Math 442

Exercise set 2, 2008 — sketch solutions

1. *(a)* $\mathrm{cl}(f(A))$ is closed in Ω_2. Thus, by 2.5, $f^{-1}(\mathrm{cl}(f(A)))$ is closed in Ω_1. But

$$f^{-1}(\mathrm{cl}(f(A))) \supseteq f^{-1}(f(A)) \supseteq A\,,$$

and so, by the definition 2.3, $f^{-1}(\mathrm{cl}(f(A))) \supseteq \mathrm{cl}\,A$, and so $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$. [2]

(b) Let U be open in Ω_3. Then $g^{-1}(U)$ is open in Ω_2 (because g is continuous), and in turn $f^{-1}(g^{-1}(U))$ is open in Ω_1. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. The result follows. [The facts about inverse images that I have assumed should be checked. Plausible statements about them are not necessarily true. For instance, $f(f^{-1}(A))$ is "usually" a proper subset of A, and $f^{-1}(f(B))$ is "usually" a proper superset of B.] [1]

2. Suppose that $\mathcal{U}$ is an open cover of A. For each $U \in \mathcal{U}$, there is a subclass of $\mathcal{B}$, $\mathcal{B}[U]$, such that $U = \bigcup_{B\in\mathcal{B}[U]} B$. Define $\mathcal{W} := \bigcup_{U\in\mathcal{U}} \mathcal{B}[U] \subseteq \mathcal{B}$. Then $A \subseteq \bigcup_{C\in\mathcal{W}} B$, and, by our assumption, there is a finite subclass $\mathcal{X}$ of $\mathcal{W}$ such that $A \subseteq \bigcup_{B\in\mathcal{X}} B$.

For each $B \in \mathcal{X} \subseteq \mathcal{W}$, *choose* some $U \in \mathcal{U}$ such that $B \in \mathcal{B}[U]$ (and, therefore, $C \subseteq U$). These sets form a finite subclass $\mathcal{V}$ of $\mathcal{U}$. It has no more elements than $\mathcal{X}$, and possibly fewer, since the same U may possibly be chosen for more than one B. (By the way, since $\mathcal{X}$ is finite, the Axiom of Choice is not needed.) Now $A \subseteq \bigcup_{U\in\mathcal{V}} U$. $\mathcal{V}$ is a finite subcovering of $\mathcal{U}$. This shows that A is compact. [3]

3. Clearly the condition must be satisfied if f is continuous. So suppose that $f^{-1}(B)$ is open for every $B \in \mathcal{B}$. Let U be any open subset of Ψ; then there is a subclass of $\mathcal{B}$, which in the previous exercise I called $\mathcal{B}[U]$, such that $U = \bigcup_{B\in\mathcal{B}[U]} B$. Then

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{B\in\mathcal{B}[U]} B\Big) = \bigcup_{B\in\mathcal{B}[U]} f^{-1}(B)\,.$$

But this is (by hypothesis) a union of open sets in Ω, so is itself open, as required. [3]

4. This is a weaker version of question 8 of set 1, and should not be here.

5. *(a)* As E_n is a finite union of closed subintervals of $[0,1]$, it is closed (and compact). Thus $\bigcap_{n=0}^{\infty} E_n$ is compact, being closed (as an intersection of closed sets) and bounded in $\mathbb{R}$. [But I have not yet shown that $E \neq \emptyset$.] [1]

(b) The length of E_0 is 1; E_1 consists of two disjoint closed intervals, each of length $\frac{1}{3}$, separated by a gap (an open interval) of length $\frac{1}{3}$; and, by induction, E_n is the union of 2^n closed intervals, each of length 3^{-n}, separated by gaps of length at least 3^{-n} (each of them will have a gap of exactly this length at one end, and of greater length at the other end, unless that end is 0 or 1).

As E is closed, it is nowhere dense if (and only if) it has empty interior. Suppose, then, that $x \in \mathrm{int}(E)$; then there is an open ball $(x-\delta, x+\delta)$ included in E, for some $\delta > 0$. However, the length of each of the closed intervals that make up E_n is 3^{-n}. Hence, if n is so large that $3^{-n} < 2\delta$, the interval $(x-\delta, x+\delta)$ must contain points not in E_n, and, therefore, not in E. The interior of E must be empty. [2]

(c) The easiest way to do this is to couple it with *(d)*, though there are other proofs.

Any point of $[0,1]$ has a ternary expansion, in which the "digits" are $0, 1, 2$. Most points have only one ternary expansion; however, those numbers that have a terminating expansion (all "digits" are 0 after some point) also have a non-terminating expansion (in which all "digits" after some point are 2). The removal of the "middle third" from E_0 means the removal of all points whose ternary expansion *must* contain a 1 in the first place (thus $0 \cdot 1000\ldots$ is *not* removed, since it is the same as $0 \cdot 0222\ldots$). Inductively, E_n consists of those points that have some ternary expansion with no 1 in the first n places, and, therefore, E is the set of points with ternary expansions expressible by 0s and 2s only.

Given such a number x in E, let x_n have the same expansion (by 0s and 2s) as x except for the nth place, which is changed to 2 if it is 0 for x and vice versa. Then, clearly, $x_n \in E$, and $x_n \to x$ (indeed, $|x_n - x| = 2.3^{-n}$). The sequence (x_n) clearly has no repeats. This proves *(c)*.

As for *(d)*, the customary proof of uncountability of $[0,1]$ can obviously be modified to work in this case. [In fact, one can map E *onto* $[0,1]$ by ϕ, where $\phi(x)$ is the number obtained by changing every 2 in the ternary expansion (without 1s) of x into a 1 and interpreting the result as a binary expansion. The existence of this surjection forces E to be uncountable. More startlingly, ϕ is continuous. So it is possible for a continuous mapping to transform a nowhere dense set into an interval.] [2,2]

6. Let Δ denote the diagonal mapping.

By definition, the product open sets constitute a base for the product topology. By qu. 3, then, it will suffice to show that, if A is a product open set in $\Omega \times \Omega$, then $\Delta^{-1}(A)$ is open in Ω. So let $A = U \times V$, where U and V are open in Ω. But

$$\Delta^{-1}(A) = \{x \in \Omega : (x,x) \in U \times V\} = \{x \in \Omega : x \in U \ \ \& \ \ x \in V\} = U \cap V ,$$

which, as the intersection of open sets, is also open. This is what is required. [3]

7. Our examples will be for $\Omega := [0,1]$, with the usual topology. Firstly, let, for $n \in \mathbb{N}$ and $0 \le t \le 1$,

$$f_n(t) := \begin{cases} 0 & \text{when } 0 \le t \le 1 - \frac{1}{n}, \\ 2n\big(t - 1 + \frac{1}{n}\big) & \text{when } 1 - \frac{1}{n} < t \le 1 - \frac{1}{2n}, \\ 2n(1-t) & \text{when } 1 - \frac{1}{2n} < t \le 1 . \end{cases}$$

Then $f_n(t) \to 0$ for every $t \in [0,1]$, but the convergence is not uniform, as $f_n\big(1 - \frac{1}{2n}\big) = 1$ for every n. However, the convergence is not monotonic. [1]

Equally, if $g_n(t) = 0$ when $t = 0$ and when $\frac{1}{n} \le t \le 1$, $g_n(t) = 1$ when $0 < t < \frac{1}{n}$, then $g_n(t)$ decreases monotonically to 0 for each t, but the convergence is not uniform, as $g_n\big(\frac{1}{2n}\big) = 1$ for all n. In this case the functions g_n are discontinuous. [1]

Finally, let $h_n(t) = t^n$. If $0 \le t < 1$, $t^n \downarrow 0$ as $n \to \infty$. For $t = 1$, $1^n = 1$ for all n. Thus the sequence does tend monotonically to a limit, but the limit is discontinuous; and the convergence is non-uniform, because, no matter how large n may be, there is some $t < 1$ such that $t^n > \frac{1}{2}$. [1]

8. As suggested, define $\phi(u) := \frac{1}{2}(a + u^2)$, where a is a fixed number in $[0,1]$. If $0 \le u < 1$, then $\phi(u) < 1$ too. If, in addition, $1 - a < (1-u)^2$, then $a > 2u - u^2$ and

$\phi(u) > \frac{1}{2}(2u - u^2 + u^2) = u$. But this means $\frac{1}{2}(a + u^2) > u \geq 0$, so that, squaring,

$$\begin{aligned} \tfrac{1}{4}(a + u^2)^2 &> u^2 && \text{and} \\ 1 - a - u^2 + \tfrac{1}{4}(a + u^2)^2 &> 1 - a\,, && \text{or} \\ (1 - \tfrac{1}{2}(a + u^2))^2 &> 1 - a\,; && \text{that is,} \\ (1 - \phi(u))^2 &> 1 - a\,. \end{aligned}$$

Consider the sequence $q_1(a) := 0$, $q_2(a) := \phi(q_1(a))$, $q_3(a) := \phi(q_2(a)), \ldots,$ $q_{n+1}(a) := \phi(q_n(a)), \ldots$. These are all polynomials in a, and the arguments above show that $q_1(a) < q_2(a) < \cdots < q_n(a) < \cdots$ and $(1 - q_1(a))^2 > \cdots > (1 - q_n(a))^2 > \cdots > 1 - a$. As $0 \leq q_n(a) < 1$ for all n, the bounded increasing sequence $(q_n(a))$ has a limit $q(a)$, and $0 \leq q(a) \leq 1$. Then $q_n(a) \to q(a)$ and $\phi(q_n(a)) \to \phi(q(a))$, so that $\phi(q(a)) = q(a)$, $\frac{1}{2}(a + q(a)^2) = q(a)$, or

$$1 - 2q(a) + q(a)^2 = 1 - a\,, \quad 1 - q(a) = \sqrt{1 - a}\,.$$

Now, for each $x \in [0, 1]$, let $p_n(x) := 1 - q_n(1 - x)$. This is a polynomial in x; it decreases monotonically as n increases, and its limit is $\sqrt{x}$ (take $a := 1 - x$ to see this). [3]

9. The function $(x - c)^3$ separates points of $[0, 1]$. Thus the self-conjugate algebra of polynomials in $(x - c)^3$ (of the form $a_0 + a_1(x - c)^3 + a_2(x - c)^6 + \cdots + a_k(x - c)^{3k}$, for $k \in \mathbb{N}$ and coefficients $a_0, a_1, \ldots, a_k$) also separates points, and contains all the constant functions. By the Stone-Weierstrass theorem, any continuous function $f : [0, 1] \longrightarrow \mathbb{C}$ may be uniformly approximated by such polynomials. It is clear that their first and second derivatives vanish at c. [3]

10. The class A of all polynomials whose terms are of even degree greater than 5 is a subalgebra of $C([0, 1]; \mathbb{C})$. It separates the points of $[0, 1]$, as x^6 does. So the Stone-Weierstrass theorem applies. [3]

11. Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in C$. By hypothesis, there exist numbers K and L such that $|f(t)| \leq K$ and $|g(t)| \leq L$ for all $t \in \mathbb{R}$. Therefore, $|\lambda f(t) + \mu g(t)| \leq |\lambda| K + |\mu| L$ for all t. Similarly, for any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |s - t| < \delta_1 &\Longrightarrow |f(s) - f(t)| < \frac{\epsilon}{1 + |\lambda| + |\mu| + K + L}\,, \\ |s - t| < \delta_2 &\Longrightarrow |g(s) - g(t)| < \frac{\epsilon}{1 + |\lambda| + |\mu| + K + L}\,. \end{aligned}$$

Take $\delta := \min(\delta_1, \delta_2)$, and then

$$\begin{aligned} |s - t| < \delta \Longrightarrow |(\lambda f + \mu g)(s) &- (\lambda f + \mu g)(t)| \\ &= |\lambda(f(s) - f(t)) + \mu(g(s) - g(t))| \\ &\leq |\lambda||f(s) - f(t)| + |\mu||g(s) - g(t)| \\ &\leq \frac{|\lambda|\epsilon}{1 + |\lambda| + |\mu| + K + L} + \frac{|\mu|\epsilon}{1 + |\lambda| + |\mu| + K + L} < \epsilon\,. \end{aligned}$$

So $\lambda f + \mu g$ is also bounded and uniformly continuous.

Similarly, $|f(t)g(t)| \leq KL$ for all t, and

$$\begin{aligned}|s-t|<\delta \Longrightarrow |f(s)g(s)-f(t)g(t)| &\leq |f(s)||g(s)-g(t)|+|g(t)||f(s)-f(t)| \\ &\leq K|g(s)-g(t)|+L|f(s)-f(t)| \\ &\leq \frac{K\epsilon}{1+|\lambda|+|\mu|+K+L}+\frac{L\epsilon}{1+|\lambda|+|\mu|+K+L}<\epsilon\,;\end{aligned}$$

so fg is also bounded and uniformly continuous. Hence C is an algebra of functions on $\mathbb{R}$.[3]

Finally, we must show that it is a Banach space with respect to the supremum norm. Let (f_n) be a Cauchy sequence in C. Then, for each $t \in \mathbb{R}$, the sequence $(f_n(t))$ is Cauchy in $\mathbb{R}$, and must converge to a limit $f(t)$. Firstly, there exists $M \in \mathbb{N}$ such that

$$m, n \geq M \Longrightarrow \sup\{|f_m(t)-f_n(t)| : t \in \mathbb{R}\} < 1\,,$$

so that, for any t, $|f_m(t)-f_n(t)|<1$ when $m, n \geq M$. Thus, taking $m = M$ and letting $n \to \infty$, $|f_M(t)-f(t)| \leq 1$, and $|f(t)| \leq 1+\sup\{|f_M(s)| : s \in \mathbb{R}\}$. It follows that f is necessarily bounded.

Take $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \Longrightarrow \sup\{|f_m(t)-f_n(t)| : t \in \mathbb{R}\} < \tfrac{1}{3}\epsilon\,,$$

and, as before, $|f_N(t)-f(t)| \leq \frac{1}{3}\epsilon$ for all t. As f_N is uniformly continuous, there exists $\delta > 0$ such that $|f_N(s)-f_N(t)| < \frac{1}{3}\epsilon$ whenever $|s-t|<\delta$. Hence, if $|s-t|<\delta$,

$$|f(s)-f(t)| \leq |f_N(s)-f(s)|+|f_N(s)-f_N(t)|+|f_N(t)-f(t)| < \tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon=\epsilon\,.$$

Thus, the limit f is both bounded and uniformly continuous, so $f \in C$. Finally, $f_n \to f$ in the norm on C; I omit this argument, which is almost a repetition of that above. [3]

12. The analogue of Stone's theorem does not hold. Let A be, for instance, the class of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy the following curious condition:

there exists a constant $\beta[f]$ such that $\{t \in \mathbb{R} : f(t) \neq \beta[f]\}$ is compact.

What this means is that, for very large positive or negative t, f takes the single value $\beta[f]$; it is "asymptotically constant".

If $f \in A$, then $f-\beta[f]$ is zero except on a compact set; thus it is uniformly continuous and bounded, and so is f. Hence, $A \subseteq C$. Furthermore, A is a subalgebra of C and separates points. That A contains the constant functions is obvious. However, the function $q \in C$ defined by: $q(t)=t$ for $-1 \leq t \leq 1$, $q(t)=-1$ if $t \leq -1$, $q(t)=1$ if $t \geq 1$, cannot be uniformly approximated more closely than at distance 1 by any $f \in A$. (The difficulty is that there must be large negative and positive values of t for which $f(t)=\beta$, and β cannot be closer than 1 to -1 and to 1 simultaneously.) [3]