Math 442

Exercise set 1, 2008 — sketch solutions

{Well, they are actually far fuller solutions than I'd ever expect anyone to present.}

1. Suppose, firstly, that f is metrically continuous at $x \in \Omega_1$. The definition (1) of 1.5 may be rephrased as

$$(\forall \epsilon > 0)(\exists \delta > 0) \quad y \in B_{\Omega_1}(x;\delta) \Longrightarrow f(y) \in B_{\Omega_2}(f(x);\epsilon)\,, \tag{1}$$

and this in turn is equivalent to

$$(\forall \epsilon > 0)(\exists \delta > 0) \quad f(B_{\Omega_1}(x;\delta)) \subseteq B_{\Omega_2}(f(x);\epsilon)\,. \tag{2}$$

We must prove that, for any metrically open set U in Ω_2 containing $f(x)$, there is a metrically open set V containing x and such that $f(V) \subseteq U$.

So, let U be a metrically open set in Ω_2 with $f(x) \in U$. By the definition of "metrically open set", there is some $\epsilon > 0$ such that $B_{\Omega_2}(f(x);\epsilon) \subseteq U$. Applying (2), there exists some $\delta > 0$ such that $f(B_{\Omega_1}(x;\delta)) \subseteq B_{\Omega_2}(f(x);\epsilon)$. Take $V := B_{\Omega_1}(x;\delta)$; this is a metrically open set in Ω_1 (there is a remark after 3.6 about this — but see below for a proof), obviously contains x, and

$$f(V) = f(B_{\Omega_1}(x;\delta)) \subseteq B_{\Omega_2}(f(x);\epsilon) \subseteq U\,.$$

So we have constructed a metrically open set V containing x and satisfying $f(V) \subseteq U$.

On the other hand, suppose that, for any metrically open set U containing $f(x)$, there is a metrically open set V containing x and such that $f(V) \subseteq U$. Let $\epsilon > 0$. Then $B_{\Omega_2}(f(x);\epsilon)$ is a metrically open subset of Ω_2 that contains $f(x)$ (again, see below), and so, by our assumption, there is a metrically open set V such that $x \in V$ and $f(V) \subseteq B_{\Omega_2}(f(x);\epsilon)$. However, as V is metrically open and contains x, there is (by the definition of a metrically open set) some $\delta > 0$ such that $B_{\Omega_1}(x;\delta) \subseteq V$. And then

$$f(B_{\Omega_1}(x;\delta)) \subseteq B_{\Omega_2}(f(x);\epsilon)\,.$$

We have, therefore, shown that (1), or equivalently (2), is satisfied.

[To fill in the gap, let me now show that, in any metric space (Ω, d), an open metric ball $B_\Omega(x;r)$ is a metrically open set. Let y be any point of $B_\Omega(x;r)$. That means that $d(x,y) < r$. Take $\delta := r - d(x,y) > 0$. If $z \in B_\Omega(y;\delta)$, then

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \delta = d(x,y) + r - d(x,y) = r\,,$$

and so $z \in B_\Omega(x;r)$. Consequently, $B_\Omega(y;\delta) \subseteq B_\Omega(x;r)$. So any point y of $B_\Omega(x;r)$ is the centre of a metric ball of positive radius included in $B_\Omega(x;r)$, and this is what we means by saying $B_\Omega(x;r)$ is metrically open.]

Now, suppose that f is metrically continuous at each point of Ω_1, and let W be a metrically open subset of Ω_2. I wish to show that $f^{-1}(W)$ is metrically open in Ω_1. For this, I must show that, for any $x \in f^{-1}(W)$, there is a metric ball of positive radius about x which is included in $f^{-1}(W)$.

Since $x \in f^{-1}(W)$, $f(x) \in W$, and, as W is metrically open, there is some $\epsilon > 0$ such that $B_{\Omega_2}(f(x);\epsilon) \subseteq W$. As f is metrically continuous at x, there is $\delta > 0$ such that

$$f(B_{\Omega_1}(x;\delta)) \subseteq B_{\Omega_2}(f(x);\epsilon) \subseteq W$$

as at (2). Thus $B_{\Omega_1}(x;\delta) \subseteq f^{-1}(W)$. But this is precisely what we wished to show.

On the other hand, suppose that $f^{-1}(W)$ is metrically open in Ω_1 for every metrically open set W of Ω_2. Take any $x \in \Omega_1$. If $\epsilon > 0$, $B_{\Omega_2}(f(x);\epsilon)$ is a metrically open set in Ω_2, and so, by our assumption, $f^{-1}(B_{\Omega_2}(f(x);\epsilon))$ is metrically open in Ω_1. It certainly contains x, and so there exists some $\delta > 0$ such that $B_{\Omega_1}(x;\delta) \subseteq f^{-1}(B_{\Omega_2}(f(x);\epsilon))$. As this argument applies for any $\epsilon > 0$, it follows that f is metrically continuous at x.

2. *(a)* Suppose that f is not uniformly continuous on K. Then

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in K)(\exists y \in \Omega))(d_\Omega(x,y) < \delta \;\;\&\;\; d_\Psi(f(x),f(y)) \geq \epsilon).$$

(This is the formal negation of the definition of uniform continuity on K.) In particular, for this ϵ, there will be for any natural number n a point $x_n \in K$ and a point $y_n \in \Omega$ such that

$$d_\Omega(x_n,y_n) < \frac{1}{n} \;\;\&\;\; d_\Psi(f(x_n,y_n) \geq \epsilon.$$

But now, as K is sequentially compact, there is a subsequence $(x_{n(k)})_{k=1}^\infty$ of (x_n) that converges to a point $x \in K$. Then $y_{n(k)} \to x$ too, since for each k

$$d_\Omega(y_{n(k)},x) \leq d_\Omega(x_{n(k)},x) + d_\Omega(x_{n(k)},y_{n(k)}) \to 0;$$

and, also for each k,

$$d_\Omega(x_{n(k)},y_{n(k)}) < \frac{1}{n(k)} \leq \frac{1}{k} \quad \text{and} \quad d_\Psi(f(x_{n(k)}),f(y_{n(k)})) \geq \epsilon.$$

By renumbering $n(k)$ as k, we may as well suppose that for each k

$$d_\Omega(x_k,y_k) < \frac{1}{k} \quad \text{and} \quad d_\Psi(f(x_k),f(y_k)) \geq \epsilon. \tag{3}$$

However, f is continuous at x, and consequently there is some $\delta > 0$ such that, if $z \in \Omega$ and $d_\Omega(x,z) < \delta$, then $d_\Psi(f(x),f(z)) < \frac{1}{2}\epsilon$. For sufficiently large k,

$$d_\Omega(x,x_k) < \delta \quad \& \quad d_\Omega(x,y_k) < \delta,$$

and then $d_\Psi(f(x),f(x_k)) < \frac{1}{2}\epsilon$ and $d_\Psi(f(x),f(y_k)) < \frac{1}{2}\epsilon$; hence, for such large k,

$$d_\Psi(f(x_k),f(y_k)) \leq d_\Psi(f(x),f(x_k)) + d_\Psi(f(x),f(y_k)) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.$$

This contradicts (3), and so establishes the theorem. [There is a more elegant proof than this, using the definition of compactness, but it is perhaps less easy to invent. As so often, a proof by contradiction allows you to play around aimlessly until a contradiction appears.]

(b) Suppose that $\mathcal{F}$ is not uniformly equicontinuous on K. Then

$$(\exists\epsilon > 0)(\forall\delta > 0)(\exists x \in K)(\exists y \in \Omega)(d_\Omega(x,y) < \delta \ \ \& \ \ (\exists f \in \mathcal{F})(d_\Psi(f(x), f(y)) \geq \epsilon))\,.$$

(This is the formal negation of the definition of uniform equicontinuity on K.) In particular, for this ϵ, there will be, for any natural number n, points $x_n \in K$ and $y_n \in \Omega$ and a function $f_n \in \mathcal{F}$ such that

$$d_\Omega(x_n, y_n) < \frac{1}{n} \ \ \& \ \ d_\Psi(f_n(x_n), f_n(y_n)) \geq \epsilon\,. \tag{4}$$

As in *(a)* above, we may pass to a subsequence and assume in (4) that $x_n \to x \in K$ and $y_n \to x$. But $\mathcal{F}$ is equicontinuous at x, so the same contradiction with (4) arises. [As in *(a)*, there is also a more elegant proof.]

3. Let $\epsilon > 0$. Then (by equicontinuity) there is some $\delta > 0$ such that

$$d_\Omega(x,y) < \delta \Longrightarrow (\forall n \in \mathbb{N}) \ \ d_\Psi(f_n(x), f_n(y)) < \epsilon\,.$$

Letting $n \to \infty$, so that $f_n(x) \to f(x)$ and $f_n(y) \to f(y)$, we find that

$$d_\Omega(x,y) < \delta \Longrightarrow d_\Psi(f(x), f(y)) \leq \epsilon\,,$$

which shows that f is continuous at x.

4. Let $\epsilon > 0$. By hypothesis, there are some $\delta' > 0$ and some $n \in \mathbb{N}$ such that

$$d_\Omega(a,y) < \delta' \ \ \& \ \ n \geq N \Longrightarrow d_\Psi(f_n(y), f(y)) < \tfrac{1}{3}\epsilon\,.$$

But also, f_N is continuous at a; so there exists $\delta'' > 0$ such that

$$d_\Omega(a,y) < \delta'' \Longrightarrow d_\Psi(f_N(a), f_N(y)) < \tfrac{1}{3}\epsilon\,.$$

Hence, if $d_\Omega(x,y) < \delta := \min(\delta', \delta'') > 0$,

$$\begin{aligned} d_\Psi(f(x), f(y)) &\leq d_\Psi(f(x), f_N(x)) + d_\Psi(f_N(x), f_N(y)) + d_\Psi(f_N(y), f(y)) \\ &< \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon\,. \end{aligned}$$

This shows that f is continuous at a.

5. Suppose $y(x) \neq 0$; then the equation may be rewritten

$$\frac{d}{dx}\big(3y^{1/3}\big) = 1\,.$$

Thus, on any interval on which y does not vanish, the solution must be of the form

$$3y^{1/3} = t - C \quad \text{or} \quad y = \big(\tfrac{1}{3}(t - C)\big)^3\,.$$

Suppose that $-\infty \le C_1 \le 0 \le C_2 \le \infty$. Then we can define for $t \in (-\infty, \infty)$

$$\phi(t) = \begin{cases} \left(\frac{1}{3}(t - C_1)\right)^3 & \text{when } t \le C_1 \,, \\ 0 & \text{when } C_1 < t < C_2 \,, \\ \left(\frac{1}{3}(t - C_2)\right)^3 & \text{when } C_2 \le t \,. \end{cases}$$

It is easily checked that ϕ is a solution of the differential equation, satisfying the initial condition $\phi(0) = 0$. (The derivatives $\phi'(C_1)$ and $\phi'(C_2)$ both exist and are 0.) Thus there are infinitely many solutions, given by the various choices of C_1 and C_2.

There is no contradiction of Theorem 2.7, because the function $y^{2/3}$ does not satisfy any Lipschitz condition in the neighbourhood of the origin; indeed,

$$\frac{y^{2/3} - 0^{2/3}}{y - 0} = y^{-1/3}$$

tends to ∞ as $y \downarrow 0$ (and to $-\infty$ as $y \uparrow \infty$).

[It is worth pointing out that the method of solution I used amounts to substituting $y = r^3$, so that the equation becomes $3r^2\, dr/dx = r^2$, or $dr/dx = \frac{1}{3}$, when $r \ne 0$. But the whole difficulty is in what happens when $r = 0$.]

6. For each $a \in A$, $\{x \in \Omega : f(x) < f(a) + 1\}$ is a neighbourhood of a in Ω. Let $U(a)$ be an open set in Ω such that $a \in U(a) \subseteq \{x \in \Omega : f(x) < f(a) + 1\}$. Then $\{U(a) : a \in A\}$ is an open covering of A. As A is compact, there is a finite subcovering, which we may list as $\{U(a_i) : 1 \le i \le n\}$. Take $K := 1 + \max_{1 \le i \le n} f(a_i)$. Then, if $y \in A$, there is some i, $1 \le i \le n$, such that $y \in U(a_i) \subseteq \{x \in \Omega : f(x) < f(a_i) + 1\}$, and so

$$f(y) < f(a_i) + 1 \le K \,.$$

This proves that $f(A)$ is bounded above.

Suppose, however, that its supremum on A, $\Lambda := \sup\{f(x) : x \in A\}$, is not attained; that is, $f(a) < \Lambda$ for each $a \in A$. Then, for each $b \in A$, $f(b) < \Lambda$, and

$$\{x \in \Omega : f(x) < f(b) + \tfrac{1}{2}(\Lambda - f(b))\}$$

is a neighbourhood of b in Ω, and includes an open neighbourhood $V(b)$ of b in Ω. Thus $\{V(b) : b \in A\}$ is an open covering of A, which must have a finite subcovering $\{V(b_j) : 1 \le j \le m\}$. Let $\lambda := \max_{1 \le j \le m} f(b_j)$. Then $\lambda < \Lambda$, and, for any $y \in A$, there is some j such that $y \in V(b_j)$, so that $f(y) < f(b_j) + \frac{1}{2}(\Lambda - f(b_j)) \le \frac{1}{2}(\lambda + \Lambda)$. This, however, is absurd, since it implies that $\frac{1}{2}(\lambda + \Lambda)$ is an upper bound for $f(A)$, whereas Λ was defined to be the least upper bound (and $\frac{1}{2}(\lambda + \Lambda) < \Lambda$). The conclusion must be that Λ is an attained supremum: there is some $a \in A$ such that $f(a) = \sup\{f(x) : x \in A\}$.

7. Take any $a \in \Omega$ and any $\epsilon > 0$. By the definition of the infimum, there is $f \in \mathcal{F}$ such that $f(a) < g(a) + \frac{1}{2}\epsilon$. As f is continuous, $V := \{x \in \Omega : f(x) < f(a) + \frac{1}{2}\epsilon\}$ is an open neighbourhood of a. But, if $x \in V$,

$$g(x) \le f(x) < f(a) + \tfrac{1}{2}\epsilon < g(a) + \epsilon\,.$$

So $V \subseteq \{x \in \Omega : g(x) < g(a) + \epsilon\}$; this proves that $\{x \in \Omega : g(x) < g(a) + \epsilon\}$ is a neighbourhood of a in Ω. As a and ϵ are arbitrary, g is upper semicontinuous on Ω.

8. For clarity, let D denote the "product metric":

$$D((a,b),(x,y)) \coloneqq d(a,x) + d(b,y)\,.$$

Then, for any $(a,b),(x,y) \in \Omega \times \Omega$,

$$d(a,b) \le d(a,x) + d(x,y) + d(b,y) = d(x,y) + D((a,b),(x,y))\,,$$

so that $d(a,b) - d(x,y) \le D((a,b),(x,y))$. The same must be true if x and y are interchanged, and so

$$|d(a,b) - d(x,y)| \le D((a,b),(x,y))\,.$$

This proves not only that d is uniformly continuous with respect to D, but even that it is "Lipschitz with Lipschitz constant 1" — but of course the specific choice of metric D on the product is involved in that.

9. Many examples are possible both in *(a)* and in *(b)*.

(a) Let $\Omega \coloneqq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, with the topology given by the usual metric in $\mathbb{C}$, $d(z_1,z_2) \coloneqq |z_1 - z_2|$. Let the mapping f be defined by, for instance,

$$(\forall z \in \mathbb{T}) \quad f(z) \coloneqq az\,,$$

where a is itself a complex number of modulus 1 other than 1 itself, such as i. Then

$$d(f(z_1), f(z_2)) = d(z_1, z_2)$$

for any $z_1, z_2 \in \mathbb{T}$, but f has no fixed point. [Other compact metrizable topological groups could be used instead of $\mathbb{T}$, which is just the simplest example that is not in some vague sense silly. A perfectly correct, but rather "silly", example would be to take $\Omega \coloneqq \{a,b\}$, just a two-point space, and to let f be the "flip" that interchanges a and b.]

(b) A simple example is

$$\Omega \coloneqq [0,1]\,, \quad f : \Omega \longrightarrow \Omega : t \mapsto t - \tfrac{1}{4}t^2\,.$$

Here $|f(s) - f(t)| = \left|s - t - \tfrac{1}{4}(s^2 - t^2)\right| = |s-t|\left|1 - \tfrac{1}{4}(s+t)\right| < |s-t|$ for $s \ne t$ (since $0 < 1 - \tfrac{1}{4}(s+t) < 1$), but on the other hand, for $s \ne t$,

$$\frac{|f(s) - f(t)|}{|s-t|} = 1 - \tfrac{1}{4}(s+t)$$

may be arbitrarily close to 1 if s and t are close enough to 0. So f is *not* a contraction mapping.

(c) The mapping $\Omega \longrightarrow \mathbb{R} : x \mapsto d(f(x), x)$ is continuous, because it is

$$\begin{array}{ccccccc} \Omega & \longrightarrow & \Omega \times \Omega & \longrightarrow & \Omega \times \Omega & \longrightarrow & \mathbb{R} : \\ x & \longrightarrow & (x,x) & \longrightarrow & (f(x),x) & \longrightarrow & d(f(x),x)\,, \end{array}$$

where each step is continuous. (I omit the proofs of these facts.) Thus it attains its (non-negative) infimum β: $\beta = d(f(y), y)$ for some $y \in \Omega$. If $\beta > 0$, then $f(y) \neq y$, and $d(f(f(y)), f(y)) < d(f(y), y) = \beta$, which is absurd. Hence $\beta = 0$; but this means precisely that $f(y) = y$. So there is a fixed point of f.

If $f(y) = y$ and $f(z) = z$ and $y \neq z$, then

$$d(f(y), f(z)) < d(y, z) = d(f(y), f(z)),$$

which is also absurd.

Again, there are other proofs; but this one is perhaps the most transparent.