\right\} < \epsilon$. For each specific index i,

$$m, n \geq N(\epsilon) \Longrightarrow \left|\xi_i^{(m)} - \xi_i^{(n)}\right| < \epsilon\,,$$

so that $(\xi_i^{(m)})_m$ is a numerical Cauchy sequence, with limit ξ_i. Define $x := (\xi_i)$. We show $x \in c_0$. First, let $m \to \infty$, to deduce that for each particular i

$$n \geq N(\epsilon) \Longrightarrow \left|\xi_i - \xi_i^{(n)}\right| \leq \epsilon\,. \tag{1}$$

For any $\epsilon > 0$, take $k := N(\frac{1}{2}\epsilon)$. Then, as $x^{(k)} \in c_0$, there exists M (depending on k and on ϵ, and so on ϵ) such that $i \geq M \Longrightarrow \left|\xi_i^{(k)}\right| < \frac{1}{2}\epsilon$. Hence, if $i \geq M$, (8) yields

$$|\xi_i| \leq \left|\xi_i^{(k)}\right| + \left|\xi_i - \xi_i^{(k)}\right| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon\,.$$

This shows that $\xi_i \to 0$ as $i \to \infty$, so that $x \in c_0$. And now (8) shows

$$n \geq N(\epsilon) \Longrightarrow \sup\left\{\left|\xi_i - \xi_i^{(n)}\right| : i \in \mathbb{N}\right\} \leq \epsilon\,,$$

which means that $x^{(n)} \to x$ in c_0 as $n \to \infty$.

2. Let $\|\|_0$ denote the norm in c_0 and $\|\|_1$ the norm in l^1. If $(\xi_n) \in c_0$, then, for each specific index k, $|\xi_k| \leq \|(\xi_n)\|_0 := \sup|\xi_n|$. Thus, $\sum|\xi_n\eta_n|$ converges by comparison with $\sum|\eta_n|$, so that $\phi((\xi_n))$ makes sense. It is clearly a linear functional on c_0. Moreover,

$$\left|\sum \xi_n\eta_n\right| \leq \sum|\xi_n\eta_n| \leq \sum \|(\xi_n)\|_0|\eta_n| = \|(\xi_n)\|_0\|(\eta_n)\|_1\,, \tag{2}$$

for any $(\xi_n) \in c_0$ and $(\eta_n) \in l^1$ .This shows that ϕ is a bounded linear functional on c_0, and that its norm in c_0' does not exceed $\|(\eta_n)\|_1$. Let us define $\Psi : l^1 \longrightarrow c_0'$ by the formula (2). We have just shown that Ψ is defined, and that $\|\Psi((\eta_n))\|_0' \leq \|(\eta_n)\|_1$ for any $(\eta_n) \in l^1$, where $\|\|_0'$ denotes the dual norm in c_0'. It is obvious that Ψ is a linear map $l^1 \longrightarrow c_0'$. What we have to show is that it is *isometric* and *surjective*.

Suppose that $\psi \in c_0'$. Define $e_n \in c_0$ to be the sequence whose terms are all zero except for the nth, which is 1, and let $\eta_n := \psi(e_n)$ for each n. If $\eta_n \neq 0$, let $\theta_n := \overline{\eta}_n/|\eta_n|$; otherwise, let $\theta_n := 1$. Define $t_N := \sum_{n=1}^{N}\theta_n e_n \in c_0$, that is, the sequence whose terms are θ_n for $1 \leq n \leq N$ and 0 thereafter.Then $\|t_N\|_0 = 1$ for every $N \in \mathbb{N}$, and

$$\sum_{n=1}^{N}|\eta_n| = \sum_{n=1}^{N} \theta_n\eta_n = \sum_{n=1}^{N} \theta_n\psi(e_n) = \psi(t_N)\,,$$

which is, therefore, real and non-negative; however, as ψ is a *bounded* linear functional,

$$\psi(t_N) = |\psi(t_N)| \le \|\psi\|_0'\|t_N\|_0 = \|\psi\|_0'$$

for all n. Thus, $\sum_{n=1}^{N}|\eta_n| \le \|\psi\|_0'$ for all N. It follows that $\sum_{n=1}^{\infty}|\eta_n|$ converges, so that $(\eta_n) \in l^1$, and $\|(\eta_n)\|_1 \le \|\psi\|_0$. I claim that $\phi := \Psi((\eta_n)) = \psi$.

Suppose $x := (\xi_n) \in c_0$. For $N \in \mathbb{N}$, let $x_N := \sum_{n=1}^{N} \xi_n e_n$. Then

$$\|x - x_N\|_0 = \sup\{|\xi_n| : n > N\} \to 0$$

as $N \to \infty$, by the definition of c_0. Thus $x_N \to x$ as $N \to \infty$. But

$$\phi(x_N) = \sum_{n=1}^{N} \xi_n\eta_n = \sum_{n=1}^{N} \xi_n\psi(e_n) = \psi\Big(\sum_{n=1}^{N} \xi_n e_n\Big) = \psi(x_N)$$

for each N, and both ψ (by hypothesis) and ϕ (by (b)) are continuous; hence

$$\phi(x) = \lim_{N\to\infty} \phi(x_N) = \lim_{N\to\infty} \psi(x_N) = \psi(x)\,.$$

This proves that Ψ is surjective, since ψ is the image under Ψ of (η_n). We saw at the end of the last paragraph that $\|(\eta_n)\|_1 \le \|\psi\|_0$, and at the end of the first paragraph that $\|\phi\|_0' \le \|(\eta_n)\|_1$. It follows that Ψ is, in fact, an isometry. This implies that it is one-to-one, and, as it is surjective, it is an isometric isomorphism.

3. Take $\eta_n := 2^{-n}$. Then $(\eta_n) \in l^1$ and $\|(\eta_n)\|_1 = 1$. But then, for any non-zero $x := (\xi_n) \in c_0$, $|\sum_{n=1}^{\infty}\xi_n\eta_n| \le \sum 2^{-n}|\xi_n|$, and this must be less than $\|x\|_0 = \sup|\xi_n|$; indeed, the only way it could equal $\|x\|_0$ would be if $|\xi_n| = \|x\|_0$ for all n, which is impossible because $|\xi_n| \to 0$ as $n \to \infty$. [All that is required is that infinitely many terms of (η_n) should be non-zero. The question is, in effect, whether the dual norm in c_0' is "attained" on the unit sphere in c_0, and our conclusion is that the linear functionals which attain their norm on the sphere are those which vanish on all but finitely many e_n. 13.2 says, by contrast, that any vector in a normed space E attains its norm on the unit sphere in E'.]

4. If $x := (\xi_n) \in l^1$ and $\phi := (\eta_n) \in l^\infty$, the sum $\sum_{n=1}^{\infty} \xi_n\eta_n$ converges absolutely by comparison with $\sum|\xi_n| \sup|\eta_n|$, and $|\sum \xi_n\eta_n| \le \sum|\xi_n| \sup|\eta_n| = \|\phi\|_\infty\|x\|_1$, where $\|\|_\infty$ denotes the norm in l^∞. So (2) defines a bounded linear mapping $\Phi : l^\infty \longrightarrow (l^1)'$, and its operator-norm $\|\Phi\|$ (with respect to $\|\|_\infty$ and the dual norm $\|\|_1'$ in $(l^1)'$) does not exceed 1. This is much as in question 2. Let e_n be defined as before, and note that $e_n \in l^1$.

Suppose that $\psi \in (l^1)'$, and as before define $\eta_n := \psi(e_n)$ for each n, $\phi := (\eta_n)$. For given $x = (\xi_n) \in l^1$, and any $N \in \mathbb{N}$, define $x_N := (\xi_1, \xi_2, \dots, \xi_N, 0, 0, 0, \dots)$:

$$x_N = \sum_{n=1}^{N} \xi_n e_n\,.$$

Hence, $\|x_N - x\| = \sum_{n=N+1}^{\infty}|\xi_n| \to 0$ as $N \to \infty$. ψ being continuous, $\psi(x_N) \to \psi(x)$ as $N \to \infty$. But $\psi(x_N) = \sum_{n=1}^{N} \xi_n\eta_n$; so, for any $x = (\xi_n) \in l^1$,

$$\sum \xi_n\eta_n \text{ converges} \quad \text{and} \quad \psi(x) = \sum \xi_n\eta_n\,. \tag{3}$$

Since $\|e_n\|_1 = 1$ in ℓ^1 for each n, and ψ is a bounded linear functional, $|\eta_n| = |\psi(e_n)| \le \|\psi\|_1'$ for each n. Hence, $(\eta_n) \in l^\infty$, and (3) shows that $\Phi((\eta_n)) = \psi$, so that Φ is onto $(l^1)'$. We have already seen that Φ does not increase norms. It remains to show that, for every $(\eta_n) \in l^\infty$, $\|\Phi((\eta_n))\|_1' \ge \|(\eta_n)\|_\infty$, which will ensure that Φ is both isometric and one-to-one.

Suppose, then, that $(\eta_n) \ne 0$ in l^∞. For any $\epsilon \in (0,1)$, there exists some N such that

$$(1-\epsilon)\|(\eta_n)\|_\infty < |\eta_N| = |\Phi((\eta_n))(e_N)|\,,$$

and, as $\|e_N\|_1 = 1$, it follows that $\|\Phi((\eta_n))\|_1' > (1-\epsilon)\|(\eta_n)\|_\infty$. But, ϵ being arbitrary, one deduces that $\|\Phi((\eta_n))\|_1' \ge \|(\eta_n)\|_\infty$, as required.

If c_0 were reflexive, this would mean that, to every continuous linear functional A on l^1, there would be $(\alpha_n) \in c_0$ such that, for every $(\eta_n) \in l^1$, $A((\eta_n)) = \sum \alpha_n \eta_n$. But this is false, because, for instance, the linear functional $A((\eta_n)) := \sum \eta_n$ (corresponding to the sequence $(1,1,1,\dots) \in l^\infty \setminus c_0$) is not of this form.

5. If $\phi \in E'$, define $R\phi \in F'$ by restriction: $(\forall x \in F)\ (R\phi)(x) = \phi(x)$. (Equivalently, R is the mapping dual to the inclusion $F \longrightarrow E$). Then the kernel K of R consists of all those bounded linear functionals on E that vanish on F (this is called the *annihilator* of F and may be written as F^0), and we have an induced mapping $\widehat{R} : E'/F^0 \longrightarrow F'$ defined in the usual way: if $\widehat{\phi} \in E'/F^0$ is the equivalence class of $\phi \in E'$, then $\widehat{R}\widehat{\phi} := R\phi$, which is the same for all choices of ϕ giving the same equivalence class. The claim is that $\widehat{R}$ is bijective and isometric.

Let $\tau \in E'/F^0$. Then recall the definition of the quotient norm $\|\|^\wedge$:

$$\|\tau\|^\wedge := \inf\{\|\phi\| : \phi \in E' \ \&\ \widehat{\phi} = \tau\}\,.$$

It is obvious that $\|R\phi\| \le \|\phi\|$ for any $\phi \in E'$:

$$\begin{aligned}\|R\phi\| &:= \sup\{|\phi(x)| : x \in F \ \&\ \|x\| \le 1\}\\ &\le \sup\{|\phi(x)| : x \in E \ \&\ \|x\| \le 1\} = \|\phi\|\,.\end{aligned}$$

Consequently, $\left\|\widehat{R}\widehat{\phi}\right\| = \|R\phi\| \le \|\phi\|$ for any $\phi \in E'$, and taking the infimum over a whole equivalence class yields $\left\|\widehat{R}\widehat{\phi}\right\| \le \left\|\widehat{\phi}\right\|^\wedge$.

Conversely, suppose that $\psi \in F'$. The Hahn-Banach theorem (in the form 13.1) tells us that there exists $\phi \in E'$ such that $R\phi = \psi$ and $\|\phi\| = \|\psi\|$. Certainly $\widehat{R}\widehat{\phi} = \psi$; this proves that $\widehat{R}$ is surjective; but also, from above,

$$\|\phi\| = \|\psi\| = \|R\phi\| = \left\|\widehat{R}\widehat{\phi}\right\| \le \left\|\widehat{\phi}\right\|^\wedge \le \|\phi\|\,,$$

so that all these inequalities are equalities. This shows that $\widehat{R}$ is isometric, and so one-to-one.

Let $\pi : E \longrightarrow E/F : x \mapsto [x]$ be the quotient map, which does not increase norms. Then, for any $\alpha \in (E/F)'$, $\Phi(\alpha) := \alpha \circ \pi \in E'$, and $\|\Phi(\alpha)\| \le \|\alpha\|$. Φ is obviously linear (it is the dual map to π).

Given $\epsilon > 0$, there exists $\xi \in E/F$ such that $\|\xi\| = 1$ and $|\alpha(\xi)| > \|\alpha\| - \epsilon$. But also, there exists $x \in E$ such that $[x] = \xi$ and $\|x\| < 1 + \epsilon$. Hence,

$$|\Phi(\alpha)(x)| = |\alpha(\xi)| > \|\alpha\| - \epsilon$$

and $\|\Phi(\alpha)\| \geq \dfrac{\|\alpha\| - \epsilon}{1+\epsilon}$. As ϵ is arbitrary, this shows that $\|\Phi(\alpha)\| \geq \|\alpha\|$. This completes the proof that Φ is isometric; so it is an isometric isomorphism of $(E/F)'$ with a closed subspace of E'. (The subspace in question is easily seen to be F^0.)

Finally, suppose that E is reflexive. Then F' is identified with E'/F^0, and $(E'/F^0)'$ is identified with a subspace of E'', which is identified with the subspace F^{00} consisting of all elements of E that are carried into 0 by all elements of E' that vanish on F. However, the Hahn-Banach theorem tells us (why?) that $F^{00} = F$. So in fact F'' is identified with F itself, and F is reflexive. [This is a rather abbreviated argument. I should really chase the identifications to show that they make the bidual map correspond to the identity of F; but this is pretty obvious in principle, even if writing it down precisely is tedious.]

6. Let $\|\|$ be the norm in E. Define a norm $\|\|_{\oplus}$ on $F \oplus F_1$ by

$$\|(x,y)\|_{\oplus} := \|x\| + \|y\|.$$

This makes $F \oplus F_1$ into a Banach space (an easy exercise, which I omit). But the mapping $S : F \oplus F_1 \longrightarrow E : (x,y) \mapsto x+y$ is a linear isomorphism (a familiar fact from algebra), and $\|S(x,y)\| \leq \|(x,y)\|_{\oplus} = \|x\| + \|y\|$. Thus S is a continuous surjection between Banach spaces, and, by the open mapping theorem, it is open; so it has a continuous inverse. [There are many possible choices for the norm on $F \oplus F_1$. The one I used, which might be called the l^1 norm, makes the argument very simple.]

7. Suppose that F is closed and complemented, with the closed "complement" F_1, so that $F \oplus F_1 \longrightarrow E : (f, f_1) \mapsto f + f_1$ is a continuous isomorphism Q with continuous inverse Q^{-1} (see the last question). Evidently $(f, f_1) \mapsto f$ is a continuous linear mapping π of $F \oplus F_1$ onto F. Let $T := i \circ \pi \circ Q^{-1}$, where $i : F \longrightarrow E$ is the inclusion mapping.

T is certainly a bounded linear map, and $T(E) = F$. And, given $f \in F$,

$$Tf = i \circ \pi \circ Q^{-1} f = i \circ \pi(f, 0) = i(f) = f,$$

so that, for any $x \in E$, $T^2 x = T(Tx) = Tx$. Thus $T^2 = T$.

Conversely, suppose that T is a bounded linear idempotent mapping $E \longrightarrow E$, such that $T(E) = F$. Define $F_1 := \ker T = T^{-1}\{0\}$. Certainly F_1 is a closed linear subspace of E. Suppose $x \in F \cap F_1$; then, as $x \in F = T(E)$, there is some $y \in E$ such that $x = Ty$, and $Tx = T^2 y = Ty = x$. But, as $x \in F_1 = \ker T$, $Tx = 0$. So $x = 0$. Hence, $F \cap F_1 = \{0\}$. Now, take any $u \in E$. Then $u = u - Tu + Tu$, and here

$$T(u - Tu) = Tu - T^2 u = 0,$$

as $T^2 = T$; thus, $u - Tu \in F_1$; whilst $Tu \in F$. Therefore, $F + F_1 = E$. Finally, suppose $y_n = Tx_n \in F$ and $y_n \to y$. Then $Ty_n \to Ty$, as T is continuous; however, $Ty_n = T^2 x_n = Tx_n = y_n$ for each n, and so $y_n \to Ty$ too. Hence $y = Ty \in F$. This proves that F must be closed in E. (Notice that $F = T(E) = \ker(I - T)$, whilst $F_1 = \ker T = (I - T)(E)$.)

8. If E is a reflexive normed space, then so is E'. Indeed, let $\langle , \rangle$ denote the dual pairing of E with E', $\langle , \rangle'$ the pairing of E' with E'', and $\langle , \rangle''$ the pairing of E'' with

E''' ; and let $J_1 : E' \longrightarrow E'''$ be the bidual mapping. For $x \in E$, $\phi \in E'$,

$$\langle x, \phi \rangle = \langle \phi, Jx \rangle' = \langle Jx, J_1\phi \rangle'' . \tag{4}$$

If J is onto (and so an isometric isomorphism), then, for any $X \in E''$,

$$\langle J^{-1}X, \phi \rangle = \langle \phi, X \rangle' = \langle X, J_1\phi \rangle'' ,$$

from which $J_1 = (J^{-1})'$ (the so-called contragredient of J). But this shows J_1 is also an isomorphism, with inverse $J' : E''' \longrightarrow E'$.

If, then, l^1 is reflexive, so is l^∞ (as the dual of l^1; see question 4). But then c_0 (as a closed subspace of l^∞) is also reflexive (see question 5), and it is not (question 4). So neither l^1 nor l^∞ can be reflexive.

9. From (4), $\langle x, \phi \rangle = \langle x, J'J_1\phi \rangle$ for any $x \in E$ and $\phi \in E'$. Consequently $J'J_1$ is the identity on E'. This ensures that $J'(E''') = E'$, and that $J_1J' : E''' \longrightarrow E'''$ is an (isometric) idempotent:

$$(J_1J')(J_1J') = J_1(J'J_1)J' = J_1 I_{E'} J' = J_1J' ,$$

where $J_1J'(E''') = J_1(E')$. By question 7, $J_1(E')$ is complemented in E'''. [The complement is the kernel of J_1J', which, as J_1 is injective, is the kernel of J'.]

10. This is almost a copy of question 2, with some small additional complications.