AN ANALOGUE OF THE ERDŐS-STONE THEOREM FOR FINITE GEOMETRIES

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ABSTRACT. For a set G of points in $PG(m-1,q)$, let $ex_q(G;n)$ denote the maximum size of a collection of points in $PG(n-1,q)$ not containing a copy of G, up to projective equivalence. We show that

$$\lim_{n\to\infty} \frac{ex_q(G;n)}{|PG(n-1,q)|} = 1 - q^{1-c},$$

where c is the smallest integer such that there is a rank-$(m-c)$ flat in $PG(m-1,q)$ that is disjoint from G. The result is an elementary application of the density version of the Hales-Jewett Theorem.

1. INTRODUCTION

Note that if M is a rank-$(r-c+1)$ flat of $PG(r-1,q)$, then $|M| = \frac{q^{r-c+1}-1}{q-1}$ and each rank-m flat of $PG(r-1,q)$ intersects M in a flat of rank at least $m-c+1$. Our main result is the following:

Theorem 1.1 (Main Theorem)**.** *For each prime-power q, all integers $m > c \geq 0$, and any real number $\epsilon > 0$, there is an integer $R = R_{1.1}(m,q,c,\epsilon)$ such that, if $n > R$ and G is a set of points in $PG(n-1,q)$ with $|G| \leq (1-\epsilon)\left(\frac{q^{n-c+1}-1}{q-1}\right)$, then there exists a rank-m flat F of $PG(n-1,q)$ such that $rank(F \cap G) \leq m-c$.*

We were motivated by a problem in extremal matroid theory posed by Kung [7]; the matroidal origins of the problem are reflected in our terminology which we briefly review below.

Let $\mathbb{F}$ be a finite field of order q and let V be a rank-r vector space over $\mathbb{F}$. A *rank-k flat* of $PG(r-1,\mathbb{F})$ is a $(k+1)$-dimensional subspace of V; the *points* are the rank-1 flats; the *lines* are the rank-2 flats; and the *hyperplanes* are the rank-$(r-1)$ flats. Technically the projective geometry depends on the particular vector space V; to make this explicit, we write $PG(V)$ for the projective geometry given by V.

Date: March 9, 2012.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. matroids, Erdős-Stone Theorem, Bose-Burton Theorem, Hales-Jewett Theorem, projective geometries, critical exponent.

We refer to a set H of points in $PG(r-1,\mathbb{F})$, for some r, as a *geometry over* $\mathbb{F}$ and we define $rank(H)$ to be the rank of the flat spanned by H. If H and G are geometries over $\mathbb{F}$, then there are vector spaces V_1 and V_2 over $\mathbb{F}$ so that H is a spanning set of points in $PG(V_1)$ and G is a spanning set of points in $PG(V_2)$. We say that H *is a restriction of* G or that G *contains* H, if there is a rank-preserving projective transformation from V_2 to a vector space V_2' containing V_1 so that H is contained in the image of G.

For a geometry H over $\mathbb{F}$ and positive integer n, we let $ex_q(H;n)$ denote the maximum number of points in a rank-n geometry over $\mathbb{F}$ not containing H.

For integers $0 \le c \le m$, let F be a rank-$(m-c)$ flat of $PG(m-1,q)$ and let $G(m-1,q,c)$ be the geometry obtained by restricting $PG(m-1,q)$ to the complement of F; thus $G(m-1,q,m) = PG(m-1,q)$ and $G(m-1,q,1) = AG(m-1,q)$, the rank-m affine geometry over $GF(q)$. The *critical exponent* of H over $GF(q)$, written $c(H;q)$, is the minimum c such that H is contained in $G(r(M)-1,q,c)$. The critical exponent was introduced by Crapo and Rota [3] and is related to the chromatic number of a graph.

The following result, which is an easy corollary of Theorem 1.1, was all but conjectured by Kung [7].

Theorem 1.2. *Let $\mathbb{F}$ be a finite field of order q. If H is a geometry over $\mathbb{F}$ with with critical exponent $c > 0$, then*

$$\lim_{n\to\infty} \frac{ex_q(H;n)}{|PG(n-1,q)|} = 1 - q^{1-c}.$$

This theorem bears a striking resemblance to the following theorem of Erdős and Stone [4]. For a graph H, let $ex(H;n)$ denote the maximum number of edges in a simple n-vertex graph that does not contain a subgraph isomorphic to H. The *chromatic-number*, $\chi(G)$, of a graph G is the minimum number of colours needed to colour the vertices so that no two adjacent vertices get the same colour.

Theorem 1.3 (Erdős-Stone Theorem)**.** *For any graph H with chromatic-number $\chi \ge 2$,*

$$\lim_{n\to\infty} \frac{ex(H;n)}{\binom{n}{2}} = 1 - \frac{1}{\chi - 1}.$$

2. Old results

In this section we briefly review related results. Note that $G(n-1,q,m-1)$ does not contain $PG(m-1,q)$; Bose and Burton [2] showed that $G(n-1,q,m-1)$ is extremal among geometries not containing $PG(m-1,q)$.

Theorem 2.1. *Let $\mathbb{F}$ be a field of order q and m and r be integers with $n \geq m \geq 0$. Then*

$$ex_q(PG(m-1,\mathbb{F});\, n) = |G(n-1,q,m-1)|.$$

Bonin and Qin [1] determine $ex_q(H;n)$ exactly for several other interesting families of geometries.

Our main result is an easy application of the following deep result due to Furstenberg and Katznelson [5, Theorem 9.10] in 1985.

Theorem 2.2. *For each field $\mathbb{F}$ of order q, integer $m \geq 2$, and real number $\epsilon > 0$, there is an integer $R = R_{2.2}(m,q,\epsilon)$ such that,*

$$ex_q(AG(m-1,\mathbb{F});\, n) < \epsilon|PG(n-1,q)|$$

for all $n > R$.

This result can be obtained as an easy application of the density version of the multidimensional Hales-Jewett theorem, also proved by Furstenberg and Katznelson [6], in 1991, using ergodic theory. An easier proof was later obtained via the polymath project [8]. The "easier proof" is still, however, more than 30 pages long. Bonin and Qin [1] have a much simpler proof of Theorem 2.2 in the case that $q = 2$.

3. New results

We start with a proof of Theorem 1.1; for convenience we restate it in a complementary form. (The equivalence between the two statements is easy and is left to the reader.)

Theorem 3.1 (Reformulation of Theorem 1.1)**.** *For any integers $m > c \geq 1$ and real number $\epsilon > 0$, there is an integer $R = R_{3.1}(m,q,c,\epsilon)$ such that,*

$$ex_q(G(m-1,q,c);n) < (1-q^{1-c}+\epsilon)|PG(n-1,q)|,$$

for all $n > R$.

Proof. Let $m > c \geq 1$ be integers and let $\epsilon > 0$ be a real number. The proof is by induction on c; the case that $c = 1$ follows directly from Theorem 2.2. Assume that $c > 1$ and that the result holds for $c-1$.

Let $r = R_{2.2}(m-c+1,q,\epsilon/2)$, let t be sufficiently large so that $q^{1-c}(q^r-1) \leq \frac{\epsilon}{2}(q^n-q^r)$ for all $n > t$, and define

$$R_{3.1}(m,q,c,\epsilon) = \max(t, R_{3.1}(r,q,c-1,q^{2-c}-q^{1-c})).$$

Now let $n > R_{3.1}(m, q, c, \epsilon)$ and let M be a restriction of $PG(n-1, q)$ with $|M| \geq (1 - q^{1-c} + \epsilon)|PG(n-1, q)|$.

By the inductive assumption, M has a $G(r-1, q, c-1)$-restriction. Thus there are flats $F_0 \subseteq F_1$ of $PG(n-1, q)$ such that $rank(F_1) = r$, $rank(F_0) = r - c + 1$, and $F_1 - F_0 \subseteq M$. Let $F_0^c \subseteq F_1$ be a rank-$(c-1)$ flat that is disjoint from F_0.

Note that, by our definition of t,

$$|M \setminus F_1| \geq \left(1 - q^{1-c} + \tfrac{\epsilon}{2}\right) |PG(n-1, q) - F_1|.$$

So by an elementary averaging argument, there exists a rank-$(r+1)$ flat F_2 containing F_1 such that

$$|M \cap (F_2 - F_1)| \geq \left(1 - q^{1-c} + \tfrac{\epsilon}{2}\right) |F_2 - F_1| = (1 - q^{1-c})q^r + \tfrac{\epsilon}{2} q^r.$$

We want to find a rank-m flat $F \subseteq F_2$ such that $F_0^c \subseteq F \not\subseteq F_1$ and $F - F_1 \subseteq M$. If F satisfies these conditions, then $rank(F \cap F_0) = m - c$ and, hence, the restriction of M to F contains $G(m-1, q, c)$.

Let $S = (F_2 - F_1) \cap M$. For a flat F of $PG(n-1, q)$ and point $e \notin F$, we let $F + e$ denote the flat spanned by $F \cup \{e\}$. Let $e \in F_2 - F_1$ and let $Q = (F_0 + e) - F_0$. Now, for each $f \in Q$, let $S_f = (F_0^c + f) \cap S$. Note that $(S_f : f \in Q)$ partitions S and $|S_f| \leq q^{c-1}$. Finally, let Q_1 be the set of all $f \in Q$, such that $|S_f| = q^{c-1}$.

All vectors in $Q - Q_1$ extend to at most $q^{c-1} - 1$ elements in S, so

$$\begin{aligned}
(q^{c-1} - 1)q^{r-c+1} + |Q_1| &\geq |S| \\
&\geq \left(1 - q^{1-c} + \tfrac{\epsilon}{2}\right) q^r \\
&= (q^{c-1} - 1)q^{r-c+1} + \tfrac{\epsilon}{2} q^r.
\end{aligned}$$

Thus $|Q_1| \geq \frac{\epsilon}{2} q^r$. By Theorem 2.2, there is a subset Q_2 of Q_1 such that $Q_2 \cong AG(m-c, q)$. Let F be the flat of $PG(n-1, q)$ spanned by F_0^c and Q_2. Thus F has rank m, $F_0^c \subseteq F$, and, since $Q_2 \subseteq Q_1$, $F - F_1 \subseteq M$. So the restriction of M to $F - F_0$ gives $G(m-1, q, c)$. $\square$

We can now prove Theorem 1.2, which we restate here for convenience.

Corollary 3.2. *Let $\mathbb{F}$ be a finite field of order q. If H is a geometry over $\mathbb{F}$ with with critical exponent $c > 0$, then*

$$\lim_{n \to \infty} \frac{ex_q(H; n)}{|PG(n-1, q)|} = 1 - q^{1-c}.$$

Proof. Observe that H is a restriction of $G(r(N) - 1, q, c)$ but it is not a restriction of $G(n-1, q, c-1)$. Then, by Theorem 3.1, for all $\epsilon > 0$

and all sufficiently large n,

$$\frac{q^n}{q^n-1}(1-q^{1-c}) = \frac{|G(n-1,q,c-1)|}{|PG(n-1,q)|} \leq \frac{ex_q(H;n)}{|PG(n-1,q)|} \leq 1-q^{1-c}+\epsilon,$$

so the result holds. □

The value of $R_{3.1}(m,q,c,\epsilon)$ provided by Theorem 3.1 depends on that of $R_{2.2}(m,q,\epsilon)$, for which the bounds in [8] are Ackermann-like for all $q > 2$. In the binary case, however, the main theorem of [1] implies that the relatively small function $R_{2.2}(m,2,\epsilon) = 2^{m-2}\lceil 1-\log_2 \epsilon\rceil$ will satisfy Theorem 2.2. From this, one can derive from the proof that $R_{3.1}(m,2,c,\epsilon) = T_c(m+d)$ will satisfy Theorem 3.1, where $d = \lceil \log_2 \lceil (2-\log_2 \epsilon)\rceil\rceil$, and T_c is the tower function recursively defined by $T_0(s) = s$ and $T_i(s) = T_{i-1}(2^s)$ for all $i > 0$.

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