# THE NUMBER OF POINTS IN A MATROID WITH NO $n$-POINT LINE AS A MINOR

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Abstract. For any positive integer $l$ we prove that if $M$ is a simple matroid with no $(l+2)$-point line as a minor and with sufficiently large rank, then $|E(M)| \leq \frac{q^{r(M)}-1}{q-1}$, where $q$ is the largest prime power less than or equal to $l$. Equality is attained by projective geometries over $\mathrm{GF}(q)$.

## 1. Introduction

Kung [5] proved the following theorem.

**Theorem 1.1.** *For any integer $l \geq 2$, if $M$ is a simple matroid with no $U_{2,l+2}$-minor, then $|E(M)| \leq \frac{l^{r(M)}-1}{l-1}$.*

The above bound is tight in the case that $l$ is a prime power and $M$ is a projective geometry. In fact, among matroids of rank at least 4, projective geometries are the only matroids that attain the bound; see [5]. Therefore, the bound is not tight when $l$ is not a prime power. We prove the following bound that was conjectured by Kung [5,4].

**Theorem 1.2.** *Let $l \geq 2$ be a positive integer and let $q$ be the largest prime power less than or equal to $l$. If $M$ is a simple matroid with no $U_{2,l+2}$-minor and with sufficiently large rank, then $|E(M)| \leq \frac{q^{r(M)}-1}{q-1}$.*

The case where $l = 6$ was resolved by Bonin and Kung in [2].

We will also prove that the only matroids of large rank that attain the bound in Theorem 1.2 are the projective geometries over $\mathrm{GF}(q)$; see Corollary 4.2.

A matroid $M$ is *round* if $E(M)$ cannot be partitioned into two sets of rank less than $r(M)$. We prove Theorem 1.2 by reducing it to the following result.

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**Theorem 1.3.** *For each prime power $q$, there exists a positive integer $n$ such that, if $M$ is a round matroid with a $PG(n-1,q)$-minor but no $U_{2,q^2+1}$-minor, then $\epsilon(M) \le \frac{q^{r(M)}-1}{q-1}$.*

For any integer $l \ge 2$, there is an integer $k$ such that $2^{k-1} < l \le 2^k$. Therefore, if $q$ is the largest prime power less than or equal to $l$, then $l < 2q$. So, to prove Theorem 1.2, it would suffice to prove the weaker version of Theorem 1.3 where $U_{2,q^2+1}$ is replaced by $U_{2,2q+1}$. With this in mind, we find the stronger version somewhat surprising.

We further reduce Theorem 1.3 to the following result.

**Theorem 1.4.** *For each prime power $q$ there exists an integer $n$ such that, if $M$ is a round matroid that contains a $U_{2,q+2}$-restriction and a $PG(n-1,q)$-minor, then $M$ contains a $U_{2,q^2+1}$-minor.*

The following conjecture, if true, would imply all of the results above.

**Conjecture 1.5.** *For each prime power $q$, there exists a positive integer $n$ such that, if $M$ is a round matroid with a $PG(n-1,q)$-minor but no $U_{2,q^2+1}$-minor, then $M$ is $GF(q)$-representable.*

The conjecture may hold with $n = 3$ for all $q$. Moreover, the conjecture may also hold when "round" is replaced by "vertically 4-connected".

## 2. Preliminaries

We assume that the reader is familiar with matroid theory; we use the notation and terminology of Oxley [6]. A rank-1 flat in a matroid is referred to as a *point* and a rank-2 flat is a *line*. A line is *long* if it has at least 3 points. The number of points in $M$ is denoted $\epsilon(M)$.

Let $M$ be a matroid and let $A, B \subseteq E(M)$. We define $\sqcap_M(A,B) = r_M(A) + r_M(B) - r_M(A \cup B)$; this is the *local connectivity* between $A$ and $B$. This definition is motivated by geometry. Suppose that $M$ is a restriction of $\mathrm{PG}(n-1,q)$ and let $F_A$ and $F_B$ be the flats of $\mathrm{PG}(n-1,q)$ that are spanned by $A$ and $B$ respectively. Then $F_A \cap F_B$ has rank $\sqcap_M(A,B)$. We say that two sets $A, B \subseteq E(M)$ are *skew* if $\sqcap_M(A,B) = 0$.

We let $\mathcal{U}(l)$ denote the class of matroids with no $U_{2,l+2}$-minor. Our proof of Theorem 1.2 relies heavily on the following result of Geelen and Kabell [3, Theorem 2.1].

**Theorem 2.1.** *There is an integer-valued function $\alpha(l,q,n)$ such that, for any positive integers $l,q,n$ with $l \ge q \ge 2$, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) \ge \alpha(l,q,n)q^{r(M)}$, then $M$ contains a $PG(n-1,q')$-minor for some prime-power $q' > q$.*

The following result is an important special case of Theorem 1.4.

**Lemma 2.2.** *If $M$ is a round matroid that contains a $U_{2,q+2}$-restriction and a $PG(2,q)$-restriction, then $M$ has a $U_{2,q^2+1}$-minor.*

*Proof.* Suppose that $M$ is a minimum-rank counterexample. Let $L, P \subseteq E(M)$ such that $M|L = U_{2,q+2}$ and $M|P = \mathrm{PG}(2,q)$. If $M$ has rank 3, then we may assume that $E(M) = P \cup \{e\}$. Since $M|P$ is modular, $e$ is in at most one long line of $M$. Then, since $|P| = q^2+q+1$, we have $\epsilon(M/e) \geq q^2+1$ and, hence, $M$ has a $U_{2,q^2+1}$-minor. This contradiction implies that $r(M) > 3$. Since $M$ is round, there is an element $e$ that is spanned by neither $L$ nor $P$. Now $M/e$ is round and contains both $M|L$ and $M|P$ as restrictions. This contradicts our choice of $M$. □

The base case of the following lemma is essentially proved in [3, Lemma 2.4].

**Lemma 2.3.** *Let $\lambda \in \mathbb{R}$. Let $k$ and $l \geq q \geq 2$ be positive integers, and let $A$ and $B$ be disjoint sets of elements in a matroid $M \in \mathcal{U}(l)$ with $\sqcap_M(A,B) \leq k$ and $\epsilon_M(A) > \lambda q^{r_M(A)}$. Then there is a set $A' \subseteq A$ that is skew to $B$ and satisfies $\epsilon_M(A') > \lambda l^{-k} q^{r_M(A')}$.*

*Proof.* By possibly contracting some elements in $B - \mathrm{cl}_M(A)$, we may assume that $A$ spans $B$ and thus that $r_M(B) = \sqcap_M(A,B)$. When $k = 1$, this means $B$ has rank 1. We resolve this base case first.

Let $e$ be a non-loop element of $B$. We may assume that $A$ is minimal with $\epsilon_M(A) > \lambda q^{r_M(A)}$, and that $E(M) = A \cup \{e\}$. Let $W$ be a flat of $M$ not containing $e$, such that $r_M(W) = r(M) - 2$. Let $H_0, H_1, \ldots, H_m$ be the hyperplanes of $M$ containing $W$, with $e \in H_0$. The sets $\{H_i - W : 1 \leq i \leq m\}$ are a disjoint cover of $E(M) - W$. Additionally, the matroid $\mathrm{si}(M/W)$ is isomorphic to the line $U_{2,m+1}$, so we know that $m \leq l$.

By the minimality of $A$, we get $\epsilon_M(H_0 \cap A) \leq \lambda q^{r(M)-1}$, so

$$\epsilon_M(A - H_0) > \lambda(q-1)q^{r(M)-1}.$$

Since the hyperplanes $H_1, \ldots, H_m$ cover $E(M) - H_0$, a majority argument gives some $1 \leq i \leq m$ such that

$$\epsilon_M(H_i \cap A) \geq \frac{1}{m}\epsilon_M(A - H_0) > \frac{\lambda}{l}(q-1)q^{r(M)-1}.$$

Setting $A' = A \cap H_i$ gives a set of the required number of points that is skew to $e$ and therefore to $B$, which is what we want.

Now suppose that the result holds for $k = t$ and consider the case that $k = t+1$. Let $A$ and $B$ be disjoint sets of elements in a matroid

$M$ with $\sqcap_M(A,B) \leq t+1$ and $\epsilon_M(A) > \lambda q^{r_M(A)}$. As mentioned earlier, we have $r_M(B) = \sqcap_M(A,B) \leq t+1$. Let $e$ be any non-loop element of $B$. By the base case, there exists $A' \subseteq A$ that is skew to $\{e\}$ and satisfies $\epsilon_M(A') > \lambda l^{-1} q^{r_M(A')}$. Since $e \notin \mathrm{cl}_M(A')$ and $r_M(B) \leq t+1$, we have $\sqcap_M(A', B) \leq t$. Now the result follows routinely by the induction hypothesis. □

The following two results are used in the reduction of Theorem 1.2 to Theorem 1.3.

**Lemma 2.4.** *Let $f(k)$ be an integer-valued function such that $f(k) \geq 2f(k-1) - 1$ for each $k \geq 1$ and $f(1) \geq 1$. If $M$ is a matroid with $\epsilon(M) \geq f(r(M))$ and $r(M) \geq 1$, then there is a round restriction $N$ of $M$ such that $\epsilon(N) \geq f(r(N))$ and $r(N) \geq 1$.*

*Proof.* We may assume that $M$ is not round and, hence, there is a partition $(A,B)$ of $E(M)$ such that $r_M(A) < r(M)$ and $r_M(B) < r(M)$. Clearly $r_M(A) \geq 1$ and $r_M(B) \geq 1$. Inductively we may assume that $\epsilon_M(A) < f(r_M(A))$ and $\epsilon_M(B) < f(r_M(B))$. Thus $\epsilon(M) \leq \epsilon(M|A) + \epsilon(M|B) \leq f(r_M(A)) + f(r_M(B)) - 2 \leq 2f(r(M)-1) - 2 < f(r(M))$, which is a contradiction. □

**Lemma 2.5.** *Let $q \geq 4$ and $t \geq 1$ be integers and let $M$ be a matroid with $\epsilon(M) \geq \frac{q^{r(M)}-1}{q-1}$ and $r(M) \geq 3t$. If $M$ is not round, then either $M$ has a $U_{2,q^2+2}$-minor or there is a round restriction $N$ of $M$ such that $r(N) \geq t$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$.*

*Proof.* Let $s = r(M)$ and let $f(k) = \left(\frac{q}{2}\right)^{s-k}\left(\frac{q^k-1}{q-1}\right)$. For any $k \geq 1$,

$$\begin{aligned} f(k+1) &= \left(\frac{q}{2}\right)^{s-k-1}\left(\frac{q^{k+1}-1}{q-1}\right) \\ &> \left(\frac{q}{2}\right)^{s-k-1}\left(q\frac{q^k-1}{q-1}\right) \\ &= 2f(k). \end{aligned}$$

Moreover $f(1) \geq 1$ and $\epsilon(M) \geq f(r(M))$. Then, by Lemma 2.4, there is a round restriction $N$ of $M$ such that $r(N) \geq 1$ and $\epsilon(N) \geq f(r(N))$. Since $M$ is not round, $r(N) < r(M) = s$ and, hence, $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$.

We may assume that $r(N) < t$. Therefore, since $s \geq 3t$ and $q \geq 4$,

$$\begin{aligned}
\epsilon(N) &\geq f(r(N)) \\
&= \left(\frac{q}{2}\right)^{s-r(N)} \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&\geq \left(\frac{q}{2}\right)^{2t} \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&\geq q^t \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&> q^{r(N)} \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&\geq \left(\frac{q^{r(N)}+1}{q+1}\right) \left(\frac{q^{r(N)}-1}{q-1}\right) \\
&= \left(\frac{q^{2r(N)}-1}{q^2-1}\right).
\end{aligned}$$

Therefore, by Theorem 1.1, $M$ has a $U_{2,q^2+2}$-minor, as required. $\square$

## 3. The main results

We start with a proof of Theorem 1.4, which we restate here.

**Theorem 3.1.** *There is an integer-valued function $n(q)$ such that, for each prime power $q$, if $M$ is a round matroid that contains a $U_{2,q+2}$-restriction and a $PG(n(q)-1,q)$-minor, then $M$ has a $U_{2,q^2+1}$-minor.*

*Proof.* Recall that the function $\alpha(l,q,n)$ was defined in Theorem 2.1. Let $q$ be a prime power, let $\alpha = \alpha(q^2-1, q-1, 3)$. Let $n$ be an integer that is sufficiently large so that $\left(\frac{q}{q-1}\right)^n > \alpha q^5(q-1)^2$. We define $n(q) = n$. Suppose that the result fails for this choice of $n(q)$ and let $M$ be a minimum-rank counterexample. Thus $M$ is a round matroid having a line $L$, with at least $q+2$ points, and a minor $N$ isomorphic to PG$(n-1,q)$, but $M \in \mathcal{U}(q^2-1)$.

Suppose that $N = M/C \setminus D$ where $C$ is independent. If $e \in C - L$, then $M/e$ is round, contains the line $L$, and has $N$ as minor — contrary to our choice of $M$. Therefore $C \subseteq L$ and, hence, $r(M) \leq r(N) + 2 \leq n+2$.

Let $X = E(M) - L$. By our choice of $n$, we have $\epsilon(M|(X-D)) \geq \frac{q^n-1}{q-1} - (q^2+1) = q^3\frac{q^{n-3}-1}{q-1} + q \geq q^{n-1} > q^4\alpha(q-1)^{n+2} \geq q^4\alpha(q-1)^{r_M(X)}$. By Lemma 2.3, there is a flat $F \subseteq X - D$ of $M$ that is skew to $L$ and satisfies $\epsilon(M|F) \geq \alpha(q-1)^{r_M(F)}$. Since $F$ is skew to $L$, $F$ is also skew to $C$. Therefore $M|F = N|F$ and hence $M|F$ is GF($q$)-representable. Then, by Theorem 2.1, $M|F$ has a PG$(2,q)$-minor. Therefore there is a set $Y \subseteq F$ such that $(M|F)/Y$ contains a PG$(2,q)$-restriction. Now $M/Y$ is round, contains a $(q+2)$-point line, and contains a PG$(2,q)$-restriction. Then, by Lemma 2.2, $M$ has a $U_{2,q^2+1}$-minor. □

Now we will prove Theorem 1.3 which we reformulate here. The function $n(q)$ was defined in Theorem 3.1.

**Theorem 3.2.** *For each prime power $q$, if $M$ is a round matroid with a PG$(n(q)-1,q)$-minor but no $U_{2,q^2+1}$-minor, then $\epsilon(M) \le \frac{q^{r(M)}-1}{q-1}$.*

*Proof.* Let $M$ be a minimum-rank counterexample. By Lemma 2.2, $r(M) > n(q)$. Let $e \in E(M)$ be a non-loop element such that $M/e$ has a PG$(n-1,q)$-minor. Note that $M/e$ is round. Then, by the minimality of $M$, $\epsilon(M/e) \le \frac{q^{r(M)-1}-1}{q-1}$. By Theorem 3.1, each line of $M$ containing $e$ has at most $q+1$ points. Hence $\epsilon(M) \le 1 + q\epsilon(M/e) \le 1+q\left(\frac{q^{r(M)-1}-1}{q-1}\right) = \frac{q^{r(M)}-1}{q-1}$. This contradiction completes the proof. □

We can now prove our main result, Theorem 1.2, which we restate below.

**Theorem 3.3.** *Let $l \ge 2$ be a positive integer and let $q$ be the largest prime power less than or equal to $l$. If $M$ is a matroid with no $U_{2,l+2}$-minor and with sufficiently large rank, then $\epsilon(M) \le \frac{q^{r(M)}-1}{q-1}$.*

*Proof of Theorem 1.2.* When $l$ is a prime-power, the result follows from Theorem 1.1. Therefore we may assume that $l \ge 6$ and, hence, $q \ge 5$. Recall that $n(q)$ is defined in Theorem 3.1 and $\alpha(l,q-1,n)$ is defined in Theorem 2.1. Let $n = n(q)$ and let $k$ be an integer that is sufficiently large so that $\left(\frac{q}{q-1}\right)^k \ge q\alpha(l,q-1,n)$. Thus, for any $k' \ge k$, we get $\frac{q^{k'}-1}{q-1} \ge q^{k'-1} \ge \alpha(l,q-1,n)(q-1)^{k'}$. Let $M \in \mathcal{U}(l)$ be a matroid of rank at least $3k$ such that $\epsilon(M) > \frac{q^{r(M)}-1}{q-1}$. By Lemma 2.5, $M$ has a round restriction $N$ such that we have $r(N) \ge k$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1} \ge \alpha(l,q-1,n)(q-1)^{r(N)}$. By Theorem 2.1, $N$ has a PG$(n(q)-1,q')$-minor for some $q' > q-1$. If $q' > q$, then $q'+1 \ge l+2$, so this projective geometry has a $U_{2,l+2}$-minor, contradicting our hypothesis. We may therefore conclude that $q' = q$, so $N$ has a PG$(n(q)-1,q)$-minor. Now we get a contradiction by Theorem 3.2. □

## 4. Extremal Matroids

In this section, we prove that the extremal matroids of large rank for Theorem 1.2 are projective geometries. We need the following result to recognize projective geometries; see Oxley [6, Theorem 6.1.1].

**Lemma 4.1.** *Let $M$ be a simple matroid of rank $n \geq 4$ such that every line of $M$ contains at least three points and each pair of disjoint lines of $M$ is skew. Then $M$ is isomorphic to* $\mathrm{PG}(n-1, q)$ *for some prime power $q$.*

We can now prove our extremal characterization.

**Corollary 4.2.** *Let $l \geq 2$ be a positive integer and let $q$ be the largest prime power less than or equal to $l$. If $M$ is a simple matroid with no $U_{2,l+2}$-minor, with $\epsilon(M) = \frac{q^{r(M)}-1}{q-1}$, and with sufficiently large rank, then $M$ is a projective geometry over $GF(q)$.*

*Proof.* Kung [5] proved the result for the case that $l$ is a prime-power. Therefore we may assume that $l \geq 6$ and, hence, $q \geq 5$. By Theorem 1.2, there is an integer $k_1$ such that, if $M$ is a matroid with no $U_{2,l+2}$-minor and with $r(M) \geq k_1$, then $\epsilon(M) \leq \frac{q^{r(M)}-1}{q-1}$. Recall that $n(q)$ is defined in Theorem 3.1 and $\alpha(l, q, n)$ is defined in Theorem 2.1. Let $k_2$ be large enough so that $\left(\frac{q}{q-1}\right)^{k_2} \geq q\alpha(l, q-1, n(q)+2)$, and $k = \max(k_1, k_2)$.

Let $M \in \mathcal{U}(l)$ be a simple matroid of rank at least $3k$ such that $\epsilon(M) = \frac{q^{r(M)}-1}{q-1}$. If $M$ is not round, then, by Lemma 2.5, $M$ has a round restriction $N$ such that $r(N) \geq k$ and $\epsilon(N) > \frac{q^{r(N)}-1}{q-1}$, contrary to Theorem 1.2. Hence $M$ is round.

From the definition of $k_2$, we get $\epsilon(M) \geq \alpha(l, q-1, n(q)+2)(q-1)^{r(M)}$, so by Theorem 2.1, $M$ has a $\mathrm{PG}(n(q)+1, q)$-minor. Therefore, by Theorem 3.1, each line in $M$ has at most $q+1$ points. Consider any element $e \in E(M)$. By Theorem 1.2, $\epsilon(M/e) \leq \frac{q^{r(M)-1}-1}{q-1}$. Then

$$\begin{aligned}
\epsilon(M) &\leq 1 + q\epsilon(M/e) \\
&\leq 1 + q\left(\frac{q^{r(M)-1}-1}{q-1}\right) \\
&= \frac{q^{r(M)}-1}{q-1} \\
&= \epsilon(M).
\end{aligned}$$

The inequalities above must hold with equality. Therefore each line in $M$ has exactly $q+1$ points.

If $M$ is not a projective geometry, then, by Lemma 4.1, there are two disjoint lines $L_1$ and $L_2$ in $M$ such that $\sqcap_M(L_1, L_2) = 1$. Let $e \in L_1$. Then $L_2$ spans a line with at least $q+2$ points in $M/e$. Since $M$ has a PG$(n(q)+1, q)$-minor, $M/e$ contains a PG$(n(q)-1, q)$-minor; see [1, Lemma 5.2]. This contradicts Theorem 3.1. $\square$

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## References

[1] J. Geelen, B. Gerards, G. Whittle, On Rota's conjecture and excluded minors containing large projective geometries, J. Combin. Theory Ser. B 96 (2006), 405-425.
[2] J.E. Bonin, J.P.S. Kung, The number of points in a combinatorial geometry with no 8-point-line minors, Mathematical essays in honor of Gian-Carlo Rota, Cambridge, MA (1996), 271-284, Progr. Math., 161, Birkhäuser Boston, Boston, MA, (1998).
[3] J. Geelen, K. Kabell, Projective geometries in dense matroids, J. Combin. Theory Ser. B 99 (2009), 1-8.
[4] J. Geelen, J.P.S. Kung, G. Whittle, Growth rates of minor-closed classes of matroids, J. Combin. Theory Ser. B 99 (2009), 420-427.
[5] J.P.S. Kung, Extremal matroid theory, in: Graph Structure Theory (Seattle WA, 1991), Contemporary Mathematics, 147, American Mathematical Society, Providence RI, 1993, pp. 21–61.
[6] J. G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

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