Math 442

Exercise set 3, 2008 — sketch solutions

1. Certainly the formula makes sense (since f and g are continuous on $[0,1]$, the integral of fg is defined) and it is obviously linear in f. So it is a linear functional on $C := C([0,1];\mathbb{K})$. It only remains to show that it is also bounded. However,

$$(\forall f \in C) \quad \left|\int_{[0,1]} g(t)f(t)\,dt\right| \le \sup\{|f(t)g(t)| : t \in [0,1]\} \le \|f\|\|g\|,$$

so that it is indeed bounded, with a bound not exceeding $\|g\|$.

Take any $a \in [0,1]$. The functional $\phi : f \mapsto f(a)$ is also linear and (trivially) bounded; indeed, $\|\phi(f)\| = |f(a)| \le \|f\|$ for every $f \in C$. Suppose that there exists $g \in C$ such that $\phi(f) = \int_{[0,1]} g(t)f(t)\,dt$ for every $f \in C$.

Suppose that $0 \le b \le 1$ and $b \ne a$. I claim that $g(b) = 0$. Indeed, if $g(b) \ne 0$, construct $f \in C$ by the formula

$$f(t) := \begin{cases} \left(1 - \frac{|t-b|}{|b-a|}\right)\overline{g(t)} & \text{when } t \in [0,1] \text{ and } |t-a| \le |b-a|, \\ 0 & \text{when } t \in [0,1] \text{ and } |t-a| > |b-a|. \end{cases}$$

Then $f \in C$ and

$$\int_{[0,1]} f(t)g(t)\,dt = \int_{0\le t\le 1,\, |t-a|\le|b-a|} \left(1 - \frac{|t-b|}{|b-a|}\right)|g(t)|^2\,dt > 0$$

(as g is continuous and $g(b) \ne 0$, the integrand is positive on some nondegenerate interval containing a). But $\phi(f) = f(a) = 0$, by the construction. This contradiction shows $g(b) = 0$. [There are many other possible proofs, of course.]

Since g is zero at every point of $[0,1]$ other than a itself, and is continuous, it is zero throughout, and cannot correspond to the obviously nonzero functional ϕ.

2. For $a, b \in \mathbb{C}$,

$$\begin{aligned} |a+b|(1+|a|)(1+|b|) &= |a+b| + |a||a+b| + |b||a+b| + |a||b||a+b| \\ &\le |a| + |b||a||a+b| + |b||a+b| + |a||b||a+b| + 2|a||b| + 2|a||b||a+b| \\ &= |a|(1+|b|)(1+|a+b|) + |b|(1+|a|)(1+|a+b|). \end{aligned}$$

Hence, dividing by $(1+|a|)(1+|b|)(1+|a+b|)$,

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|},$$

and, if $f, g, h \in C := C([0,1];\mathbb{K})$, the inequality

$$\frac{|f(x)-h(x)|}{1+|f(x)-g(x)|} \le \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} + \frac{|h(x)-g(x)|}{1+|h(x)-g(x)|} \tag{1}$$

holds for each $x \in \Omega$. On integration, this gives the triangle inequality for d. It is obvious that $d(\{f\},\{g\}) = 0$ only if $f = g$ a.e.

Suppose that $f, g, h, k \in C$. Then (1) gives with suitable changes of notation

$$\frac{|(f(x)+g(x))-(h(x)+k(x))|}{1+|(f(x)+g(x))-(h(x)+k(x))|} \le \frac{|f(x)-h(x)|}{1+|f(x)-h(x)|} + \frac{|g(x)-k(x)|}{1+|g(x)-k(x)|}$$

for each x, and on integration we find that

$$d(f+g, h+k) \le d(f,h) + d(g,k)\,,$$

which shows that addition is (uniformly) continuous with respect to d:

$$d(f,h) < \tfrac{1}{2}\epsilon \;\;\&\;\; d(g,k) < \tfrac{1}{2}\epsilon \Longrightarrow d(f+g, h+k) < \epsilon\,.$$

That was easy, but scalar multiplication is harder. Suppose $\lambda \in \mathbb{K} \setminus \{0\}$, $f \in C \setminus \{0\}$, and $\epsilon > 0$. Choose a natural number K such that $|\lambda| + 1 \le K$, $\delta_0 := \frac{1}{2}\epsilon/K$ and $\delta_1 := \min(\frac{1}{2}\epsilon/\|f\|, 1)$, and then, if $d(f,g) < \delta_0$ and $|\lambda - \mu| < \delta_1$, (1) gives [notice that $a/(1+a)$ increases with a, for $a \ge 0$]

$$\begin{aligned}\frac{|\lambda f(x)-\mu g(x)|}{1+|\lambda f(x)-\mu g(x)|} &\le \frac{|\lambda-\mu||f(x)|}{1+|\lambda-\mu||f(x)|} + \frac{|\mu||f(x)-g(x)|}{1+|\mu||f(x)-g(x)|}\\ &\le |\lambda-\mu||f(x)| + \frac{(|\lambda|+1)|f(x)-g(x)|}{1+(|\lambda|+1)|f(x)-g(x)|}\\ &\le \delta_1\|f\| + \frac{K|f(x)-g(x)|}{1+K|f(x)-g(x)|}\\ &\le \tfrac{1}{2}\epsilon + K\frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\end{aligned}$$

(by induction from (1), since K is a natural number). Now integrate:

$$d(\lambda f, \mu g) \le \tfrac{1}{2}\epsilon + K d(f,g) = \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon\,.$$

The cases where $\lambda f = 0$ are similar but simpler, and I omit them. [I don't claim this is the only possible argument — far from it.]

3. Much the same proofs as in the last question apply. We show that, for $a, b \in \mathbb{C}$,

$$\min(|a+b|, 1) \le \min(|a|, 1) + \min(|b|, 1)\,. \tag{2}$$

Now $|a+b| \le |a| + |b|$; if $|a+b| \le 1$, $|a| \le 1$, $|b| \le 1$, the desired inequality reduces to this, and if $|a| > 1$ it becomes $|a+b| \le 1 + \min(|b|, 1)$, which is no less true. (Likewise if $|b| > 1$.) On the other hand, if $|a+b| > 1$, then $1 < |a| + |b|$, and (2) holds if the minima are $|a|, |b|$, whilst if either minimum is 1 it holds anyway. That d' is a metric, and that addition is continuous, follow as before. Although the proof of continuity of scalar multiplication is not quite identical, the differences are in the formulæ and not in the ideas.

4. It is clear that $\dfrac{a}{1+a} \le \min(a,1)$ for any $a \ge 0$. (If $a = 0$, both sides are 0. If $0 < a \le 1$, the left-hand side is less than a and the right-hand side is a. If $a > 1$, the left side is less than $\frac{1}{2}$ and the right is 1.) But also, $\min(a,1) \le \dfrac{2a}{1+a}$; for, if $0 < a \le 1$, $a + a^2 \le 2a$, whilst if $a > 1$, $1 < 2a/(1+a)$. Thus, for any $f \in C$,

$$d(f,0) := \int_0^1 \frac{|f(x)|}{1+|f(x)|}\,dx \le \int_0^1 \min(|f(x)|,1)\,dx = d'(f,0)$$
$$\le 2\int_0^1 \frac{|f(x)|}{1+|f(x)|}\,dx = 2d(f,0)\,,$$

from which the result follows easily. (Thanks to Tiantian for a much simplified argument.]

5. Firstly, let $(E, \|\|)$ be a normed space, and suppose $f_n \to 0$ in E. For any $\epsilon > 0$, there is some N such that $n \ge N \implies \|f_n\| < \epsilon$. Thus, if $n \ge N$,

$$\left\| \frac{f_{n+1} + f_{n+2} + \cdots + f_{2n}}{n} \right\| \le \frac{\|f_{n+1}\| + \cdots + \|f_{2n}\|}{n} < \epsilon\,.$$

Thus $g_n \to 0$, where, for each $n \in \mathbb{N}$, $g_n := \dfrac{f_{n+1} + f_{n+2} + \cdots + f_{2n}}{n}$.

Now define a sequence (f_n) in C as follows. f_1 is the function that is constantly 1. If $k \in \mathbb{N}$ and $2^{k-1} < n < 2^k$, say $n = 2^{k-1} + i$, then f_n is defined for $t \in [0,1]$ by

$$f_n(t) = \begin{cases} 0 & \text{if } |t - 2^{1-k}i| \ge 2^{1-k}\,, \\ 2^{k-1}(1 - 2^{k-1}|t - 2^{1-k}i|) & \text{otherwise.} \end{cases}$$

Finally, define $f_{2^k}(t) = 2^{k-1}(1 - 2^{k-1}t)$ if $0 \le t < 2^{1-k}$, $f_{2^k}(t) = 2^{k-1}(1 - 2^{k-1}(1-t))$ if $1 - 2^{1-k} < t \le 1$, and $f_{2^k}(t) = 0$ otherwise. [Think of this graphically; the terms for $2^{k-1} < n < 2^k$ give $2^{k-1} - 1$ teeth of height 2^{k-1} and width 2^{2-k}, and the term $n = 2^k$ is a pair of half-teeth at the ends.] Hence,

$$\sum\nolimits_{k=1}^{2^{k-1}} f_{n+k}(t) = 2^{k-1} \quad \text{for all } t \in [0,1]\,.$$

[Adjacent teeth add up exactly to a constant height.] But this implies that the means

$$\frac{1}{2^{k-1}} \sum\nolimits_{k=1}^{2^{k-1}} f_{n+k}$$

are all the same as f_1, and do not tend to 0. [This is the *subsequence*, indexed by powers of 2, of the sequence (g_n) of all the means. But of course (g_n) cannot tend to 0 either.]

On the other hand, $d'(f_n, 0) < 2.2^{1-k}$ for $2^{k-1} < n \le 2^k$, because f_n takes value 0 except on an interval, or on the union of two intervals, of total length 2^{2-k}, and the integrand in the definition of d' cannot exceed 1 (and is less at some points of the intervals in question). Thus, in fact, $d'(f_n, 0) < 4/n$ for all n, so that $f_n \to 0$.

6. Define (as usual) e_n to be the sequence in c_{00} that has all terms zero except the nth, which is 1. If one has a linear relation amongst the e_n,

$$\sum\nolimits_{r=1}^{q} \lambda_r e_{n(r)} = 0$$

(where the indices $n(r)$ are all distinct), then the $n(r)$th term of the sequence on the left is

λ_r, and it follows that $\lambda_r = 0$ for each r. Thus the set $\{e_1, e_2, \dots\}$ is linearly independent. On the other hand, if $x = (\xi_1, \xi_2, \xi_3, \dots) \in c_{00}$, and $\xi_r = 0$ for $r > R$, then clearly

$$x = \sum\nolimits_{r=1}^{R} \xi_r e_r .$$

So $\{e_1, e_2, e_3, \dots\}$ is an algebraic basis for c_{00}.

7. Suppose $x, y \in E$ and $x \neq y$. Then $y - x \neq 0$. As $\{0\}$ is closed, there is an open set U such that $0 \in U$ and $0 \notin y - x + U$. By 7.3, there is a balanced open set V such that $0 \in V$ (any non-empty balanced set must contain 0) and $V + V \subseteq U$. Now $x + V, y + V$ are open sets containing x, y respectively, and I assert that $(x+V) \cap (y+V) = \emptyset$.

Indeed, suppose that $z \in (x+V) \cap (y+V)$. Then $z - x \in V$ and $z - y \in V$, which implies $x - z \in V$, since V is balanced; so $x - y = (y - z) + (z - x) \in V + V \in U$, and

$$0 = (y - x) + (x - y) \in y - x + U .$$

This contradicts the definition of U. We conclude that $(x+V) \cap (y+V) = \emptyset$. This shows that E satisfies the definition of a Hausdorff space.

8. Let K denote the intersection of all the open sets containing the origin; let $\lambda, \mu \in \mathbb{K}$ and $x, y \in K$. Take any open set W containing 0. As E is a topological vector space, there is an open set W' containing 0 such that the mapping

$$\phi : (\xi, \eta) \mapsto \lambda\xi + \mu\eta : E \times E \longrightarrow E$$

carries $W' \times W'$ into W. But $x, y \in K \subseteq W'$, so $\lambda x + \mu y \in W$; and as W could be any open set containing 0, this shows $\lambda x + \mu y \in K$. So K is a vector subspace.

Suppose $x \notin K$. Then there exists an open set U containing 0 such that $x \notin U$, and (again by continuity) there is another, V, such that $V - V := \{a - b : a, b \in V\} \subseteq U$. (You could refer here to Lemma 6.3.) But then $(x+V) \cap V = \emptyset$ (for otherwise there would exist $v_1, v_2 \in V$ such that $x + v_1 = v_2$ and $x = v_2 - v_1 \in U$). Hence, $(x+V) \cap K = \emptyset$. As $x + V$ is an open set containing x, this shows that any point x not in K is contained in an open set disjoint from K; so K is closed.

If E is Hausdorff, and $x \neq 0$, there is an open set Q containing 0 such that $x \notin Q$. So $x \notin K$; and K can consist only of the single element 0. Conversely, if $K = \{0\}$, and if $y \neq z$ in E, then $y - z \notin K$ and, taking $x := y - z$ in the argument of the previous paragraph, there is an open set V containing 0 such that $(y - z + V) \cap V = \emptyset$, or, equivalently, $(y+V) \cap (z+V) = \emptyset$. Thus E is Hausdorff.

9. Suppose that there is an open set U in $\mathbb{K}$ which is neither $\emptyset$ nor $\mathbb{K}$. If $a \in U$, then $U - a$ is an open set containing 0 and not equal to $\mathbb{K}$. By 6.3, there is a balanced open set V such that $0 \in V \subseteq U - a$. Take any $b \notin V$, and any $c \in V \setminus \{0\}$. Then, as V is balanced,

$$\{\mu b : |\mu| \geq 1\} \cap V = \emptyset, \quad V \supseteq \{\lambda c : |\lambda| \leq 1\},$$

which clearly says that $B(0; |c|) \subseteq V \subseteq B(0; |b|)$. Since κV is also an open set containing 0, for any $\kappa > 0$, it follows that all the balls $B(0; \kappa|b|)$ with respect to the modulus in $\mathbb{K}$ contain an open set containing 0, and that all balanced open sets containing 0 also contain a ball about 0 with respect to the modulus. This clearly means that the given topology is in fact the usual topology.