General Formulation of the Singularity Locus for a 3-dof Regional Manipulator

Peter Donelan and Andreas Müller

Abstract— **The analysis of singularities is a central aspect in the design of robotic manipulators. Such analyses are usually based on the use of geometric parameters like DH parameters. However, the manipulator kinematics is naturally described using the concept of screws and twists, associated to Lie groups and algebras. These give rise to general and coordinate-invariant singularity conditions on the manipulator geometry. In this setting no restrictions are imposed onto the type of joints, as it is the case when using DH parameters. In this paper a single closed-form equation is presented that gives a complete description of the singularity locus of an arbitrary regional manipulator in terms of two joint variables and all design parameters, expressed by joint screw coordinates, together with the coordinates for the wrist centre. Some examples are reported, and it is shown that the expression can be used to analyse bifurcations in the singularity locus. The simple form of the condition should make it useful for practical design as well as for a deeper understanding of singularities.**

I. INTRODUCTION

The aim of this paper is to develop the analysis of 3-dof serial manipulator kinematics using Lie group and Lie algebra methods applied to the group $SE(3)$ of Euclidean isometries. In particular, we consider regional manipulators, that is to say a 3-dof serial manipulator (e.g. in Fig. 1) together with a choice of coupler point or wrist centre, although the methodology can be extended to arbitrary serial manipulators. It allows greater generality than most other approaches so that the dependence of the kinematics on all of (i) the manipulator architecture, (ii) the choice of wrist centre and (iii) the joint variables is captured. We derive a closed form expression for the kinematic mapping and singular locus of a general regional manipulator and are then able to analyse how this varies in terms of the design parameters and hence to identify bifurcations.

The importance of regional manipulators arises initially from the fact that wrist-partitioned 6-dof serial manipulators have solvable inverse kinematics, in the sense that while the general 6-dof serial manipulator gives rise to a degree 16 equation for its inverse kinematics, this special case has degree four [1]. The kinematic mapping involves only three joint variables and the wrist centre moves in 3 dimensions, The singularities of such manipulators are of particular interest and they are the subject of considerable literature, [2]–[12] being a small selection. The singular locus

P. Donelan is with the School of Mathematics, Statistics & Operations Research, Victoria University of Wellington, Wellington 6140, New Zealand peter.donelan@msor.vuw.ac.nz

A. Müller is with the Chair of Mechanics and Robotics, University Duisburg–Essen, 47057 Duisburg, Germany andreas.muller@ieee.org

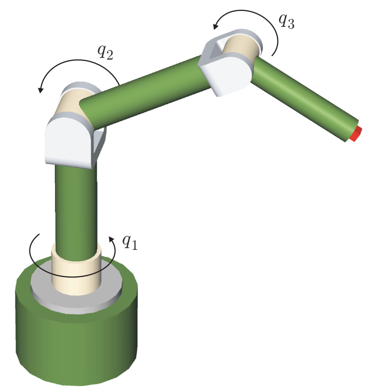

Fig. 1. A 3R regional manipulator

is invariant with respect to the first joint variable and so can be represented in the 2-dimensional space of variables for the second and third joints, while the singular image in space coordinates is invariant under the action of the first joint so that, away from its screw axis, the kinematic mapping effectively reduces to a mapping from plane to plane. The relatively low dimensions of the spaces involved mean that the singularity analysis is resaonably tractable. In these dimensions, the only stable singularities (in the topological sense) are simple folds occurring along curves and isolated cusps on those curves [13], [14].

It is consequently practicable to bring into consideration design parameters for regional manipulators. Most authors prefer to use Denavit–Hartenberg parameters here. These have several advantages, especially for manipulators with only revolute or prismatic joints. They provide a minimal set of parameters for describing a serial manipulator, are invariant under motion of the joints of the manipulator and are practical and widely used. On the other hand, they do depend on an appropriate choice of coordinate system in each component of the manipulator and the parametrisation possesses singularities. Almost all studies of regional manipulators assume the joints are revolute and often impose additional properties, typically requiring axes to be parallel or orthogonal. These assumptions are reasonable in that they reflect the large majority of devices in industrial use.

The alternative approach we employ here is to use screw coordinates. In this formulation, given a home configuration for the manipulator, each joint can be described by a non-zero 6-vector in the space of twists, that is the Lie algebra of the Euclidean group. This approach enables the kinematic mapping to be written very naturally as a product of exponentials [15]. Then the theory of Lie groups and Lie algebras can be used to derive both completely general and quite specific results about singularities. Products of exponentials have been proposed as a basis for kinematic calibration and control [16]–[18]. The importance of the Lie group and Lie algebra structures for manipulators has been established, for example in the study of mobility [19]. We contend that they also throw light on the singularity locus. By taking this broad viewpoint, it is possible to determine how the singularity locus varies under perturbations of parameters both in design and manufacturing tolerance, though here we simply show how to derive the general formulation and illustrate by a variety of cases.

II. EUCLIDEAN GROUP AND ITS LIE ALGEBRA

The relevant background material on the Euclidean group is presented here (se [20], [21] for a more thorough treatment). Consider a serial spatial manipulator with k 1-dof joints, which may be revolute, helical or prismatic. Given a choice of reference frame in the link and the ambient space that coincide in a nominated home configuration, one represents the ith joint, $i = 1, \dots, k$, by a non-zero twist $X_i \in \mathfrak{se}(3)$ belonging to the Lie algebra of the Euclidean motion group $SE(3)$. Note that X_i is determined up to a non-zero scalar multiple, so can be uniquely represented by a screw—an element of the projective space of the Lie algebra.

The twists X_i can be represented by 6-vectors $(\boldsymbol{\omega}_i, \mathbf{v}_i)$, comprised of two 3-vectors corresponding to infinitesimal rotation and translation, or by a 4×4 matrix, partitioned as

$$\left(\begin{array}{c|c} \widetilde{\omega}_i & \mathbf{v}_i \\ \hline \mathbf{0}^t & 0 \end{array}\right). \tag{1}$$

Here $\widetilde{\omega}_i$ is the skew-symmetric matrix associated to $\boldsymbol{\omega}_i \in \mathbb{R}^3$ as follows:

$$\boldsymbol{\omega}_i = \begin{pmatrix} \omega_{i1} \\ \omega_{i2} \\ \omega_{i3} \end{pmatrix} \leftrightarrow \widetilde{\omega}_i = \begin{pmatrix} 0 & -\omega_{i3} & \omega_{i2} \\ \omega_{i3} & 0 & -\omega_{i1} \\ -\omega_{i2} & \omega_{i1} & 0 \end{pmatrix}. \tag{2}$$

The pitch of X_i is a joint-invariant: $\boldsymbol{\omega}_i \cdot \mathbf{v}_i / \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_i$. If $\boldsymbol{\omega}_i \cdot \mathbf{v}_i = 0$ then X_i is revolute , while if $\boldsymbol{\omega}_i = \mathbf{0}$, then it is prismatic. However we make no prior assumption about the values of the pitches.

The continuous motion associated with the joint is given by the one-parameter subgroup $\exp(q_i X_i)$, where q_i denotes the joint variable. Under a simultaneous change of base and end-effector coordinates $g \in SE(3)$, the motion transforms by conjugacy and the twist via its derivative, the *adjoint* action of the Lie group on its Lie algebra:

$$X_i \mapsto \mathrm{Ad}\,(g)(X_i) = g X_i g^{-1}. \tag{3}$$

A simple way to realise this action is to represent both Euclidean isometries and twists as 6×6 matrices. Following [21], if $g \in SE(3)$ can be written as $g = (R, \mathbf{t}) \in SO(3) \times_s \mathbb{R}^3$ (where $\times_s$ denotes semi-direct product), then g is represented by the 6×6 partitioned matrix

$$\left(\begin{array}{c|c} R & O \\ \hline \widetilde{\mathbf{t}} R & R \end{array}\right) \tag{4}$$

with the skew-symmetric matrix $\widetilde{\mathbf{t}}$ related to $\mathbf{t}$ as in (2). The corresponding representation of $(\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3)$ is as

$$\left(\begin{array}{c|c} \widetilde{\omega} & O \\ \hline \widetilde{\mathbf{v}} & \widetilde{\omega} \end{array}\right), \tag{5}$$

where $\widetilde{\mathbf{v}}$ and $\widetilde{\omega}$ are skew-symmetric.

Differentiating Ad in (3) with respect to $g \in SE(3)$ gives the anti-symmetric adjoint representation of the Lie algebra $\mathfrak{se}(3)$ on itself, the *Lie bracket* in the Lie algebra:

$$\mathrm{ad}\,(X)(Y) = [X, Y] \tag{6}$$

In the matrix representation (5), this is simply the commutator $[Y, X] = YX - XY$; the Lie bracket is a measure of the failure of X and Y to commute. In twist coordinates,

$$[(\boldsymbol{\omega}_1, \mathbf{v}_1), (\boldsymbol{\omega}_2, \mathbf{v}_2)] = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2)\,. \tag{7}$$

It is a theorem of (connected) Lie groups (see, for example, [22]) that:

$$\mathrm{Ad}\,(\exp(qX)) = \mathrm{Exp}(q\,\mathrm{ad}\,(X)) = \sum_{n=0}^{\infty} \frac{q^n}{n!} (\mathrm{ad}\,X)^n, \tag{8}$$

where Exp acts as an operator on the Lie algebra. Powers of $\mathrm{ad}\,X$ act as nested Lie brackets.

III. KINEMATIC MAPPINGS AND PRODUCT OF EXPONENTIALS

The kinematic mapping of the serial manipulator can be written as the product of exponentials (placing the reference frame at the wrist center in the home configuration) [15], [20], [21]:

$$f(q_1, \dots, q_k) = \exp(q_1 X_1) \cdots \exp(q_k X_k), \tag{9}$$

For a regional manipulator $k = 3$. Denote with $\mathbf{c} \in \mathbb{R}^3$ the position vector of the wrist-centre expressed in the frame of the the third link. The kinematic mapping for the wrist centre describes the motion of the wrist centre in spatial coordinates, $f_{\mathbf{c}} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$f_{\mathbf{c}}(q_1, q_2, q_3) = \exp(q_1 X_1) \exp(q_2 X_2) \exp(q_3 X_3) \cdot \mathbf{c}. \tag{10}$$

This can be considered as the composition of the manipulator kinematic mapping f in (9) with the 'evaluation' map $\epsilon_{\mathbf{c}} : SE(3) \to \mathbb{R}^3$ defined by the action of the group on the wrist-centre $\mathbf{c}$: $\epsilon_{\mathbf{c}}(g) = g \cdot \mathbf{c}$, $g \in SE(3)$. That is, $f_{\mathbf{c}} = \epsilon_{\mathbf{c}} \circ f$.

A. Geometric Jacobian in Terms of Exponentials

A kinematic mapping f has a singularity at $\mathbf{q}$ when the rank of its Jacobian matrix $Jf(\mathbf{q})$ drops below its maximum possible value, which is the smaller of the dimensions k of the joint-space and n of the configuration space. Strictly, the Jacobian is a matrix representation of the linear mapping (defined independent of any particular parametrisation of the configuration space) that best approximates the kinematic mapping for the given choice $\mathbf{q}$ of joint variables. When the configuration group is the Euclidean group or a subgroup, an analytic Jacobian can be found by taking the matrix of partial derivatives of f, for a given parametrisation of the group. The image of this linear map lies in the tangent space to the group at $f(\mathbf{q})$. Multiplication in the group, either on the left or the right, enables one to 'pull back' this image to the Lie algebra and so obtain the more familiar geometric Jacobian, whose columns are twist vectors representing the instantaneous motion possible for the manipulator at joint variables $\mathbf{q}$. The rank of the Jacobian is unchanged by this operation, which we display explicitly below.

The exponential commutes with its defining twist and its derivative is therefore:

$$\frac{d}{ds}\exp(sX) = X \cdot \exp(sX) = \exp(sX) \cdot X, \tag{11}$$

where the operations between the transformation $\exp(sX)$ and the twist X can be realised by matrix multiplication. However, it does not commute with a general twist. In order to identify the geometric Jacobian, rewrite the partial derivatives of the kinematic mapping via repeated application of the following for $i = 1, \ldots, k-1$:

$$\begin{aligned}\frac{\partial}{\partial q_i} & e^{q_i X_i} e^{q_{i+1} X_{i+1}} \\ &= e^{q_i X_i} X_i e^{q_{i+1} X_{i+1}} \\ &= e^{q_i X_i} e^{q_{i+1} X_{i+1}} \mathrm{Ad}\left(e^{-q_{i+1} X_{i+1}}\right)(X_i) \\ &= e^{q_i X_i} e^{q_{i+1} X_{i+1}} \mathrm{Exp}\left(-q_{i+1} \mathrm{ad}\, X_{i+1}\right) X_i, \end{aligned} \tag{12}$$

using (8) and the linearity of ad for the last step. It follows that the ith column of the analytic Jacobian, $i = 1, \ldots, k$, has the form:

$$f(q_1, \ldots, q_k) \cdot \mathrm{Exp}(-q_k \mathrm{ad}\, X_k) \cdots \mathrm{Exp}(-q_{i+1} \mathrm{ad}\, X_{i+1}) X_i \tag{13}$$

The ith column of the geometric Jacobian, resulting from left multiplication by $[f(\mathbf{q})]^{-1}$, is therefore:

$$\mathrm{Exp}(-q_k \mathrm{ad}\, X_k) \cdots \mathrm{Exp}(-q_{i+1} \mathrm{ad}\, X_{i+1}) X_i. \tag{14}$$

This is the so-called body Jacobian [20], and (14) is the expression for the ith joint twist expressed in end-effector coordinates. One could similarly use right multiplication to obtain the expression in base coordinates (or indeed in coordinates for any of the links) that gives the co-called spatial Jacobian. There is some advantage in working in end-effector coordinates for regional manipulators since then the coordinates of the wrist centre are fixed.

It is a simple observation that if adjacent twists X_i, X_{i+1} commute, that is $[X_i, X_{i+1}] = 0$, then $\mathrm{Exp}(-q_{i+1} \mathrm{ad}\, X_{i+1}) X_i = X_i$ so that the twists remain dependent in all configurations (this includes linear dependence, i.e. $X_{i+1} = \gamma X_i$, $\gamma \in \mathbb{R}$). Hence, we shall assume that no two consecutive twists commute.

Expressions of the form (14) can be expanded using the series form of (8). In particular, if $k = 3$, the relevant value for a regional manipulator, then, using the antisymmetry of the bracket, we obtain for the first and second columns of the geometric Jacobian:

$$\begin{aligned} X_1 - q_2[X_2, X_1] - q_3[X_3, X_1] + \tfrac{1}{2} q_2^2 [X_2, [X_2, X_1]] \\ + q_2 q_3 [X_3, [X_2, X_1]] + \tfrac{1}{2} q_3^2 [X_3, [X_3, X_1]] + O(3) \end{aligned} \tag{15a}$$

$$X_2 - q_3[X_3, X_2] + \tfrac{1}{2} q_3^2 [X_3, [X_3, X_2]] + O(3) \tag{15b}$$

and the third column is simply X_3.

B. A Closed Form for the Geometric Jacobian

Selig [21], [23] shows how to express the exponential of an adjoint representation, $\mathrm{Exp}(\mathrm{ad}\, X)$ where $X = (\boldsymbol{\omega}, \mathbf{v})$ and $\boldsymbol{\omega} \neq \mathbf{0}$, as a fourth degree expression in $\mathrm{ad}\, X$ with coefficients depending on $\boldsymbol{\omega}$. The derivation follows [24], by writing $\mathrm{ad}\, X$ in terms of a spectral decomposition derived from its minimal polynomial $x^5 + 2r^2 x^3 + r^4 x$, where $r = \|\boldsymbol{\omega}\|$. In the case where the twist X is multiplied by the joint variable q, by choosing to represent the joint by a twist with $r = 1$, the resulting closed form expression can be written as

$$\mathrm{Exp}(t\, \mathrm{ad}\, X) = \sum_{i=0}^{4} b_i(t) (\mathrm{ad}\, X)^r \tag{16a}$$

where

$$\begin{aligned} b_0(t) &= 1 \\ b_1(t) &= \tfrac{1}{2}(3 \sin t - t \cos t) \\ b_2(t) &= \tfrac{1}{2}(4 - 4 \cos t - t \sin t) \\ b_3(t) &= \tfrac{1}{2}(\sin t - t \cos t) \\ b_4(t) &= \tfrac{1}{2}(2 - 2 \cos t - t \sin t). \end{aligned} \tag{16b}$$

Note that $b_r(t) = t^r/r! + O(t^5)$, $r = 1, \ldots, 4$. In the infinite pitch case $\boldsymbol{\omega} = \mathbf{0}$, the adjoint representation is nilpotent, its minimal polynomial is simply x^2 and

$$\mathrm{Exp}(t\, \mathrm{ad}\, X) = I + t\, \mathrm{ad}\, X, \tag{16c}$$

independent of the choice of X. So in this case we must set $b_0(t) = 1$, $b_1(t) = t$ and $b_i(t) = 0$ for $i \geq 2$.

There is a useful alternative derivation of the coefficient functions. If A is any square matrix with minimal polynomial $x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$, then A^m and higher powers can be written in terms of $I, A, \ldots, A^{m-1}$ so that

$$e^{tA} = \sum_{i=0}^{m-1} b_i(t) A^i \tag{17}$$

for some functions $b_i(t)$, $i = 0, \ldots, m-1$. Since $(d/dt)(e^{tA}) = A e^{tA}$, then $(d^r/dt^r)(e^{tA}) = A^r e^{tA}$. Differentiating (17) r times and substituting in the minimal polynomial gives:

$$\sum_{r=0}^{m} \left(\sum_{i=0}^{m-1} a_r b_i^{(r)} A^i \right) = 0. \tag{18}$$

Reordering of summation yields:

$$\sum_{i=0}^{m-1} \left(\sum_{r=0}^{m} a_r b_i^{(r)} \right) A^i = 0 \tag{19}$$

and equating coefficients of A^i results in

$$\sum_{r=0}^{m} a_r b_i^{(r)} = 0 \tag{20}$$

for each $i = 0, \ldots, m-1$. Thus each coefficient function $b_i(t)$ is a solution of a single differential equation (20) of degree m, corresponding to the initial conditions $b_i^{(j)} = \delta_{ij}$ (Kronecker delta). In our case, the relevant equation for which the functions (16b) are solutions is, in the finite pitch case:

$$b^{(5)} + 2b''' + b' = 0.$$

Finally, applying (16) repeatedly, we obtain a closed expression for twists of the form (14). For example, if X_2 and X_3 have finite pitch then (15a) and (15b) become, respectively:

$$\sum_{j=0}^{4}\sum_{i=0}^{4} b_i(q_2)b_j(q_3)[X_3^j X_2^i X_1] \tag{21a}$$

$$\sum_{k=0}^{4} b_k(q_3)[X_3^k X_2] \tag{21b}$$

where $[X_3^p X_2^n X_1^m]$ denotes the twist given by the nested bracket:

$$\underbrace{[X_3,[\cdots,[X_3,}_{p}\underbrace{[X_2,[\cdots,[X_2,}_{n}\underbrace{[X_1,[\cdots,[X_1,X_1]}_{m}\cdots].$$

Since the bracket is anti-symmetric, this clearly vanishes if the rightmost non-zero superscript is greater than 1, but that does not arise in our expression. The expressions (21) are simply linear combinations of twists.

IV. MANIPULATOR SINGULARITIES

We now consider regional manipulators, in the broad sense of a manipulator having a series of three arbitrary 1-dof joints together with a choice of wrist centre $\mathbf{c}$ in the end link. This setting accounts for the majority of industrial manipulators. The corresponding kinematic mapping $f_{\mathbf{c}} : \mathbb{R}^3 \to \mathbb{R}^3$ is given in (10). Following the analysis of instantaneous singular sets in [25], recall from section III that $f_{\mathbf{c}} = \epsilon_{\mathbf{c}} \circ f$ so that applying the chain rule, $Jf_{\mathbf{c}}(\mathbf{q}) = J\epsilon_{\mathbf{c}}(f(\mathbf{q})) \circ Jf(\mathbf{q})$. It follows that the rank of the Jacobian $Jf_{\mathbf{c}}(\mathbf{q})$ is less than 3, so that $f_{\mathbf{c}}$ has a singularity at $\mathbf{q}$, if and only if one of the following occurs:

i. $\operatorname{rank} Jf(\mathbf{q}) < 3$ so that f itself has a singularity at $\mathbf{q}$ or
ii. the kernel $J\epsilon_{\mathbf{c}}(f(\mathbf{q}))$ has non-trivial intersection with the image of $Jf(\mathbf{q})$.

In case (i), the joint twists are linearly dependent in the Lie algebra. In case (ii), the kernel of the derivative of $\epsilon_{\mathbf{c}}$ at the identity is precisely the 3-dimensional subspace of pitch-zero twists (lines or revolute joints) whose axes pass through $\mathbf{c}$. This provides a condition for determining singularity. A singularity occurs at $\mathbf{q}$ when the 6×6 matrix given by $Jf(\mathbf{q})$, augmented by columns representing 3 twists spanning the subspace, has zero determinant. It is straightforward to verify that the resulting condition has the form:

$$g_{\mathbf{X},\mathbf{c}}(\mathbf{q}) := \det\left(\begin{array}{c|c} A(\mathbf{q}) & I \\ \hline B(\mathbf{q}) & \widetilde{\mathbf{c}} \end{array}\right) = 0 \tag{22}$$

where A, B are 3×3 blocks of $Jf(\mathbf{q})$ in its geometric form. In (22) the 3 twists are the aligned with the reference frame in the last link hence the 3×3 identity matrix in the top right corner.

An immediate consequence is that, for fixed $\mathbf{q}$ not a singularity of f itself, (22) determines a quadric in the coordinates c_1, c_2, c_3 of the wrist centre. The reason the condition is not cubic is that $\widetilde{\mathbf{c}}$ is skew-symmetric. This was initially observed by Stanišić and Engelberth [3]. Subsequently Cocke et al. [25] catalogued the nature of the quadric in terms of types of 3-system, the 3-dimensional space in the Lie algebra spanned by the instantaneous joint twists.

The first 3 columns of the matrix in (22) can be written in the form (21). Since these are linear sums of twists, the determinant function $g_{\mathbf{X},\mathbf{c}}$ can be expanded as a sum of terms

$$\sum_{k=0}^{4}\sum_{j=0}^{4}\sum_{i=0}^{4} \gamma_{\mathbf{X},\mathbf{c}}^{ijk} b_i(q_2)b_j(q_3)b_k(q_3) \tag{23}$$

where $\gamma_{\mathbf{X},\mathbf{c}}^{ijk}$ is the determinant of a 6×6 matrix partitioned like that in (22) and whose first three columns are $[X_3^j X_2^i X_1], [X_3^k X_2]$ and X_3 respectively.

The equation $g_{\mathbf{X},\mathbf{c}} = 0$ gives a complete description of the singularity locus of a regional manipulator in terms of both the joint variables q_2, q_3 and of all the design parameters as expressed by twist coordinates for the joints X_1, X_2, X_3 and coordinates for the wrist centre (c_1, c_2, c_3). It is readily computable using, for example Maple. In the following section some examples of the singularity locus are given.

V. BIFURCATIONS

The choice of examples has been made primarily to illustrate the flexibility of the formulation, and demonstrate ways in which the singularity locus can bifurcate in respect to various design parameters. Bifurcations may affect the number of aspects (connected components of the non-singular set). They can be identified as values of design parameters at which the gradient vector of the $g_{\mathbf{X},\mathbf{c}}$ vanishes for some choice of (q_2, q_3) on the singular locus. In the terminology introduced by [4] and adopted by [6], [8], the bifurcations correspond to non-generic manipulators.

A. Orthogonal 3R manipulator

These have mutually orthogonal revolute joints with non-zero link lengths and offset along the second joint [9]. We take as design parameters:

$$X_1 = (0,0,1,0,0,0), \quad X_2 = (1,0,0,0,-1,\tfrac{1}{2}),$$
$$X_3 = (0,0,1,-2,1,0)$$

so that $d_2 = \frac{1}{2}, d_3 = \frac{3}{2}$ and $r_2 = 1$. The remaining DH parameters are associated to the choice of wrist centre and this is allowed to vary along the line $\mathbf{c} = (0, 0, t)$. Fig. 2 shows the bifurcation in the singular locus through the value $t = -1$. For this value, (23) gives the equation:

$$3\cos q_2 \cos^2 q_3 + \tfrac{17}{2}\cos q_2 \cos q_3 \sin q_3 + 2\cos^2 q_3 + \tfrac{3}{4}\cos q_2 \cos q_3 - 6\cos q_2 \sin q_3 + \tfrac{3}{2}\cos q_3 \sin q_3 - \cos q_2 - \tfrac{3}{2}\sin q_3 - \tfrac{3}{4}\cos q_3 = 1$$

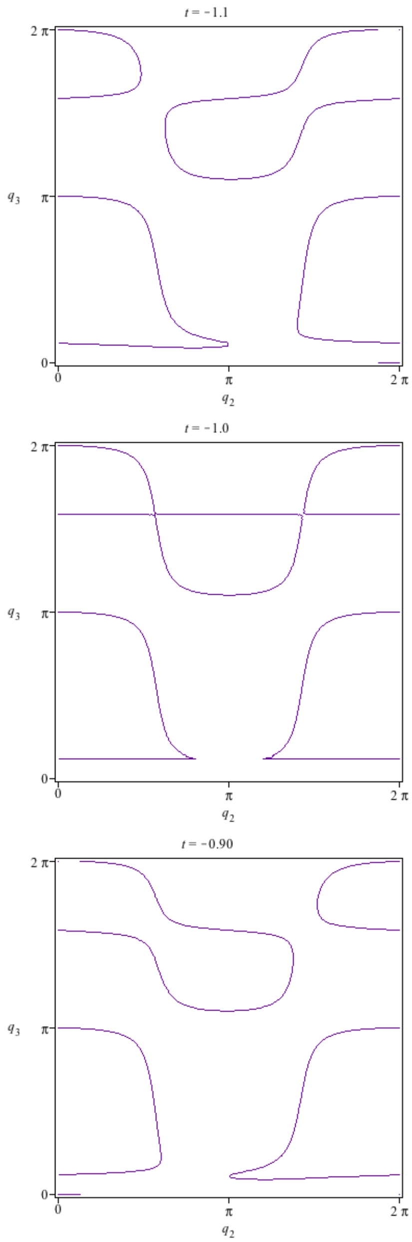

Fig. 2. Singularity locus bifurcation for an orthogonal 3R manipulator with respect to wrist centre

B. Non-orthogonal 3R manipulator

In this example we retain X_1 and X_2 but allow X_3 to vary in one parameter while remaining revolute:

$$X_3(t) = (t/\sqrt{1+2t^2}, t/\sqrt{1+2t^2}, 1/\sqrt{1+2t^2}, -1, 1, 0)$$

The wrist centre is fixed at $\mathbf{c} = (0, 0, 1)$. The bifurcation between $t = -0.48$ and -0.47 is shown in Fig. 3 and represents a change in homotopy type of the locus.

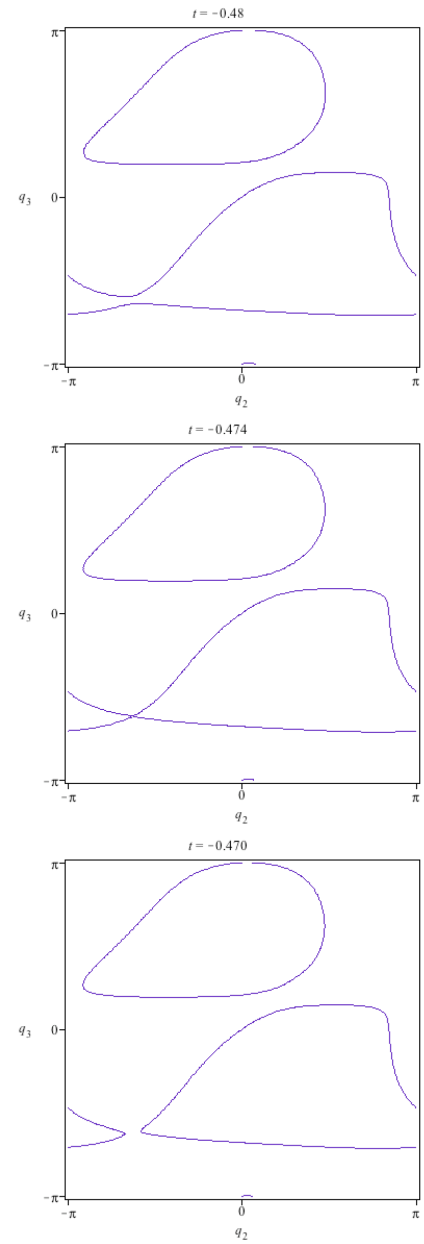

Fig. 3. Singularity locus bifurcation for a non-orthogonal 3R manipulator with respect to joint 3

C. RHR manipulator

In this example we retain X_1, X_3 from Example V-A and wrist centre $\mathbf{c} = (0, 0, 1)$ but consider what happens if there is variation in the pitch h of X_2:

$$X_2(h) = \left(1, 0, 0, h, -1, \tfrac{1}{2}\right)$$

A number of bifurcations occur in the singular locus, Fig. 4. Note that the locus is no longer periodic in q_2 and this is an example where the new formula (23) extends that used in previous work.

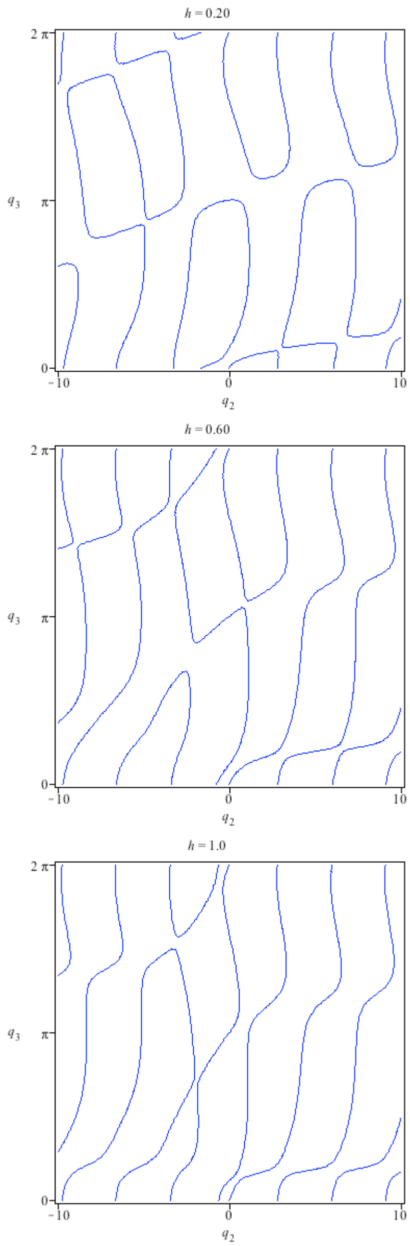

Fig. 4. Singularity locus bifurcation for HRR manipulator with respect to pitch parameter

VI. CONCLUSION

By using results of the theory of Lie groups and their algebras, we have obtained a single closed-form equation for expressing the singularity locus of a 3-dof regional manipulator with arbitrary 1-dof joints. A number of examples show that this can be useful in experimenting with variations in the design parameters, in particular for exploring bifurcations in the singularity locus. At this stage, we have simply used the equation as a computational graphing tool. However, the relatively simple form of the general equation suggest it may be useful for deeper analysis, along the lines of [12], in which questions of genericity of the singularity locus are addressed. Moreover, the method lends itself to extension to general n-dof serial manipulators and potentially to parallel manipulators where each leg has a kinematic mapping expressible as a product of exponentials.

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