Trajectory Singularities for a Class of Parallel Motions

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Abstract. A rigid body, three of whose points are constrained to move on the coordinate planes, has three degrees of freedom. Bottema and Roth [2] showed that there is a point whose trajectory is a solid tetrahedron, the vertices representing corank 3 singularities. A theorem of Gibson and Hobbs [9] implies that, for general 3-parameter motions, such singularities cannot occur generically. However motions subject to this kind of constraint arise as interesting examples of parallel motions in robotics and we show that, within this class, such singularities can occur stably.

Mathematics Subject Classification (2000). Primary 70B15; Secondary 57R45.

Keywords. screw system, trajectory singularity, parallel motion.

1. Introduction

The presence of singularities in the motion of a robot manipulator presents both difficulties in terms of control and potential advantages in task performance. Understanding the singularity types that arise is valuable therefore for optimal manipulator design. While it is true that engineers frequently use special design geometry to bring about specific outcomes, nevertheless stable phenomena are still desirable if one is to allow some tolerance in the manufacture and operation of a manipulator. In this paper, the authors explore the occurrence of certain highly degenerate singularities in a class of parallel manipulators from the perspective of stability.

The International Organisation for Standardisation (ISO) [13] defines a *manipulator* to be:

> a machine, the mechanism of which usually consists of a series of segments, jointed or sliding relative to one another, for the purpose of grasping and/or moving objects (pieces or tools) usually in several degrees of freedom. It may be controlled by an operator, a programmable

electronic controller, or any logic system (for example cam device, wired, etc.)

Of special interest is the motion of the *end-effector*, the grasping component or component to which the operative tool is attached, which may be represented as a function of the joint variables.

From the point of view of kinematics we are interested in the geometry of the manipulator, rather than its control. The class of *parallel* manipulators is distinguished from *serial* manipulators in that they include closed chains of linked or jointed components. Characteristic advantages of parallel manipulators are that they generally can handle heavier payloads and have greater accuracy. Their inverse kinematics—determining joint variables from knowledge of the configuration of the end-effector—is generally easier than their forward kinematics.

In [6], the authors showed that trajectory singularities of tracing points in the end-effector of a manipulator form *instantaneous singular sets* whose geometry is determined by the screw system of the manipulator at that configuration. The Hunt–Gibson class of screw system [10] also provides first-order information, i.e. the corank, for the singular trajectories.

In this paper, the implications are explored for a particular class of parallel manipulators, namely those for which a given set of k points is each constrained to lie on a surface, in particular, the case $k = 3$. We will call such a device a *k-point mechanism.* The contact of one point of a rigid body with the surface of another constitutes the simplest kind of joint. Although, in the terminology of mechanisms, it is a higher pair, in many cases it is possible to synthesise the resulting motion using lower pairs (joints with contacting surfaces). Each joint of this kind imposes, in general, one constraint, or the loss of one degree of freedom, on the moving body.

The motivating example is due initially to Darboux. He considered the 3 degree-of-freedom rigid-body motion, generated by three points in the body being constrained to lie in three planes in general position, and whether a fourth point could move in a plane, which he answered in the negative. Bottema and Roth [2] analysed this motion more carefully in the case that the planes are mutually orthogonal. They showed that if the triangle formed by the constrained points is acute, then there is a point in the body whose trajectory is a solid tetrahedron. Such a situation arises when the normals to the constraining planes at the points of contact mutually intersect. The point of intersection in the moving body lies at the vertex of its tetrahedral trajectory and is at a corank 3 singularity.

Two practical manipulators that incorporate 3-point mechanisms are:

Remote Centre Compliance Device: a device attached to a robot-arm end-effector to facilitate peg-in-hole insertion tasks. The device was invented by Watson, Nevins and Whitney [14, 16, 17] . It is not actively controlled but passively responsive to forces and torques at the end-effector tip, where the peg is held. In a simplified form, three rigid rods of equal length connect the vertices of an equilateral triangle in the base to a similar smaller triangle in the the end-effector, by means of ball joints. Hence, the three joints in the end-effector are effectively constrained to move on the surfaces of spheres. In its relaxed configuration the axes of the rods intersect at the end-effector tip. This component handles rotation of the peg in response to torques.

HVRam mechanism: designed for control of telescope mirror focussing. The device is analysed in detail by Carretero *et al* [3, 4]. Three hydraulically extensible (P joint) arms in the plane of the base are each connected by revolute (R) joints, with axes perpendicular to the arms in the same plane, to legs of fixed and equal length. These in turn are connected to the mirror by ball (S) joints, thus forming a 3-PRS architecture. The optimal positioning of the component arms is the subject of the second paper, but in its simplest form, they are symmetrically placed at angles of $2\pi/3$. The three joints in the mirror (end–effector) are constrained to move on three planes which intersect in the focal axis of the mirror when in its home configuration (mirror parallel to the base plane). The associated singularities mean that the tracking motion of the mirror is an order of magnitude smaller than the input through extension of the hydraulic legs, leading to superior control.

Section 2 of this paper describes the mathematical formulation for analysing rigid-body motions, together with basic results on screw systems and instantaneous singular sets, including the Genericity Theorem. Section 3 establishes the relationship between the configuration of contact normals for a 3-point mechanism and the associated screw system. The key result in section 4 is to establish conditions under which high-corank singularities appear stably. Although the theorem disproves a natural genericity hypothesis for this class of parallel motions, the phenomenon explains why such motions provide a valuable class from the point of view of mechanical advantage and control. Application of the result to the examples above appears in Section 5.

2. Motions, screw systems and ISSs

2.1. Motions and trajectories

We shall restrict our attention to those spatial motions for which the underlying joint space, encoding all the feasible combinations of joint variables for a given manipulator, is a smooth manifold. In practice, for most design geometries, the joint space is in fact an algebraic variety, and is smooth for almost all choices of design parameters (e.g. component lengths). Let $SE(3)$ denote the Euclidean isometry group $SE(3)$, combining rotations and translations in the semi-direct product $SO(3) \ltimes \mathbb{R}^3$. It is a 6-dimensional Lie group. By assigning orthonormal coordinate frames to the rigid body (moving coordinates) and the ambient space (fixed coordinates), configurations of the body can be represented by elements of the group.

Definition 2.1. A **spatial rigid-body motion** is a smooth function $\lambda : M \to SE(3)$, where M, the joint space of the motion, is a manifold. The rank d of (the derivative of) λ at a given configuration $x \in M$ is the **infinitesimal degree of freedom at** x. The maximum value of the infinitesimal degrees of freedom over M is the **degree of freedom** of the motion.

The rigid-body motion of the end-effector of a mechanical devices is frequently called a *kinematic mapping*, but we stick to the terminology of motions as it confers greater generality. A rigid-body motion $\lambda : M \to SE(3)$ may be represented by $\lambda(x) = (A(x), \mathbf{a}(x))$, where $A(x) \in SO(3)$ and $\mathbf{a}(x) \in \mathbb{R}^3$. Given a point $\mathbf{w} \in \mathbb{R}^3$ of the rigid body (in moving coordinates), the *trajectory* of $\mathbf{w}$ is determined by the action of the $SE(3)$ on $\mathbb{R}^3$, that is by the function

$$\tau_w : M \to \mathbb{R}^3, \qquad \tau_w(x) = \lambda(x).\mathbf{w} = A(x)\mathbf{w} + \mathbf{a}(x). \tag{2.1}$$

Note that τ_w can be thought of as a composition of the action with the motion λ itself. That is, if $ev_w : SE(3) \to \mathbb{R}^3$ is the map

$$ev_w(A, \mathbf{a}) = A\mathbf{w} + \mathbf{a}, \tag{2.2}$$

then $\tau_w = ev_w \circ \lambda$. It is valuable to regard the trajectories as forming a family parametrised by $\mathbf{w} \in \mathbb{R}^3$:

$$\tau : M \times \mathbb{R}^3 \to \mathbb{R}^3; \qquad \tau(x, \mathbf{w}) = \tau_w(x). \tag{2.3}$$

Given positive integers r and k, there is an induced multijet extension ${}_rj^k\tau_w : M^{(r)} \to {}_rJ^k(M, \mathbb{R}^3)$, where $M^{(r)}$ is the manifold of r-tuples of distinct points in M. Since τ depends smoothly on $\mathbf{w}$, there is also a three-parameter family of multijets:

$${}_rj_1^k\tau : M^{(r)} \times \mathbb{R}^3 \longrightarrow {}_rJ^k(M, \mathbb{R}^3),$$

where the subscript 1 indicates we are taking jets with respect to the first component only. The following Genericity Theorem [9] concerns general kinematics of rigid-body motions.

Theorem 2.2. *Let $\mathcal{S}$ be a finite stratification of ${}_rJ^k(M, \mathbb{R}^3)$. The set of rigid-body motions $\lambda : M \to SE(3)$ with ${}_rj_1^k\tau$ transverse to $\mathcal{S}$ is residual in $C^\infty(M, SE(3))$, endowed with the Whitney C^∞ topology.*

A relevant example is to apply the theorem to 3-dof spatial motions ($\dim M = 3$) and to consider monogerms of 1-jets ($r = k = 1$) with stratification by corank of the derivative. The corank 1 stratum has codimension 1so for a generic motion, given any tracing point $\mathbf{w} \in \mathbb{R}^3$, there would be a surface in M (or possibly no points) where the trajectory τ_w has a corank 1 singularity. There would be a surface of tracing points $w \in \mathbb{R}^3$ whose trajectories possess isolated corank 2 singularities (codimension 4 stratum), and there would be no tracing points with corank 3 singularities (codimension 9 stratum).

A working hypothesis is that such a theorem holds true for classes of motion arising from specific mechanism geometries, such as the k-point mechanisms under consideration here. The difficulty is that such classes typically depend only on finitely many parameters. So the validity of the hypothesis depends on how that finite-dimensional family sits within the infinite-dimensional space of all motions.

2.2. Screw systems

The Lie algebra $se(3)$ of the Euclidean group inherits its semi-direct product structure: $so(3) \ltimes t(3)$. The Lie algebra $so(3)$, corresponding to the rotations, consists of the skew–symmetric 3×3 matrices. That means we can write $B \in so(3)$ in the form

$$B = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

which we identify with the vector $\mathbf{u} = (u_1, u_2, u_3)$. Note that $\mathbf{u}$ spans the kernel of B, so long as $B \neq 0$. The instantaneous translations may be represented by a 3-vector $\mathbf{v} = (v_1, v_2, v_3)$. In the engineering literature, elements of the Lie algebra are referred to as *motors* and elements of the corresponding 5-dimensional projective space $Pse(3)$ are called *screws*. Thus $(\mathbf{u}, \mathbf{v})$ are referred to as *motor coordinates* and the homogeneous version $(\mathbf{u}; \mathbf{v})$ as *screw coordinates*.

Given a rigid-body motion $\lambda : M \to SE(3)$, the instantaneous motion at a configuration $x \in M$ is given by the image of the derivative of λ at x, a subspace of $T_{\lambda(x)}SE(3)$. By suitable choice of moving and fixed coordinates, we may assume this to be a subspace of the Lie algebra $se(3)$. The corresponding projective subspace in $Pse(3)$ is called a *screw system*. If λ has k-dof at x then it is a *k-system*.

The classification of screw systems was originally proposed by Hunt [12], and given a firm mathematical basis by Gibson and Hunt [10]. They introduced the pencil of pitch quadrics $Q_h = 0$, where $h \in \mathbb{R} \cup \{\infty\}$ is the *pitch*. Explicitly, Q_h is a quadratic form, given in motor coordinates by:

$$Q_h(\mathbf{u}, \mathbf{v}) = \mathbf{u}.\mathbf{v} - h(\mathbf{u}.\mathbf{u}), \quad h \neq \infty; \quad Q_\infty(\mathbf{u}, \mathbf{v}) = \mathbf{u}.\mathbf{u} \tag{2.4}$$

Since the forms are homogeneous, the corresponding varieties (quadrics) are well-defined in screw coordinates. The special case $Q_0 = 0$ corresponds to the classical Klein quadric of lines in projective 3-space, where we identify a pure rotation with its axis; in this case, the coordinates correspond to Plücker line coordinates of the axis (see, for example, [15]). We note for later reference that by viewing a line as an intersection of a pencil of planes, it may also be represented by its *dual* Plücker coordinates $(\mathbf{u}^*; \mathbf{v}^*) = (\mathbf{v}, \mathbf{u})$. Application of duality enables geometric assertions about points and lines to be transformed into statements about planes and lines, and vice versa.

The quadric $Q_\infty = 0$, corresponding to infinitesimal translations, degenerates to a (projective) plane. Thus the family S_h of sets of screws of each pitch h, in the screw system S, is just its pencil of intersections with the hypersurfaces $Q_h = 0$— in the case of 3-systems, this gives a pencil of conics, as had been recognised by Ball [1]. Each pitch quadric $Q_h = 0$, $h \neq \infty$, has a pair of rulings: by the α-planes corresponding to all screws of pitch h whose axis passes through a given point, and by the β-planes corresponding to all screws whose axis lies in a given plane.

The quadratic forms (2.4) have associated bilinear forms; in particular, for a pair of motors $\mu_i = (\mathbf{u}_i, \mathbf{v}_i)$, $i = 1, 2$, we have $Q_0(\mu_1, \mu_2) = \frac{1}{2}(\mathbf{u}_1.\mathbf{v}_2 + \mathbf{u}_2.\mathbf{v}_1)$. Given a k-system S, define the *reciprocal* $(6-k)$-system

$$S^\perp = \{\$ \,|\, Q_0(\$, \$') = 0, \forall \$' \in S\}.$$

The broad classification of 3-systems is on the following basis. Type I systems do not lie wholly in a single pitch quadric and therefore they intersect the pitch quadrics in a pencil of conics as indicated above. Type II systems are those contained within a single pitch quadric. The subtypes A, B, C, D distinguish the projective dimension of intersection with Q_∞: subtype A denoting empty intersection, up to subtype D denoting a 2-dimensional intersection. Within the type I systems, further distinction is made on the basis of the projective type of the pencil of conics.

Further refinement of the classification was provided in [7, 8] where screw systems were placed in the context of the Lie group approach to rigid body motions. The Hunt–Gibson classification can be derived from equivalence under the action induced on the Grassmannian of screw systems of a given dimension by the adjoint action of the Lie group on its Lie algebra. The pitch arises as the fundamental invariant of the projectivised adjoint action. The benefit of this approach is that the classes of screw systems arise as submanifolds of a Grassmannian manifold. The codimensions and adjacencies were determined in [8] and the stratification was shown to be Whitney regular.

The intersection of the screw system with Q_0 is also of significance both mathematically and from the engineering point of view. In the fine classification of 3-systems in Table 1 from [8], superscripts refer to the signs of the principal pitches and other moduli.

The Thom Transversality Theorem ensures that motions in a residual set have 1-jet transverse to these stratifications. Therefore, generically, one would expect to encounter, for example, a smooth surface (codimension 1 manifold) in the 3-dimensional jointspace M where the screw system is of type IA_1^{+0-}, since this stratum has codimension 1, but not one of type IIA^0, as it has codimension 6.

2.3. Instantaneous singular sets

Given a motion $\lambda : M \to SE(3)$ and a configuration $x \in M$, define the *instantaneous singular set at* x to be

$$I(\lambda, x) = \{\mathbf{w} \in \mathbb{R}^3 \,|\, x \text{ is a singular point of } \tau_w\} \tag{2.5}$$

where the trajectory function τ_w associated to λ was defined in (2.1). Given a tracing point $\mathbf{w} \in \mathbb{R}^3$, let A_w denote the α-plane in Q_0 consisting of lines (or screws of pitch zero) through $\mathbf{w}$. We have the following central theorem, stated here for 3-dof motions. The general result appears in [6].

Broad class	Intermediate class	Codim	Fine classes
IA_1	IA_1	0	IA_1^{+++}, IA_1^{++-}, IA_1^{+--}, IA_1^{---}
	IA_1^0	1	IA_1^{++0}, IA_1^{+0-}, IA_1^{0--}
IA_2	IA_2	2	$IA_2^{(++)+}$, $IA_2^{+(++)}$, $IA_2^{(++)-}$ $IA_2^{+(--)}$, $IA_2^{(--)-}$, $IA_2^{-(--)}$
	IA_2^0	3	$IA_2^{(++)0}$, $IA_2^{+(00)}$, $IA_2^{(00)-}$, $IA_2^{0(--)}$
IB_0	IB_1	1	IB_0^+, IB_0^-
	IB_1^0	2	IB_0^0
	IB_2	2	$IB_0^{0,+}$, $IB_0^{0,-}$
	IB_2^0	3	$IB_0^{0,0}$
IB_3	IB_3	3	IB_3^{++}, IB_3^{+-}, IB_3^{--}
	IB_3^0	4	IB_3^{+0}, IB_3^{0-}
IC	IC	4	IC^+
	IC^0	5	IC^0
IIA	IIA	5	IIA^+, IIA^-
	IIA^0	6	IIA^0
IIB	IIB	5	IIB^+, IIB^-
	IIB^0	6	IIB^0
IIC	IIC	6	IIC^+, IIC^-
	IIC^0	7	IIC^0
IID		9	no finer subtypes

TABLE 1. Classification of 3–systems.

Theorem 2.3. *Let $\lambda : M \to SE(3)$ be a 3-dof motion and let S be the screw system at $x \in M$. A tracing point $\mathbf{w}$ belongs to $I(\lambda, x)$ if and only if $S \cap A_w$ is non-empty.*

Elementary corollaries of Theorem 2.3 are:

1. $I(\lambda, x)$ depends only on the associated screw system S—it is a first-order invariant of the motion. We may therefore write $I(S)$ for the ISS associated in this way to the screw system S.
2. The projective dimension of $S \cap A_w$ plus one is the corank of the singularity of τ_w at x.
3. A point $\mathbf{w}$ is in $I(S)$ if and only if S contains a screw of pitch zero, and $\mathbf{w}$ lies on its axis. Hence, $I(S)$ is *ruled*, in the sense that for any point $\mathbf{w} \in I(S)$, there is a line through $\mathbf{w}$ contained in $I(S)$.
4. The geometric form of this ISS is determined by the intersection of the screw system with Q_0 which, in turn, can be determined from the fine classification

of 3-systems. These are given in Table 2, together with the maximum corank of any singular trajectory.

type of 3–system	intersection	ISS	max corank
IA_1^{+++}, IA_1^{---}	empty	empty	0
IA_1^{++-}, IA_1^{+--}	conic	elliptic 1–sheet hyperboloid	1
IA_1^{++0}, IA_1^{0--}	point	line	1
IA_1^{+0-}	line pair	plane pair	2
$IA_2^{(++)+}$, $IA_2^{+(++)}$	empty	empty	0
$IA_2^{(--)-}$, $IA_2^{-(--)}$	empty	empty	0
$IA_2^{(++)-}$, $IA_2^{+(--)}$	conic	circular 1–sheet hyperboloid	1
$IA_2^{(++)0}$, $IA_2^{0(--)}$	point	line	1
$IA_2^{+(00)}$, $IA_2^{(00)-}$	repeated line	plane	2
IB_0^{+}, IB_0^{-}, $IB_0^{0,+}$, $IB_0^{0,-}$	conic	hyperbolic paraboloid	1
IB_0^{0}, $IB_0^{0,0}$	line pair	plane pair	2
IB_3^{++}, IB_3^{--}	point in Q_∞	empty	0
IB_3^{+-}	line pair	parallel planes	1
IB_3^{+0}, IB_3^{0-}	line	plane	1
IC, IC^0	line	plane	1
IIA^+, IIA^-	empty	empty	0
IIA^0	α–plane	whole space	3
IIB^+, IIB^-	point in Q_∞	empty	0
IIB^0	β–plane	plane	2
IIC^+, IIC^-	line in Q_∞	empty	0
IIC^0	α–plane	whole space	1
IID	Q_∞	empty	0

Table 2. Instantaneous singular sets for 3–systems

The following theorem in [6] provides a fundamental connection between a screw system and its reciprocal and is important for the analysis of 3–point motions.

Theorem 2.4. *Let S be a 3–system, and let $S^\perp$ be the reciprocal 3–system. Then $I(S) = I(S^\perp)$.*

3. k–Point Motions

3.1. Configuration space

We formalise the notion of a k-point motion.

Definition 3.1. A **k-point motion** ($1 \leq k \leq 6$) is a rigid-body motion in which a set of points $W_1, \ldots, W_k$ of the rigid body, satisfying the condition that any subset of four or less points is affinely independent, is constrained to lie, respectively, on a set of k smooth surfaces, $N_1, \ldots, N_k$. The points W_i are the **contact points** and the surfaces N_i the **contact surfaces**. The points W_1, W_2, W_3 define the **coupler triangle** where, if $k < 3$, introduce $3 - k$ additional points W_j, $j = k+1, \ldots, 3$ so that W_1, W_2, W_3 are affinely independent.

Not all choices of contact points and surfaces result in a proper rigid-body motion, that is one for which the configuration space is a manifold. Sufficient conditions for this are established below. For clarity we shall call such a motion *regular* and otherwise *singular* (though, strictly, it is not a motion at all by our definition).

By a smooth surface, in the definition, is meant an embedded, orientable, 2-dimensional submanifold in $\mathbb{R}^3$. Globally, therefore, $N_i = \phi_i(M_i)$, where M_i is an orientable 2-dimensional manifold and $\phi_i : M_i \to N_i \subset \mathbb{R}^3$ an embedding, for each i. If one is only interested in what happens locally, then each contact surface N_i can be parametrised by a smooth function $\phi_i : U_i \to N_i$, where U_i is some open subset of $\mathbb{R}^2$. Sometimes it may be more convenient to represent the contact surfaces implicitly by $N_i = f_i^{-1}(0)$, where $f_i : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $0 \in \mathbb{R}$ a regular value, for each $i = 1, \ldots, k$.

Let the moving coordinates of the contact points W_i, $i = 1, \ldots, l$ (where $l = \min\{3, k\}$) be $\mathbf{w}_i = (w_{i1}, w_{i2}, w_{i3})$. Then the rigidity of the moving body gives rise to an equation for each pair i, j such that $1 \leq i < j \leq k$:

$$||\phi_i(x_i) - \phi_j(x_j)||^2 - ||\mathbf{w}_i - \mathbf{w}_j||^2 = 0. \tag{3.1}$$

This gives $\frac{1}{2}k(k-1)$ equations on points $(x_1, \ldots, x_k) \in M_1 \times \ldots \times M_k$, a $2k$-dimensional manifold. For $k < 3$, we add further variables, $\mathbf{z} = (z_{i1}, z_{i2}, z_{i3})$ for $i = (k+1), \ldots 3$, denoting the fixed coordinates of the unconstrained vertices of the coupler triangle, and further equations of the form:

$$\begin{aligned} ||\phi_i(x_i) - \mathbf{z}_j||^2 - ||\mathbf{w}_i - \mathbf{w}_j||^2 &= 0, \quad 1 \leq i \leq k,\ k+1 \leq j \leq 3 \\ ||\mathbf{z}_i - \mathbf{z}_j||^2 - ||\mathbf{w}_i - \mathbf{w}_j||^2 &= 0 \quad k+1 \leq i < j \leq 3. \end{aligned} \tag{3.2}$$

In the case $k \geq 5$ there is some dependence between the equations and it suffices to restrict to three equations respecting the distance between the vertices of the coupler triangle and, for each W_j, $j > 3$, the distances from W_j to each vertex of the coupler triangle. Further care is needed in the cases $k \geq 4$ as here the equations do not distinguish between the possible orientations or combinations of orientations of contact tetrahedra. Thus, we must choose the component of the solution set of equations (3.1) for which the orientation of each contact tetrahedron over the coupler triangle corresponds to that of the moving contact points.

In summary, the configuration space has the form $F_k^{-1}(0)$, with $F_k : M_1 \times \ldots \times M_k \times \mathbb{R}^{p_k} \to \mathbb{R}^{q_k}$, where p_k is the number of additional variables for non-constrained contact points, and q_k is the number of equations, taking the values given in Table 3. For a regular motion, 0 is required to be a regular value of this map. In that case dimension of the configuration space is $2k + p_k - q_k = 6 - k$ for each k.

Contacts	k	1	2	3	4	5	6
Surface variables	$2k$	2	4	6	8	10	12
Non-constraint variables	p_k	6	3	0	0	0	0
Total variables	$2k + p_k$	8	7	6	8	10	12
Equations	q_k	3	3	3	6	9	12

Table 3. Variables and equations for parametric k-point motions.

3.2. Regularity conditions

The following theorem determines sufficient conditions on contact surfaces and points for the configuration space to be a manifold and hence for the motion to be regular. A direct proof using local parametrisations is reasonably straightforward too, but here a simpler implicit surface approach is given.

Theorem 3.2. *The configuration space for a k-point motion (with contact surfaces defined implicitly) is a smooth manifold unless, in some realisable configuration, the surface normals at the contact points, thought of as screws of pitch zero, fail to span a k-system.*

Proof. Suppose the contact surfaces are defined implicitly by $N_i = f_i^{-1}(0)$, $i = 1, \ldots, k$. Let the moving coordinates of the contact points W_i, $i = 1, \ldots, k$, be $\mathbf{w}_i = (w_{i1}, w_{i2}, w_{i3})$. Then the configuration space M is defined as a subset of $SE(3)$ by the equation $G(\mu) = 0$, where the components of $G : SE(3) \to \mathbb{R}^k$ in terms of $\mu = (A, \mathbf{a}) \in SE(3)$ are:

$$G_i(A, \mathbf{a}) = f_i(A\mathbf{w}_i + \mathbf{a}), \qquad i = 1, \ldots, k. \tag{3.3}$$

M is a manifold unless, for some $\alpha \in M$, the rank of G at α is less than k. Suppose we are at such an α and the fixed contact points are $\alpha(\mathbf{w}_i) = \mathbf{x}_i$, $i = 1, \ldots, k$. By the rank formula of linear algebra, the dimension of the kernel S of the derivative $DG(\alpha)$ must be $> (6 - k)$. Note that S is just a screw system. In fact, since $M = \bigcap_{i=1}^k G_i^{-1}(0)$, we have $S = \bigcap_{i=1}^k S_i$, where S_i is the kernel of $DG_i(\alpha)$.

Now $G_i = f_i \circ ev_{w_i}$, where ev_{w_i} is defined in equation (2.2). Since ev_{w_i} has no singular points and, by assumption, 0 is a regular value of f_i, it follows that 0 is a regular value of G_i and hence S_i is a 5-system. Its elements are just those screws for which the instantaneous direction of motion of $\mathbf{w}_i$ lies in the tangent space to N_i at $\mathbf{x}_i$.

It is clear that S_i must contain the α-plane of pitch-zero screws whose axes pass through $\mathbf{x}_i$, since these fix $\mathbf{x}_i$, and also the pitch infinity screws parallel to the tangent plane. This gives us five independent screws, which therefore span S_i. It is now an easy exercise to show that the pitch-zero screw with axis normal to the tangent plane is reciprocal to S_i, and hence to $S \subseteq S_i$. Thus the contact surface normals all lie in the reciprocal screw system $S^{\perp}$ and in fact span it. Since the dimension of S is greater than $6 - k$ then $S^{\perp}$, spanned by the surface normals, must have dimension less than k. □

We spell out the import of Theorem 3.2 for 3-point motions.

Corollary 3.3. *The configuration space of a 3-point motion is a manifold unless, for some configuration, either*

1. *there is a common normal to two of the contact surfaces at the points of contact or*
2. *the three surface normals are coplanar and coincident or*
3. *the three surface normals are coplanar and parallel.*

Proof. By Theorem 3.2 we require the three surface normals in each configuration to span a 3-system. This can only fail if they span a 2-system; a 1-system is not possible as this would require the three normals to coincide, but we have assumed the contact points are affinely independent, so cannot be collinear. We have a 2-system if either two normals coincide, and hence we are in case (a), or they are (projectively) collinear as points in Q_0. In the latter case, three distinct points of a 2-system lie on Q_0 so the 2-system must be of type IIA^0 or IIB^0 [6] and projectively span a line in an α-plane (or β-plane) corresponding to a planar pencil of lines in $\mathbb{R}^3$, with either finite or infinite vertex, giving rise to cases (b) and (c) respectively. □

Theorem 3.4. *Given k contact surfaces in $\mathbb{R}^3$, $1 \leq k \leq 6$, for a residual set of k contact points in $\mathbb{R}^{3k}$ the configuration space for the resulting k-point motion is a manifold of dimension $6 - k$ or empty.*

Proof. Treat the coordinates of the contact points as variables in the equations (3.1). The resulting function can readily be shown to be a submersion, so 0 is a regular value. The result now follows by a standard transversality theorem (e.g. Golubitsky and Guillemin [11], Chapter 2, §4). □

3.3. Singularities of 3-point motions

The key to determining the instantaneous singular sets and singular trajectory types lies in the observation, in the proof of Theorem 3.2, that the normals to the contact surfaces at the points of contact lie in the reciprocal screw system at a given configuration. We have the following result.

Theorem 3.5. *For a 3-point motion, the contact surface normals lie in the ISS at each configuration.*

Proof. Let S be the 3-system of the motion in a given configuration. Then the reciprocal system $S^{\perp}$ is spanned by the surface normals. As screws these have pitch zero and so, by Theorem 2.3, the normals belong to $I(S^{\perp})$. The invariance of the ISS under reciprocity (Theorem 2.4) establishes the result. □

A similar statement is true for k-point motions, $k = 1, 2$, but more care is needed for $k \geq 4$ as a screw belonging to Q_0 does not ensure that its axis is in the ISS; rather we require the k-system to intersect an α-plane in a projective line at least.

The knowledge that any ISS of a regular 3-point motion contains three distinct lines, together with the information in Table 2, enables us to exclude immediately a number of possible screw types, namely all those for which the ISS is empty or a line. It is also possible to eliminate types $\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(00)-}$, IB_3^{+0}, IB_3^{0-} and IC. For in those cases, the ISS is a plane which would require the surface normals to be coplanar. Then, either the normals form a planar pencil in which case the configuration space is singular, or the entire 3-system lies in Q_0 and hence is of type II. Details can be found in [5].

The following lemma establishes a simple relationship between the configuration of the surface normals and screw system types.

Lemma 3.6. *Given a 3-point motion and some configuration, if the direction vectors of the surface normals in that configuration:*

1. *span* $\mathbb{R}^3$*, then the associated 3-system has type A;*
2. *span a plane, then the associated 3-system has type B;*
3. *span a line, then the associated 3-system has type C.*

A type D system is not possible.

Proof. The screw system contains a screw of infinite pitch if and only if it corresponds to an infinitesimal translation perpendicular to all the surface normal directions. The result follows. □

Further consideration of the ISSs for each type enables us to establish a precise correspondence between the screw system type and the configuration of the surface normals. This is summarised in Table 4.

It can be noted immediately that the special configurations, described in the Introduction, for the classical Darboux motion and for the RCC device, are ones in which the surface normals are coincident, so the instantaneous screw type is IIA^0. From Table 2, the trajectory of the point in question has corank 3 and every point in the moving body is instantaneously singular. This screw type has codimension 6 amongst 3-systems, so it and the corresponding corank 3 singularities should not occur generically.

type of 3–system	configuration of surface normals
IA_1^{++-}, IA_1^{+--}	3 mutually skew lines
IA_1^{+0-}	2 intersect in finite point, 3rd skew to others
$IA_2^{(++)-}$, $IA_2^{+(--)}$	3 mutually skew lines [1]
IIA^0	3 lines intersect in finite point
IB_0^+, IB_0^-	3 mutually skew lines
IB_0^0,	2 intersect in finite point, 3rd in parallel plane
$IB_0^{0,+}$, $IB_0^{0,-}$	3 mutually skew lines with common perpendicular
$IB_0^{0,0}$	2 intersect in finite point, 3rd in parallel plane with common perpendicular through intersection
IB_3^{+-}	2 parallel, 3rd in parallel plane
IIB^0	3 coplanar not meeting in a point
IIC^0	3 parallel but not coplanar

[1] The distances between each pair of lines in the direction parallel to the third are equal [5].

Table 4. Screw system types for 3-point motions

For the HVRam device, the home configuration is one in which the contact normals are coplanar but not coincident, and hence the screw system type is IIB^0. This type has codimension 6, the ISS is the plane of the mirror joints and every point in the plane has a trajectory with a corank 2 singularity. Again this is not generic in the space of all motions.

4. Stability of type II screw systems

There are several ways of perturbing a k-point motion:

- by altering the dimensions of the coupler triangle;
- by altering design parameters within a given family of contact surfaces;
- by altering the contact surfaces in a general way.

From an engineering perspective, the first two are of most interest. Mathematically, the last, of which the second is a special case, provides greatest leeway (one is perturbing in an infinite-dimensional space) and would be most likely to perturb away degenerate behaviour.

Within the class of 3-point motions, we show that certain high-codimension screw types (IIA^0 and IIB^0) may occur stably—so the class fails to satisfy the genericity hypothesis referred to in Section 2.3. Essentially, this is because the condition in

Table 4, concerning the surface normal arrangements giving rise to these screw types, has only codimension 3 among triples of lines in 3-space.

Let $N_i = \phi_i(M_i)$, $i = 1, 2, 3$, be the three contact surfaces, as in Section 3.1. We are interested in the space of triples of embeddings $\Gamma_3 = Emb(M_1, \mathbb{R}^3) \times Emb(M_2, \mathbb{R}^3) \times Emb(M_3, \mathbb{R}^3)$ endowed with the (product) Whitney C^∞ topology. Let $\hat{\Gamma}_3$ denote the open subset for which the associated 3-point motion is regular.

If $W_1W_2W_3$ is a coupler triangle, then the resulting motion is characterised by the lengths of its sides, $d_{ij} = ||\mathbf{w}_i - \mathbf{w}_j|| > 0$, $(i, j) = (1, 2), (2, 3), (3, 1)$. We denote this set of design parameters by $\delta = (d_{23}, d_{31}, d_{12}) \in \mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z > 0\}$. Given $\phi = (\phi_1, \phi_2, \phi_3) \in \hat{\Gamma}_3$, and $\delta \in \mathbb{R}^3_+$, define $F^{\phi,\delta} : M_1 \times M_2 \times M_3 \to \mathbb{R}^3$ by

$$F_k^{\phi,\delta}(x_1, x_2, x_3) = ||\phi_i(x_i) - \phi_j(x_j)||^2 - d_{ij}^2, \quad k = 1, 2, 3$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Then $M = (F^{\phi,\delta})^{-1}(0)$ is the configuration space of the corresponding 3-point motion. Treating ϕ and δ as variables gives rise to a continuous map

$$F : \hat{\Gamma}_3 \times \mathbb{R}^3_+ \to C^\infty(M_1 \times M_2 \times M_3, \mathbb{R}^3); \quad (\phi, \delta) \mapsto F^{\phi,\delta}.$$

We now characterise the configurations corresponding to the screw systems of interest. Given any $X = (x_1, x_2, x_3) \in M_1 \times M_2 \times M_3$, let $L_i(X)$ denote the normal line to $N_i = \phi_i(M_i)$ at the point $\mathbf{y}_i = \phi_i(x_i)$, $i = 1, 2, 3$. If $\mathbf{n}_i(x_i)$ denotes a smooth choice of normal vector to N_i at $\mathbf{y}_i$, then we may represent $L_i(X)$, in motor coordinates, by $(\mathbf{n}_i(x_i), \mathbf{y}_i(x_i) \times \mathbf{n}_i(x_i)) = (\mathbf{n}_i(x_i), \mathbf{v}_i(x_i))$, say.

In terms of the bilinear form Q_0, define $G_k^\phi : M_1 \times M_2 \times M_3 \to \mathbb{R}$, $k = 1, 2, 3$, by

$$G_k^\phi(X) = Q_0(L_i(X), L_j(X))$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, and then define the following map:

$$H^{\phi,\delta} : M_1 \times M_2 \times M_3 \to \mathbb{R}^6; \quad H^{\phi,\delta} = (F_1^{\phi,\delta}, F_2^{\phi,\delta}, F_3^{\phi,\delta}, G_1^\phi, G_2^\phi, G_3^\phi).$$

As for F, since the manifolds are orientable, the normal vectors may be chosen smoothly, so this can be regarded as defining a continuous map

$$H : \hat{\Gamma}_3 \times \mathbb{R}^3_+ \to C^\infty(M_1 \times M_2 \times M_3, \mathbb{R}^6); \quad (\phi, \delta) \mapsto H^{\phi,\delta}.$$

Finally, let $\rho^\phi, \sigma^\phi : M_1 \times M_2 \times M_3 \to \mathbb{R}$ be given by

$$\begin{aligned} \rho^\phi(X) &= \mathbf{n}_1(x_1).(\mathbf{n}_2(x_2) \times \mathbf{n}_3(x_3)), \\ \sigma^\phi(X) &= \mathbf{v}_1(x_1).(\mathbf{v}_2(x_2) \times \mathbf{v}_3(x_3)). \end{aligned}$$

Lemma 4.1. *Given a regular 3-point motion defined by $\phi \in \hat{\Gamma}_3$ and $\delta \in \mathbb{R}^3_+$, as above, a point $X \in M_1 \times M_2 \times M_3$ is a configuration of the motion and has:*

1. *a type IIA^0 screw system at X if and only if $X \in (H^{\phi,\delta})^{-1}(0)$ and $\rho^\phi(X) \neq 0$;*
2. *a type IIB^0 screw system at X if and only if $X \in (H^{\phi,\delta})^{-1}(0)$ and $\sigma^\phi(X) \neq 0$.*

Proof. Suppose that $H^{\phi,\delta}(X) = 0$. Then X is certainly a configuration since $F^{\phi,\delta}(X) = 0$.

It is a standard result of line geometry (see for example [10, 15]) that two lines L_1, L_2 intersect, possibly at infinity (i.e. are parallel), if and only if $Q_0(L_1, L_2) = 0$. Thus, since $G_k^{\phi}(X) = 0$, $k = 1, 2, 3$, the three surface normals either intersect or are parallel, pairwise.

Suppose now that $\rho^{\phi}(X) \neq 0$. It follows that the three direction vectors of the lines are not coplanar and, in particular, no two lines are parallel. Hence the normals intersect pairwise and, to avoid coplanarity, the three points of intersection must coincide at a single point so the corresponding screw system is type IIA^0.

Conversely, if X is a configuration and if the screw system there is type IIA^0, then the relevant conditions all hold. This proves (a).

By duality, the condition $\sigma^{\phi}(X) \neq 0$ corresponds to the three lines not being coincident. Hence they either intersect pairwise in distinct points, in which case the lines are coplanar and the screw system type IIB^0, or at least two are parallel. If all three were parallel then the motion would be singular by Corollary 3.3, contrary to hypothesis. So at most two are parallel and the third must intersect each of them, again resulting in a coplanar system of lines. □

Theorem 4.2. *Given a 3-point motion defined by $\phi \in \hat{\Gamma}_3$ and a coupler triangle with parameters $\delta = (d_{23}, d_{31}, d_{12}) \in \mathbb{R}^3_+$, suppose X is a configuration at which the screw type is IIA^0 (resp. IIB^0). If $H^{\phi,\delta} \pitchfork \{0\}$ then there are open neighbourhoods U' of $\phi \in \hat{\Gamma}_3$ and V' of $\delta \in \mathbb{R}^3_+$ such that for any $\phi' \in U'$ and $\delta' \in V'$, the corresponding 3-point motion is regular and possesses a configuration at which the screw type is IIA^0 (resp. IIB^0).*

Proof. By a standard result of transversality (see for example [11]), since $\{0\} \subset \mathbb{R}^6$ is closed, the set of maps

$$\{f \in C^{\infty}(M_1 \times M_2 \times M_3, \mathbb{R}^6) \,|\, f \pitchfork \{0\}\,\}$$

is open. Moreover the inequalities in Lemma 4.1 also define open sets in $C^{\infty}(M_1 \times M_2 \times M_3, \mathbb{R}^6)$. The theorem now follows from the lemma and the continuity of H. □

5. Applications

While Theorem 4.2 establishes sufficient conditions for stability of these screw types, we wish to establish whether they hold for the specific examples of 3–point mechanisms discussed in Section 1.

5.1. Darboux motions.

Let N_i, $i = 1,2,3$, to be three planes in general position. We may assume the planes intersect at the origin and denote unit vectors along the line of intersection of N_i, N_j by $\mathbf{r}_{ij}$ for $(i,j) = (1,2),(2,3),(3,1)$. Then each N_i may be parametrised by

$$\phi_i(u_i, v_i) = u_i\mathbf{r}_{ki} + v_i\mathbf{r}_{ij}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$. A normal vector at any point of N_i is then $\mathbf{n}_i = \mathbf{r}_{ki} \times \mathbf{r}_{ij}$.

Let $a_{ij} = \cos\theta_{ij} = \mathbf{r}_{ik}.\mathbf{r}_{kj}$, where θ_{ij} is the angle between the planes N_i, N_j. These three numbers are effectively design parameters for the motion, in addition to the side lengths d_{ij} of the coupler triangle. The feasible region for these parameters is bounded by $\theta_{12} + \theta_{23} + \theta_{31} = 2\pi$; taking cosines and applying multiple angle formulae results in the boundary condition:

$$a_{12}^2 + a_{23}^2 + a_{31}^2 - 2a_{12}a_{23}a_{31} = 1.$$

The normal line L_i, at a point $\mathbf{y}_i = \phi_i(u_i, v_i) \in N_i$, can be expressed in motor coordinates by $(\mathbf{n}_i, \mathbf{y}_i \times \mathbf{n}_i)$. Hence

$$Q_0(L_i, L_j) = \mathbf{n}_i.(\mathbf{y}_j \times \mathbf{n}_j) + \mathbf{n}_j.(\mathbf{y}_i \times \mathbf{n}_i) = (\mathbf{y}_i - \mathbf{y}_j).(\mathbf{n}_i \times \mathbf{n}_j).$$

Note that $\mathbf{n}_i \times \mathbf{n}_j$ must lie in the intersection $N_i \cap N_j$, so is a non-zero multiple of $\mathbf{r}_{ij}$. Since we are interested in the zeroes of the function $H^{\phi,\delta}$ of Lemma 4.1, we may safely ignore the constant and assume $G_k^\phi = (\mathbf{y}_i - \mathbf{y}_j).\mathbf{r}_{ij}$. $H^{\phi,\delta} = (F^{\phi,\delta}, G^\phi)$ may now be expressed in terms of the 6 design parameters a_{ij}, d_{ij} and the 6 internal variables (u_i, v_i), $i = 1,2,3$.

$$\begin{aligned} F_k^{\phi,\delta}(u_1,v_1,u_2,v_2,u_3,v_3) &= (u_i^2+v_i^2) + (u_j^2+v_j^2) + 2a_{jk}u_iv_i - \\ &\quad 2a_{jk}u_iu_j - 2a_{ij}u_iv_j - 2u_jv_i - 2a_{ki}v_iv_j + 2a_{ki}u_jv_j; \\ G_k^\phi(u_1,v_1,u_2,v_2,u_3,v_3) &= a_{jk}u_i + v_i - u_j + a_{ki}v_j, \end{aligned}$$

for $k = 1,2,3$ and (i,j,k) cyclic.

The determinant of the Jacobian may be calculated (most easily using computer algebra software) and has the form:

$$8(1 - a_{12}^2 - a_{23}^2 - a_{31}^2 + 2a_{12}a_{23}a_{31})^2(u_1u_2u_3 + v_1v_2v_3).$$

The repeated factor represents the boundary of the feasible region of design parameters established above. The last factor is proportional to the volume of the tetrahedron whose vertices are the origin and the contact points $\mathbf{y}_i = \phi_i(u_i, v_i)$, $i = 1,2,3$:

$$(\mathbf{y}_1 \times \mathbf{y}_2).\mathbf{y}_3 = (u_1u_2u_3 + v_1v_2v_3)[(\mathbf{r}_{12} \times \mathbf{r}_{23}).\mathbf{r}_{31}].$$

It follows that $H^{\phi,\delta}$ is a local diffeomorphism at a configuration with screw type IIA^0 unless the "coupler tetrahedron" collapses to become planar. This represents the boundary in the space of coupler triangles, for which there exist IIA^0 screw systems. For example, in the special case considered by Bottema and Roth, where the contact surfaces are the coordinate planes, the coupler triangle must be right-angled or acute for $H^{\phi,\delta} = 0$, and transversality fails only for the right-angled triangles (see [5]).

5.2. RCC device.

A general analysis of the case of three contact spheres has so far proved intractable. We therefore concentrate on the local situation in the standard case described in the Introduction. We may assume that the centres $\mathbf{c}_i$, $i = 1, 2, 3$, are at $(1, 0, 0)$, $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 0)$. Let the spheres have radius R and suppose the vertices of the coupler triangle lie on a circle of radius $r < 1$. Then the sides of the coupler triangle have length $\sqrt{3}r$. The spheres may by parametrised by:

$$\phi_i(u_i, v_i) = R(\cos v_i \cos u_i, \cos v_i \sin u_i, \sin v_i) + \mathbf{c}_i, \quad i = 1, 2, 3.$$

The home configuration of the device, in which the screw system is type IIA^0, is such that the contact points are at $r.\mathbf{c}_i + (0, 0, \sin v)$, where v is the common value of the parameters v_1, v_2, v_3 for which the triangle is horizontal. A simple trigonometric argument shows that $\cos v = (1 - r)/R$. Evaluating the determinant of the Jacobian of the associated map $H^{\phi,\delta}$ shows that it is non-zero, except for the special values $r = 1 \pm R$. (In fact, for these values the motion is singular as the normals are coplanar as well as coincident.) Otherwise, for any local perturbation of the spheres or the coupler triangle, there remains a type IIA^0 screw system for the perturbed motion.

5.3. HVRam device.

In this case, it is the presence of a IIB^0 screw system that confers mechanical advantage. The three contact planes may be assumed to intersect in the z–axis and hence to be parametrised by:

$$\phi_i(u_i, v_i) = u_i(a_i, b_i, 0) + v_i(0, 0, 1), \quad i = 1, 2, 3,$$

where we may take $a_i^2 + b_i^2 = 1$. One establishes easily that

$$G_k^{\phi}(u_1, v_1, u_2, v_2, u_3, v_3) = (a_i b_j - a_j b_i)(v_i - v_j)$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. It follows that the last three rows of the Jacobian of $H^{\phi,\delta}$, for any choice of coupler triangle, will have rank 2 only, so transversality fails. Indeed, it is clear that the coupler triangle can be translated vertically from the given configuration and will retain screw type IIB^0. In this case, small perturbations of the device may not possess its special property.

Acknowledgment

The authors are indebted to Dr. P. Andreae and Professor C. Wampler for introducing us to the remote centre compliance device.

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