Math 442

Exercise set 6, 2008

In various Hilbert spaces, there are standard orthonormal bases, which arose in many cases from important problems.

One of the most famous is the basis in $L^2(-1,1)$ formed by the *Legendre polynomials*, also called *Legendre coefficients* or *zonal harmonics*. They were invented explicitly by Legendre in 1785, although they had been known in substance before, and have applications in potential theory (solving Laplace's equation in spherical polars). Exercises 1–7 introduce some of their properties. There are several equivalent definitions; but, whichever you choose, it is an interesting exercise to deduce the other properties. You may find it easier to do the exercises in a different order, although I have tried to suggest a possible path to follow. Some steps are easy and obvious, whilst others are quite perplexing — the theory was developed over quite a long period by several mathematicians, in an era when heavy manipulation was the norm, so each of the various formulæ was an achievement.

1. Define, for $n = 0, 1, 2, \dots$,

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} \{(x^2 - 1)^n\} .$$

(*Rodrigues' formula* (1816) for the Legendre polynomial P_n.) Clearly P_n is a polynomial of degree n, as the nth derivative of a polynomial of degree $2n$. Calculate $P_0(x), P_1(x), P_2(x)$.

2. Show by integration by parts that, if $m > n \geq 0$,

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0 , \qquad \int_{-1}^{1} P_n(x)^2\, dx = \frac{2}{2n+1} .$$

3. Deduce from the previous exercise that $\left\{ \sqrt{n + \frac{1}{2}}\, P_n(x) : x = 0, 1, 2, \dots \right\}$ is an orthonormal *basis* in $L^2([-1,1]; \lambda)$, where λ is Lebesgue measure in $\mathbb{R}$. (You will need to make an assumption that I also made in 17.16.)

4. Show that

(a) $$\int_{-1}^{1} x P_n(x)\, P_m(x)\, dx = \begin{cases} 0 & \text{if } 0 \leq m < n-1 , \\ \dfrac{2n}{(2n+1)(2n-1)} & \text{if } m = n-1 ; \end{cases}$$

(b) $$\int_{-1}^{1} P_m(x)\, P_n'(x)\, dx = \begin{cases} 0 & \text{if } n \leq m , \\ 1 - (-1)^{m+n} & \text{if } n > m ; \end{cases}$$ and that

(c) $$\int_{-1}^{1} x P_m(x)\, P_n'(x)\, dx = \begin{cases} 0 & \text{if } n < m , \\ 1 + (-1)^{m+n} & \text{if } n > m , \\ \dfrac{2n}{2n+1} & \text{if } m = n . \end{cases}$$

over ⟶

5. Show that, for $n \geq 1$,

(a) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$,
(b) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$,
(c) $nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x)$,
(d) $(x^2-1)P_n'(x) = nxP_n(x) - nP_{n-1}(x)$.

[There are many possible proofs of these relations.]

6. Prove that $P_n(x)$ satisfies the differential equation (Legendre's differential equation of degree n)

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0\,.$$

7. Show that $P_n(x)$ may equivalently be defined as the coefficient of h^n in the expansion as a power series in h of $(1-2xh+h^2)^{-1/2}$.

8. Suppose (a,b) is a non-degenerate open interval in $\mathbb{R}$ (formal values $\pm\infty$ allowed), and $w : (a,b) \longrightarrow \mathbb{R}$ is a positive-valued continuous function on (a,b). Let $(q_n(x))_{n=0}^{\infty}$ be a sequence of polynomials in $L^2((a,b);w)$ (supposing they can exist) such that the degree of q_n is n (and q_0 is non-zero) and

$$\int_a^b q_m(x)\, q_n(x)\, w(x)\, dx = 0 \quad \text{for } m \neq n\,.$$

[This implies linear independence of $\{q_n : n = 0, 1, 2, \dots\}$.] Show that, for each n,

(a) $q_n(x)$ has exactly n distinct zeros, all in (a,b);

(b) for any real constant α, $q_{n+1} - \alpha q_n$ has $n+1$ distinct real roots, of which at least n lie in (a,b);

(c) between any two adjacent zeros of q_n, there is exactly one zero of q_{n+1} (and *vice versa*).

[Remark: *continuity* of w is a stronger condition than is really needed. It is perhaps easier to consider "changes of sign" rather than zeros.]

9. Define the *Hermite polynomial* $H_n(x)$ by the identity

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}\,.$$

Prove the "Rodrigues formula" $H_n(x) = (-1)^n \exp(x^2)\dfrac{d^n}{dx^n}(\exp(-x^2))$, and show

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x) \exp(-x^2)\, dx = \begin{cases} 0 & \text{if } m \neq n\,, \\ 2^n n! \sqrt{\pi} & \text{if } m = n\,. \end{cases}$$

Prove the relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \quad \text{and} \quad H_n'(x) = 2nH_{n-1}(x)\,.$$

[The Hermite polynomials do form an orthonormal basis in $L^2(\mathbb{R}, e^{-x^2})$, but a proof of this fact is rather non-trivial — not advanced or really difficult, but far from obvious either.]