Polynomial Invariants of Screws and Screw Systems

Peter Donelan
Victoria University of Wellington, New Zealand.
peter.donelan@vuw.ac.nz
Jon Selig
London South Bank University, U. K.
seligjm@lsbu.ac.uk

Abstract

We present a new derivation of the polynomial invariants for the adjoint action of the Euclidean group in 3 dimensions (screw invariants) using representation theory, and develop some results towards exhaustive lists of polynomial invariants for screw–systems.

Keywords: Euclidean group, polynomial invariant, representation theory, screw system

0 Introduction

Screws and screw systems are the fundamental mathematical representations of single and multi-variate infinitesimal motion of rigid bodies. Hence they are widely used in, for example, robotics and mechanism theory. The position of a rigid body with respect to some reference position can be represented by an element of the Euclidean group $SE(3)$ which is a smi-direct product of the group of orientation-preserving rotations $SO(3)$ and the translation group $\mathbb{R}^3$. $SE(3)$ is a 6–dimensional Lie group with Lie algebra $\mathfrak{se}(3)$. A *screw* is an element of the projective Lie algebra, i.e. a 1–dimensional subspace of $\mathfrak{se}(3)$ while a *screw system of degree* k (or k–system) is a k–dimensional subspace. The motion of the end–effector of a robot arm with k degrees of freedom may be represented by a kinematic mapping $\lambda : M^{(k)} \to SE(3)$, where $M^{(k)}$ is a k–dimensional manifold representing the joint freedoms of the robot: its infinitesimal capabilities in a given configuration $x \in M$ are represented by the image of the derivative of λ at x. Away from singularities of the kinematic mapping and with appropriate choice of coordinate systems so that $\lambda(x)$ is the identity in $SE(3)$, this will be a k–system.

The classification of screw systems due to Hunt and Gibson [5, 7] is based on various geometric considerations, but its mathematical foundations depend on understanding the invariants of the adjoint action of $SE(3)$. Donelan and Gibson [2, 3] showed that the ring of polynomial invariants is generated by the two classical invariants, the Klein and Killing forms. Selig [10] subsequently showed that much of the Hunt–Gibson classification could be partially recovered by distinguishing polynomial invariants on the spaces of screw systems.

Here we present a new and simpler derivation of the polynomial invariants of the adjoint action (screw invariants) using classical invariant theory and develop some results towards exhaustive lists of screw system invariants. The basic idea is that the adjoint representation contains as a subrepresentation a 2–fold copy of the standard representation of $SO(3)$. Section 1 briefly reviews the representation theory for $SO(3)$ while Section 2 shows how this applies to the adjoint action of $SE(3)$ and we prove the finite generation theorem. Section 3 presents Sylvester and Hilbert's counting argument for invariant polynomials and its application to $SE(3)$. In Sections 4 and 5 we briefly discuss implications and extensions to multi-screws and screw systems.

1 Representations of $SO(3)$

The group $SO(3)$ is defined to be the subset of elements of $GL(3)$ that preserve a given symmetric positive-definite bilinear form Q, i.e. the Euclidean inner product, on $\mathbb{R}^3$ under the standard action of $GL(3)$. The form Q is an element of the dual space $(\mathrm{Sym}^2\mathbb{R}^3)^* \cong \mathrm{Sym}^2(\mathbb{R}^{3*})$. Let $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{R}^3$ and $\{f_1, f_2, f_3\} \subset \mathbb{R}^{3*}$ its canonical dual basis, so that $f_j(e_i) = \delta_{ij}$, then Q has the form $\frac{1}{2}(f_1^2 + f_2^2 + f_3^2)$. We denote $\mathbb{R}^{3*}$ by V for now.

The k-fold symmetric powers $\mathrm{Sym}^k V$ can be identified with the space of homogeneous polynomials of degree k in f_1, f_2, f_3 and $\dim\left(\mathrm{Sym}^k V\right) = \binom{k+2}{k} = \dfrac{1}{2}(k+2)(k+1)$. There is an induced representation of $SO(3)$ on this space. Regard $\mathrm{Sym}^k V$ as a subrepresentation of the k-fold tensor product $V^{\otimes k}$ via the embedding

$$f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_k} \mapsto \sum_\sigma f_{\sigma(i_1)} \otimes \ldots \otimes f_{\sigma(i_k)}$$

where $i_j \in \{1, 2, 3\}$, $i_1 \leq i_2 \leq \cdots \leq i_k$ and the sum ranges over the permutations of $\{1, 2, 3\}$. The induced representation of $SO(3)$ on $V^{\otimes k}$ is given by:

$$G \cdot (v_1 \otimes \ldots \otimes v_k) = Gv_1 \otimes \ldots \otimes Gv_k$$

[Check: this doesn't seem quite right since $\mathfrak{so}(3) \cong \mathfrak{su}(2)$] Invariants of the group action are determined by invariants of the associated Lie algebra action and vice versa [8], so we can work with the Lie algebra representation where convenient. The associated action of the Lie algebra $\mathfrak{so}(3)$ is

$$g.(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = g.v_1 \otimes v_2 \otimes \ldots \otimes v_k + v_1 \otimes g.v_2 \otimes \ldots \otimes v_k + \cdots + v_1 \otimes v_2 \otimes \ldots \otimes g.v_k \quad (1)$$

[Say something about real & complex reps in general.] The unit infinitesimal rotations X, Y, Z about the axes e_1, e_2, e_3 form a basis for $\mathfrak{so}(3)$. However its representation theory is better understood via the complex representations of $\mathfrak{so}(3, \mathbb{C})$. The element $J_3 = Z$ (or equally X, Y) spans a Cartan subalgebra, giving rise to the decomposition of the Lie algebra into its root spaces. In this case, the roots are $\pm i$, with root spaces spanned by $J_\pm = Y \pm iX$ respectively. Explicitly,

$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}.$$

A 0-eigenvector for J_+ is $e_+ = (i, 1, 0)^t$, which together with $e_0 = J_-(e_+)$, $e_- = J_-(e_0)$ form a basis for $\mathbb{C}^3$. If e_x, e_y, e_z are the zero eigenvectors of X, Y, Z respectively then

$$\begin{array}{rclrcl} e_+ & = & ie_x + e_y & e_x & = & -\frac{1}{4}i(2e_+ - e_-) \\ e_0 & = & -2ie_z & e_y & = & \frac{1}{4}(2e_+ + e_-) \\ e_- & = & -2ie_x + 2e_y & e_z & = & \frac{1}{2}ie_0 \end{array}$$

The irreducible representations of $\mathfrak{so}(3)$ are classified by their highest weight, which can be any positive integer multiple $mi/2$ of $i/2$. Denote the corresponding representation by $L_{m/2}$. The associated representation has dimension $m+1$. However these only give rise to representations of $SO(3)$ when m is even because $SU(2)$ double covers $SO(3)$. The standard representation is of course L_1 ($m = 2$).

So suppose the eigenvectors of J_3 in the standard representation are e_+, e_0, e_- (with eigenvalues $i, 0, -i$ respectively) and the dual basis is f_+, f_0, f_-. Note that $f_+(e_-) = 1$ etc., because passing to the dual (coadjoint) representation sends J_3 to $-J_3^t$. Then $e_0 = e_z$, $e_\pm = e_x \mp ie_y$. In these coordinates $Q = \frac{1}{2}f_0^2 - f_+f_-$.

For each $k \geq 2$, the quadratic form gives rise to a contraction $\Psi^{(k)} : \mathrm{Sym}^k(V) \to \mathrm{Sym}^{k-2}(V)$ via

$$\Psi^{(k)}(e_{i_1}.e_{i_2}\ldots e_{i_k}) = \sum_{m<n} Q(e_{i_m}, e_{i_n})(e_{i_1}\ldots\hat{e}_{i_m}\ldots\hat{e}_{i_n}\ldots e_{i_k}). \tag{2}$$

The kernel of this contraction is a subrepresentation—that is, an invariant subspace, since Q is fixed by $SO(3)$—and its dimension is

$$\binom{k+2}{k} - \binom{k}{k-2} = 2k+1.$$

This derives from a general contraction construction $\mathrm{Sym}^2V^* \otimes \mathrm{Sym}^kV \to \mathrm{Sym}^{k-2}V$, by taking the fixed element $Q \in \mathrm{Sym}^2V^*$, see [4], Appendix B.3. The point for us is that Q is killed by the induced action of $\mathfrak{so}(3)$ on Sym^2V^* so is built in as a core invariant.

Lemma 1.1. *Irreducible components of Sym^kV are either in the kernel of $\Psi^{(k)}$ or are mapped isomorphically by it to a component of $\mathrm{Sym}^{k-2}V$.*

Proof. Observe that $\Psi^{(k)}$ is $SO(3)$-equivariant and hence highest weight eigenvectors are either mapped to zero or map to an eigenvector of the same weight:

$$J_3.\Psi^{(k)}(\mathbf{e}) = \Psi^{(k)}(J_3.\mathbf{e}) = \Psi^{(k)}(\lambda\mathbf{e}) = \lambda\Psi^{(k)}(\mathbf{e}).$$

□

You can see immediately that e_+^k is in the kernel of $\Psi^{(k)}$ and, by (1), the corresponding eigenvalue is $ki/2$. Thus the kernel must be the irreducible representation L_k of $SO(3)$. It follows by repeated application of the lemma that the symmetric power representations decompose as

$$\begin{aligned}\mathrm{Sym}^{2m}V &= L_{2m} \oplus L_{2m-2} \oplus \cdots \oplus L_0 \\ \mathrm{Sym}^{2m+1}V &= L_{2m+1} \oplus L_{2m-1} \oplus \cdots \oplus L_1 \end{aligned} \tag{3}$$

Polynomial invariants of the standard action of $SO(3)$ can only arise from trivial components of these, hence can only be of even degree. For $\mathrm{Sym}^2 V$, clearly these are (scalar multiples of) powers of $e_x^2 + e_y^2 + e_z^2$ spanning the orthogonal complement of the kernel of Ψ_2 with respect to Q.

2 Invariants of the adjoint action of $SE(3)$

We now consider the adjoint action of $SE(3)$. Represent elements $\mathbf{s} \in \mathfrak{se}(3)$ of the Lie algebra by 6–vector formed from a pair of 3–vectors $(\boldsymbol{\omega}^T, \mathbf{v}^T)^T$. Here $\boldsymbol{\omega}$ spans the kernel of a 3×3 infinitesimal rotation (skew-symmetric matrix) $\Omega \in \mathfrak{so}(3)$ while $\mathbf{v}$ is an infinitesimal translation.

Recall that it can be written in the form

$$\mathrm{Ad}(g) = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}$$

where $R \in SO(3)$ is a rotation and T is a skew-symmetric 3×3 translation matrix. These matrices can also be characterized as elements of $GL(6)$ that preserve both the quadratic forms $Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$ and $Q_\infty = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$, subject to some additional condition on determinants.

The corresponding action of the Lie algebra is given by

$$ad(\mathbf{s}_1)\mathbf{s}_2 = [\mathbf{s}_1, \mathbf{s}_2] = \begin{pmatrix} \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_1 \times \mathbf{v}_2 - \boldsymbol{\omega}_2 \times \mathbf{v}_1 \end{pmatrix}$$

Note that there is a subrepresentation of $SO(3)$ ($\mathfrak{so}(3)$) corresponding to two copies of its standard representation on $L_1 \oplus L_1 \cong \mathbb{R}^3 \oplus \mathbb{R}^3$. To determine the invariants of this subrepresentation, note that

$$\mathrm{Sym}^k(V \oplus V) \cong \bigoplus_{r=0}^{k} \mathrm{Sym}^{k-r}V \otimes \mathrm{Sym}^r V.$$

Then the invariants correspond to the 2-fold joint invariants of the standard representation of $\mathfrak{so}(3)$, i.e. $\boldsymbol{\omega}.\boldsymbol{\omega}$, $\boldsymbol{\omega}.\mathbf{v}$, $\mathbf{v}.\mathbf{v}$ and their polynomial combinations [4], Appendix F. We can then ask which of these invariants is additionally preserved by the action of the infinitesimal translations

$$\begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} + \mathbf{t} \times \boldsymbol{\omega} \end{pmatrix} \tag{4}$$

So the first two remain invariant. However it is not quite sufficient to ensure that those two generate all the polynomial invariants, as originally proved in [2] (and which is also a consequence of results in [9] on semi-direct products).

Theorem 2.1. *Every polynomial invariant of the adjoint action of $SE(3)$ is generated by $\boldsymbol{\omega}.\boldsymbol{\omega}$, $\boldsymbol{\omega}.\mathbf{v}$.*

Proof. Suppose $F(\boldsymbol{\omega}, \mathbf{v})$ is an $SE(3)$ invariant polynomial of degree $2k$, then it is a 2–fold invariant for the subrepresentation of $SO(3)$ and hence can be written as a polynomial G in $\boldsymbol{\omega}.\boldsymbol{\omega}$, $\boldsymbol{\omega}.\mathbf{v}$, $\mathbf{v}.\mathbf{v}$. Write this in the form

$$F(\boldsymbol{\omega}, \mathbf{v}) = G(\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v}, \mathbf{v}.\mathbf{v}) = \sum_{r=0}^{k} g_r(\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v})(\mathbf{v}.\mathbf{v})^r$$

where g_r is a polynomial of degree $\leq k-r$. Under the action of a translation (4) the g_r are fixed but $\mathbf{v}.\mathbf{v}$ can take arbitrary non-negative values. It follows that $G(x, y, z)$, thought of as a polynomial in z alone, must be constant, i.e. $G(x, y, z) = g_k(x, y)$, as required. $\square$

3 Counting Invariant Polynomials

In [6], Hilbert presents Sylvester's counting method for polynomial invariants. We illustrate this for $SO(3)$ invariants for the moment.

3.1 $\mathbf{Sym}^d(L_1)$

First observe that $\mathrm{Sym}^d(L_1)$ is generated as a vector space by vectors of the form, $e_+^{n_1} e_0^{n_2} e_-^{n_3}$ where,

$$n_1 + n_2 + n_3 = d.$$

The eigenvalue (with respect to J_3) of such a vector is,

$$n_1 - n_2 = k.$$

Any invariant must have eigenvector $k = 0$ and must also vanish under the action of the operator J_-. Now the transformation J_- takes 0-eigenvectors to eigenvectors with eigenvalue -1, so the kernel of J_-, the space of invariants, has dimension equal to the difference between dimension of the 0-eigenspace and the (-1)-eigenspace.

The subspace of 0-eigenvectors is generated by the elements $e_+^{n_1} e_0^{n_2} e_-^{n_3}$ where the exponents satisfy

$$\begin{aligned} n_1 + n_2 + n_3 &= d \\ n_1 \qquad - n_3 &= 0. \end{aligned}$$

Taking the second equation away from the first gives:

$$\begin{aligned} n_1 + n_2 + n_3 &= d \\ n_2 + 2n_3 &= d. \end{aligned} \tag{5}$$

Similarly, the eigenspace of -1 eigenvectors is generated by monomials in e_+, e_0 and e_- satisfying the equations

$$\begin{aligned} n_1 + n_2 + n_3 &= d \\ n_2 + 2n_3 &= d + 1. \end{aligned} \tag{6}$$

So we just need to count up the non-negative integer solutions to (5) and subtract the number of solutions to (6).

We distinguish two cases. First suppose the total degree d is odd, so $d = 2g + 1$ for some non-negative integer g. The 0-eigenvectors can now be enumerated as,

$$e_0^{2g+1},\ e_0^{2g-1} e_+ e_-,\ e_0^{2g-3} e_+^2 e_-^2, \ldots e_0 e_+^g e_-^g.$$

By looking at the exponent of e_+ in each term it is easy to see that there will be $g + 1$ of these eigenvectors. On the other hand the (-1)-eigenvectors are,

$$e_0^{2g} e_-,\ e_0^{2g-2} e_+ e_-^2,\ e_0^{2g-4} e_+^2 e_-^3, \ldots e_+^g e_-^{g+1}.$$

again there will be $g + 1$ of these. So there are no odd invariants in this case.

Now suppose the total degree is even, $d = 2g$. We have $g + 1$ 0-eigenvectors,

$$e_0^{2g},\ e_0^{2g-2} e_+ e_-,\ e_0^{2g-4} e_+^2 e_-^2, \ldots, e_+^g e_-^g.$$

The (-1)-eigenvectors are,

$$e_0^{2g-1}e_-,\ e_0^{2g-3}e_+e_-^2,\ e_0^{2g-5}e_+^2e_-^3,\ldots,e_0e_+^{g-1}e_-^g,$$

so there are g of these. The difference is 1, showing that there is a single degeree d invariant when d is even. This can be identified with the gth power of the single degree 2 invariant showing that the ring of invariants for L_1 is generated by the degree 2 invariant.

3.2 $\mathbf{Sym}^d(L_1 \oplus L_1)$

The arguments above can be extended to look at the $SO(3)$ invariants of the representation $L_1 \oplus L_1$. In this representation we have two species of eigenvectors, e_0, e_+, e_- and e_0', e_+', e_-'. In the above argument, every power of an eigenvector e_i^n could be replaced with from 0 to n copies of e_i'. So for a monomial, $e_+^{n_1}e_0^{n_2}e_-^{n_3}$ we have $(n_1+1)(n_2+1)(n_3+1)$ possibilities. Now we can sum these for the possible cases.
First when $d = 2g+1$, the number of 0-eigenvectors will be given by

$$\sum_{i=1}^{g+1}(2g+4-2i)i^2$$

and the number of (-1)-eigenvalues will be,

$$\sum_{i=1}^{g+1}(2g+3-2i)i(i+1).$$

Rather than evaluate these separately it is easier find the difference:

$$\sum_{i=1}^{g+1}(2g+4-2i)i^2-(2g+3-2i)i(i+1)=\sum_{i=1}^{g+1}3i^2-(2g+3)i=0,$$

using standard results for $\sum i$ and $\sum i^2$. Hence, again there are no odd invariants.
In the case $d = 2g$, the number of 0-eigenvectors is

$$\sum_{i=1}^{g+1}(2g+3-2i)i^2.$$

The number of (-1)-eigenvectors is now

$$\sum_{i=1}^{g}(2g+2-2i)i(i+1).$$

and taking the difference:

$$(g+1)^2 + \sum_{i=1}^{g} \left[(2g+3-2i)i^2 - (2g+2-2i)i(i+1)\right] = (g+1)^2 + \sum_{i=1}^{g} \left[3i^2 - 2(g+1)i\right].$$

Substitution of the standard results gives the number of invariants as

$$\frac{1}{2}(g+1)(g+2) = \binom{g+2}{2}$$

which gives three degree 2 invariants and in higher degrees precisely the number of their products (or powers). Now we know that only two of the degree 2 $SO(3)$ invariants are also invariant under the full isometry group $SE(3)$ and we also have an argument that will now show that any invariant of a single screw will be a polynomial in the two degree 2 invariants.

4 Invariants of Multi-Screws

We can use a standard argument via the Capelli identity [4] to show that any multi-screw invariant will be a polynomial in the polarised forms of the degree 2 invariants and the determinant $\det(\mathbf{s}_1, \ldots, \mathbf{s}_6)$.

This has some immediate applications to "grasp metrics". An important problem in robotics concerns the grasping of solid objects. Suppose we hold an object using a number of solid fingers. Each can apply a force along its contact normal to the surface of the object. The question is where should we place the fingers to achieve the "best" grasp? A general approach is to give some index of merit to the arrangement of fingers, and then try to maximise the index over all possible arrangements. Most attempts to do this have used indices that are not coordinate invariant and hence change when, for example, the origin is moved. Clearly what is required is an invariant, and preferably one that can be generalised to different numbers of fingers. This should be relatively straightforward to investigate on the basis of multi-screw invariants.

5 Invariants of Screw Systems

A k–system may be regarded as an element of the k-fold anti–symmetric (exterior) power of the adjoint representation. Hence, for screw system invariants we will want to consider the symmetric powers $S^\ell(\bigwedge^k \mathrm{Ad})$. In view of the previous section, we may start by analysing $\bigwedge^k(L_1 \oplus L_1)$.

λ	L_1	L'_2	L'_1	L'_0	L''_1
2		$e_+ \wedge e'_+$			
1	$e_+ \wedge e_0$	$e_+ \wedge e'_0$ $+e_0 \wedge e'_+$	$e_+ \wedge e'_0$ $-e_0 \wedge e'_+$		$e'_+ \wedge e'_0$
0	$e_+ \wedge e_-$	$e_+ \wedge e'_- + 2e_0 \wedge e'_0$ $+e_- \wedge e'_+$	$e_+ \wedge e'_-$ $-e_- \wedge e'_+$	$e_+ \wedge e'_- - e_0 \wedge e'_0$ $+e_- \wedge e'_+$	$e'_+ \wedge e'_-$
-1	$e_0 \wedge e_-$	$3e_0 \wedge e'_-$ $+3e_- \wedge e'_0$	$e_0 \wedge e'_-$ $-e_- \wedge e'_0$		$e'_0 \wedge e'_-$
-2		$6e_- \wedge e'_-$			

Table 1: Generators for the components of $\bigwedge^2(L_1 \oplus L_1)$

5.1 2–systems

From multilinear algebra and known decompositions of representations of $SO(3)$:

$$\begin{aligned}\bigwedge^2(L_1 \oplus L_1) &= (\bigwedge^2 L_1 \otimes \bigwedge^0 L_1) \oplus (\bigwedge^1 L_1 \otimes \bigwedge^1 L_1) \oplus (\bigwedge^0 L_1 \otimes \bigwedge^2 L_1) \\ &= L_1 \oplus (L_2 \oplus L_1 \oplus L_0) \oplus L_1.\end{aligned}$$

We can find bases for the components in terms of eigenvectors of J_3. Denote these by e_+, e_0, e_- and e'_+, e'_0, e'_- for the 2 copies of L_1. Then we have the following Table 5.1 (using $'$ to distinguish components) where vertical columns are generated by applying the ladder operator J_- and the rows have eigenvalue λ.

Note that we may straight away affirm the existence of a linear $SO(3)$ invariant of 2–systems. By observation, in terms of a pair of screws $\mathbf{s}_i = (\boldsymbol{\omega}_i^T, \mathbf{v}_i^T)^T$, $i = 1, 2$, the relevant form is $\boldsymbol{\omega}_1.\mathbf{v}_2 - \boldsymbol{\omega}_2.\mathbf{v}_1$. (This can also be deduced directly in coordinates using $e_+ = (i, 1, 0)^t$, $e_0 = J_- e_+ = (0, 0, 2i)^t$ and $e_- = J_- e_0 = (-2i, 2, 0)^t$ etc.) The invariant can be expressed in Plücker coordinates p_{ij}, the 2×2 minor corresponding to rows i and j of the 6×2 matrix $(\mathbf{s}_1 \quad \mathbf{s}_2)$ as $p_{14} + p_{25} + p_{36}$.

However, the decomposition is *not* preserved by the action of the translations in $SE(3)$. Using the additional elements P_+, P_3, P_- in $\mathfrak{se}(3)$ (sending e_is to e'_js), clearly the components L''_1, $(L'_2 \oplus L'_1 \oplus L'_0) \oplus L''_1$ are invariant but do not have invariant complements. In particular, there is no linear $SE(3)$ invariant.

We now look for higher degree invariants for 2–systems by considering the symmetric powers $\mathrm{Sym}^d(\bigwedge^2(L_1 \oplus L_1))$. In the light of the decomposition above, we can study these using their canonical decompositions. The decomposition

$$\mathrm{Sym}^k(L_1 \oplus L_1) \equiv \bigoplus_{r=0}^{k} \mathrm{Sym}^r(L_1) \otimes \mathrm{Sym}^{k-r}(L_1)$$

can be used to write the symmetric powers as a direct sum of tensor products of symmetric powers of the irreducible representations L_i:

$$\begin{aligned} \mathrm{Sym}^d(\textstyle\bigwedge^2(L_1 \oplus L_1)) &= \mathrm{Sym}^d(L_1 \oplus (L_2 \oplus L_1 \oplus L_0) \oplus L_1) \\ &= \bigoplus_{\sum_{j=1}^5 i_j = d} \mathrm{Sym}^{i_1} L_1 \otimes \mathrm{Sym}^{i_2} L_2 \otimes \mathrm{Sym}^{i_3} L_1 \otimes \mathrm{Sym}^{i_4} L_0 \otimes \mathrm{Sym}^{i_5} L_1 \end{aligned} \tag{7}$$

In particular we need to know something about $\mathrm{Sym}^k L_2$. Listing of basis elements in terms of their eigenvalues, shows that:

$$\mathrm{Sym}^2 L_2 \cong L_4 \oplus L_2 \oplus L_0 \tag{8a}$$
$$\mathrm{Sym}^3 L_2 \cong L_6 \oplus L_4 \oplus L_3 \oplus L_2 \oplus L_0 \tag{8b}$$
$$\mathrm{Sym}^4 L_2 \cong L_8 \oplus L_6 \oplus L_5 \oplus 2L_4 \oplus 2L_2 \oplus L_0 \tag{8c}$$
$$\mathrm{Sym}^5 L_2 \cong L_{10} \oplus L_8 \oplus L_7 \oplus 2L_6 \oplus L_5 \oplus 2L_4 \oplus L_3 \oplus 2L_2 \oplus L_0 \tag{8d}$$
$$\mathrm{Sym}^6 L_2 \cong L_{12} \oplus L_{10} \oplus L_9 \oplus 2L_8 \oplus L_7 \oplus 3L_6 \oplus L_5 \oplus 3L_4 \oplus L_3 \oplus 2L_2 \oplus 2L_0 \tag{8e}$$

It is not immediately clear how to proceed in the absence of a generating function for the decomposition. However we can try to find invariants degree by degree, starting with $d = 2$. In that case we have, from (8a):

$$\begin{aligned} &\mathrm{Sym}^2(\textstyle\bigwedge^2(L_1 \oplus L_1)) \\ &\quad = \mathrm{Sym}^2 L_2 \oplus 3\mathrm{Sym}^2 L_1 \oplus \mathrm{Sym}^2 L_0 \oplus 3(\mathrm{Sym}^1 L_2 \otimes \mathrm{Sym}^1 L_1) \\ &\qquad \oplus 3(\mathrm{Sym}^1 L_1 \otimes \mathrm{Sym}^1 L_1) \oplus \mathrm{Sym}^1 L_2 \otimes \mathrm{Sym}^1 L_0 \oplus 3(\mathrm{Sym}^1 L_1 \otimes \mathrm{Sym}^1 L_0) \\ &\quad = (L_4 \oplus L_2 \oplus L_0) \oplus 3(L_2 \oplus L_0) \oplus L_0 \oplus 3(L_3 \oplus L_2 \oplus L_1) \\ &\qquad \oplus 3(L_2 \oplus L_1 \oplus L_0) \oplus L_2 \oplus 3L_1 \\ &\quad = L_4 \oplus 3L_3 \oplus 11L_2 \oplus 9L_1 \oplus 8L_0 \end{aligned}$$

where we have also used

$$L_m \otimes L_n \equiv L_{m+n} \oplus L_{m+n-1} \oplus \cdots \oplus L_{|m-n|}.$$

Another possible approach for L_2 invariants, is to use the identity $\mathrm{Sym}^2 L_1 \cong L_2 \oplus L_0$. Thus L_2 can be identified with the space of symmetric 3×3 matrices with trace zero. The action of $\mathfrak{so}(3)$ is by commutation, so using the fact that $J_3 e_2 = 2e_2$, where $J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}$ we get:

$$e_2 = \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and can then derive the other eigenvectors by applying J_-. That immediately enables us to interpret the quadratic invariant above as the sum of the 2×2 principle minors and the cubic invariant as the determinant. Moreover we can see that these generate all invariants by Weyl's argument: the action of $SO(3)$ enables us to diagonalise any symmetric matrix, uniquely up to permutation of the diagonal elements, so any invariant must be a symmetric function in the eigenvalues. Since our matrices have trace zero, we can use the fundamental theorem of symmetric functions to derive the result.

References

[1] Ball, R. S., *A Treatise on the Theory of Screws*, Cambridge University Press, Cambridge, 1998

[2] Donelan, P. S. and Gibson, C. G., First–Order Invariants of Euclidean Motions, *Acta Applicandae Mathematicae* **24** (1991) 233–251

[3] Donelan, P. S. and Gibson, C. G., On the Hierarchy of Screw Systems, *Acta Applicandae Mathematicae*, **32** (1993) 267–296

[4] Fulton, W. and Harris, J., *Representation Theory*, Springer, New York, 1991

[5] Gibson, C. G. and Hunt, K. H., Geometry of Screw Systems I & II, *Mechanism and Machine Theory*, **25** (1990) 1–27

[6] Hilbert, D., *Theory of Algebraic Invariants*, Cambridge University Press, Cambridge, 1993

[7] Hunt, K. H. *Kinematic Geometry of Mechanisms*, Clarendon Press, Oxford, 1978

[8] Olver, P., *Classical Invariant Theory*, CUP, Cambridge, 1999

[9] Panyushev, D. I., Semi-Direct Products of Lie Algebras, Their Invariants and Representations, arXiv:math.AG/0506579

[10] Selig, J., *Geometric Fundamentals of Robotics*, Springer, New York, 2005