

Kinematic Singularities of Robot Manipulators

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1. Introduction

Kinematic singularities of robot manipulators are configurations in which there is a change in the expected or typical number of instantaneous degrees of freedom. This idea can be made precise in terms of the rank of a Jacobian matrix relating the rates of change of input (joint) and output (end-effector position) variables. The presence of singularities in a manipulator's effective joint space or work space can profoundly affect the performance and control of the manipulator, variously resulting in intolerable torques or forces on the links, loss of stiffness or compliance, and breakdown of control algorithms. The analysis of kinematic singularities is therefore an essential step in manipulator design. While, in many cases, this is motivated by a desire to avoid singularities, it is known that for almost all manipulator architectures, the theoretical joint space must contain singularities. In some cases there are potential design advantages in their presence, for example fine control, increased load-bearing and singularity-free posture change.

There are several distinct aspects to singularity analysis—in any given problem it may only be necessary to address some of them. Starting with a given manipulator architecture, manipulator kinematics describe the relation between the position and velocity (instantaneous or infinitesimal kinematics) of the joints and of the end-effector or platform. The physical construction and intended use of the manipulator are likely to impose constraints on both the input and output variables; however, it may be preferable to ignore such constraints in an initial analysis in order to deduce subsequently joint and work spaces with desirable characteristics. A common goal is to determine maximal singularity-free regions. Hence, there is a *global* problem to determine the whole locus of singular configurations. Depending on the architecture, one may be interested in the singular locus in the joint space or in the work space of the end-effector (or both). A more detailed problem is to classify the types of singularity within the critical locus and thereby to stratify the locus. A *local* problem is to determine the structure of the singular locus in the neighbourhood of a particular point. For example, it may be important to know whether the locus separates the space into distinct subsets, a strong converse to this being that a singular configuration is isolated.

Typically, there will be a number of design parameters for a manipulator with given architecture—link lengths, twists and offsets. *Bifurcation* analysis concerns the changes in both local and global structure of the singular locus that occur as one alters design parameters in a given architecture. The design process is likely to involve optimizing some desired characteristic(s) with respect to the design parameters.

The aim of this Chapter is to provide an view of the development and current state of kinematic singularity research and to survey some of the specific methodology and results in the

literature. More particularly, it describes a framework, based on the work of Müller (2006; 2009) and of the author (Donelan, 2007b; 2008), in which singularities of both serial and parallel manipulators can be understood.

2. Origins and Development

The origins of the the study of singularities in mechanism and machine research literature go back to the 1960s and relate particularly to determination of the degree of mobility via screw theory (Waldron, 1966; Baker, 1978), the study of over-constrained closed chains (Duffy & Gilmartin, 1969) and the analysis of inverse kinematics for serial manipulators. Pieper (1968) showed that the inverse kinematic problem could be solved explicitly for wrist-partitioned manipulators, typical among serial manipulator designs. Generally, this remains a major problem in manipulator kinematics and singularity analysis. In the context of control methods, Whitney introduced the Jacobian matrix (Whitney, 1969) and this has become the central object in the study of instantaneous kinematics of manipulators and their singularities—a number of significant articles appeared in the succeeding years (Paul & Shimano, 1978; Featherstone, 1982; Sugimoto et al., 1982; Waldron, 1982; Waldron et al., 1985; Litvin et al., 1986); now the literature on kinematic singularities is very extensive, numbering well over a thousand items.

Interest in parallel mechanisms also gained momentum in the 1980s. Hunt (1978) proposed the use of in-parallel actuated mechanisms, such as the Gough–Stewart platform (Dasgupta & Mruthyunjaya, 2000), as robot manipulators, given their advantages of stiffness and precision. In contrast to serial manipulators, where the forward kinematic mapping is constructible and its singularities correspond to a loss of degrees of freedom in the end-effector, for parallel manipulators, the inverse kinematics is generally more tractable and its singularities correspond to a gain in freedom for the platform or end-effector (Hunt, 1983; Fichter, 1986). While screw theory already played a role in analyzing singularities, Merlet (1989) showed that Grassmann line geometry, which could be viewed as a subset of screw theory (see Section 4.2) is sufficiently powerful to explain the singular configurations of the Gough–Stewart platform. Thereafter, a Jacobian-based approach to understanding parallel manipulator singularities was provided by Gosselin & Angeles (1990), who showed that they could experience both direct and inverse kinematic singularities and indeed a combination of these. Subsequently, the subtlety of parallel manipulator kinematics has become even more apparent, in part as a result of the development of manipulators with restricted types of mobility, such as translational and Schönflies manipulators (Di Gregorio & Parenti-Castelli, 2002; Carricato, 2005).

The difficulty in resolving the precise configuration space and singularity locus have meant that a great deal of the singularity analysis takes a localized approach—one assumes a given configuration for the manipulator and then determines whether it is a singular configuration. It may also be possible to determine some local characteristics of the locus of singularities. This is remarkably fruitful: by choosing coordinates so that the given configuration is the identity, it is possible to reason about necessary conditions for singularity in terms of screws and screw systems. The difficulty that arises in deducing the global structure of the singularity locus is that there is no straightforward way to solve the necessary inverse kinematics. A good deal of progress can be made in some problems using Lie algebra and properties of the closure subalgebra of a chain (open or closed). This approach can be found in (Hao, 1998; Rico et al., 2003) but it appears to fail for the mechanisms dubbed “paradoxical” by Hervé (1978).

A number of authors have sought to apply methods of mathematical singularity theory to the study of manipulator singularities, for example Tchoń (1991); Pai & Leu (1992); Gibson (1992);

Karger (1996). A recent survey of this approach can be found in (Donelan, 2007b). There are several salient features. Firstly, the kinematic mapping is explicitly recognized as a function between manifolds, though it may not be given explicit form. Secondly, singularities may be classified not only on the basis of their kinematic status but also in terms of intrinsic characteristics of the mapping. For example, the rank deficiency (corank) of the kinematic mapping is a simple discriminator. More subtle higher-order distinctions can be made that distinguish between the topological types of the local singularity locus and enable it to be stratified. Thirdly, it provides a precise language and machinery for determining *generic* properties of the kinematics.

Following the results of Merlet (1989), another approach has been to use geometric algebra, especially in the analysis of parallel manipulator kinematics and singularities. It is a common theme that singular configurations correspond to special configurations of points, lines and planes associated with a manipulator—for example coplanarity of joint axes or collinearity of spherical joints. Such conditions can be expressed as simple equations in the appropriate algebra. Some examples of recent successful application of these techniques are White (1994); Wolf et al. (2004); Torras et al. (1992); Ben-Horin & Shoham (2009).

3. Manipulator Architecture and Mobility

A robot manipulator is assumed to consist of a number of rigid components (links), some pairs of which are connected by joints that are assumed to be Reuleaux lower pairs, so representable by the contact of congruent surfaces in the connected pair of links (Hunt, 1978). These include three types with one degree of freedom (dof): revolute R, prismatic P and helical or screw H (the first two correspond to purely rotational and purely translational freedom respectively) and three types having higher degree of freedom: cylindrical C with 2 dof, planar E and spherical S each with 3 dof. Some manipulators include universal U joints consisting of two R joints with intersecting axes, also denoted (RR).

The *architecture* of a manipulator is essentially a topological description of its links and joints: it can be determined by a graph whose vertices are the links and whose edges represent joints (Müller, 2006). A *serial manipulator* is an open chain consisting of a sequence of pairwise joined links, the initial (base) and final (end-effector) links only being connected to one other link. If the initial and final links of a serial manipulator are connected to each other, the result is a *simple closed chain*. This is the most basic example of a *parallel manipulator*, that is a manipulator whose topological representation includes at least one cycle or loop. Note that manipulators such as multi-fingered robot hands are neither serial nor parallel—their graph is a tree and the relevant kinematics are likely to concern the simultaneous placement of each finger.

Associated with the architecture is a combinatorial invariant, the (*full-cycle*) *mobility* μ of the manipulator, which is its total internal or relative number of degrees of freedom. This is given by the formula of Chebyshev–Grübler–Kutzbach (CGK) (Waldron, 1966; Hunt, 1978):

$$\mu = n(l - 1) - \sum_{i=1}^k (n - \delta_i) = \sum_{i=1}^k \delta_i - n(k - l + 1) \quad (1)$$

where n is the number of degrees of freedom of an unconstrained link ($n = 6$ for spatial, $n = 3$ for planar or spherical manipulators), k is the number of joints, l the number of links and δ_i the dofs of the i th joint. The first expression represents the difference between the total freedom of the links and the constraints imposed by the joints. The second version emphasizes that

the mobility is the difference between the total joint dofs and the number of constraints as expressed by the dimension of the cycle space of the associated graph (Gross & Yellen, 2004). A specific manipulator requires more information, determining the variable design parameters inherent in the architecture. The formula (1) is *generic* (Müller, 2009): there may be specific realizations of an architecture for which the formula does not give the true mobility. For example, the Bennett mechanism consists of 4 links connected by 4 revolute joints into a closed chain and is designed so that the axes of the joints lie pairwise on the two sets of generators of a hyperboloid. The CGK formula gives $\mu = 6 \times (4 - 1) - \sum_{i=1}^4 (6 - 1) = -2$ but in fact the mechanism is mobile with 1 dof. Instances of an architecture in which (1) underestimates the true mobility are termed *over-constrained*. In other cases, there are specific configurations in which μ does not coincide with the infinitesimal freedom of the manipulator. This has given rise to a search for a more universal formula that takes into account the non-generic cases, see for example (Gogu, 2005). It is precisely the discrepancies that arise that correspond to forms of singularity that are explored in subsequent sections.

4. The Kinematic Mapping

In the robotics literature, the Jacobian matrix for a serial manipulator is the linear transformation that relates joint velocities to end-effector velocities. Explicitly, suppose the joint variables are $\mathbf{q} = (q_1, \dots, q_k)$ and the end-effector's position is described by coordinates $\mathbf{x} = (x_1, \dots, x_n)$. Thus, k is the total number (or total degrees of freedom, if any of the joints have ≥ 2 dof) of the joints, while n is the dimension of the displacement space of the end-effector, typically either $n = 3$ for planar or spherical motion or $n = 6$ for full spatial motion. The *kinematic mapping* is a function $\mathbf{x} = f(\boldsymbol{\theta})$ that determines the displacement of the end-effector corresponding to given values of the joint variables. Then for a time-dependent motion described by $\mathbf{q} = \mathbf{q}(t)$, at a configuration $\mathbf{q}_0 = \mathbf{q}(t_0)$ say,

$$\dot{\mathbf{x}}(t_0) = Jf(\mathbf{q}_0)\dot{\mathbf{q}}(t_0) \quad (2)$$

where $J = Jf(\mathbf{q}_0)$ is the $n \times k$ matrix of partial derivatives of f with ij th entry $\partial f_i / \partial q_j$. It is important to note that the linear relation expressed by (2) holds infinitesimally; the Jacobian matrix is itself dependent on the joint variables. In many practical situations, for example in control algorithms, the requirement is to find an inverse matrix for J , which is only possible when $k = n$ and the determinant of the Jacobian is non-zero. In the case $k > n$, the kinematics are said to be *redundant* and one may seek a pseudo-inverse. In the case of wrist-partitioned serial manipulators, the Jacobian itself partitions in a natural way and so the singularity loci of such manipulators can also be analyzed relatively simply (Stanišić & Engleberth, 1978). It is not essential to consider the time-dependence of a motion: from the point of view of the manipulator's capabilities, the Jacobian is determined by the choice of coordinates for joints and end-effector so the properties of interest are those that are invariant under change of coordinates. This is made clearer if the domain and range of the kinematic mapping f are properly defined.

4.1 Displacement groups

The range is the set of rigid displacements of the end-effector. The rigidity of the links of a manipulator, including its end-effector, means that their motion in space is assumed to be isometric (distance between any pair of points is preserved) and orientation-preserving. Assuming the ambient space to be Euclidean, the resulting set of possible displacements is the

spatial Euclidean (isometry) group $SE(3)$ (Murray et al., 1994; Selig, 2005). Composition of displacements and the existence of an inverse displacement ensure that, mathematically, this set is a group. Moreover it can be, at least on a neighbourhood of every point, given Euclidean coordinates, and hence forms a Lie group having compatible spatial or topological (differentiable manifold) and algebraic (group) structures. The number of independent coordinates required is the dimension of the Lie group and $SE(3)$ is 6-dimensional. It is, via choices of an origin and 3-dimensional orthonormal coordinates in the ambient space and in the link, isomorphic (that is to say topologically and algebraically equivalent to) to the semi-direct product $SO(3) \times \mathbb{R}^3$, where the components of the product correspond to the orientation-preserving rotations about a fixed point (the origin) and translations, respectively.

For planar manipulators, analogously, the Euclidean isometry group is the 3-dimensional group $SE(2) \cong SO(2) \times \mathbb{R}^2$. Every element of the rotation group $SO(2)$ can be identified may be written, with respect to given orthonormal basis for \mathbb{R}^2 , in the form:

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (3)$$

where θ measures the angle of counter-clockwise rotation. In this case, θ is a coordinate for the 1-dimensional group $SO(2)$ which can, in fact, be used globally, though it is not one-to-one. Indeed, it is clear that $SO(2)$ is, in a topological sense, the same as a unit circle, denoted S^1 and θ and $\theta + 2p\pi$ represent the same point for any integer p . Topologically, $SO(3)$ is 3-dimensional projective space. Coordinates may be chosen in a number of ways. The dimensions correspond locally to the rotation about each of three perpendicular axes, or one can use Euler angles. Alternatively, there is a 2:1 representation by points of a 3-dimensional sphere, and unit quaternions or Euler–Rodrigues parameters are often used.

Regarding the joint variables, the motion associated with each Reuleaux pair corresponds to a subgroup of $SE(3)$ of the same dimension as its degrees of freedom (Hervé, 1978). In particular R, P and H joints can be represented by one-dimensional subgroups of the Euclidean group. For an R joint, the subgroup is topologically S^1 , while for P and H joints the group is \mathbb{R} . For an S joint, the subgroup is (a copy of) $SO(3)$; for C and E joints the subgroups are equivalent to $S^1 \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ respectively. Depending on the architecture of the manipulator, its joint variables therefore lie in a product of components, each of the form either S^1 , \mathbb{R} or $SO(3)$, and this product Q , say, forms the theoretical domain of the kinematic mapping. As mentioned previously, however, there are in practice almost certainly restrictions on the joint variables so that the actual domain is some subset of the theoretical joint space. The set Q is also a manifold for which the joint variables give coordinates, say $\mathbf{q} = (q_1, \dots, q_k)$, at least locally though not necessarily in a one-to-one manner for the whole space at once.

The kinematic mapping has the form of a function $f : Q \rightarrow G$, where Q is the joint space and G the displacement group for the end-effector, are well-defined manifolds. Local coordinates enable f to be expressed explicitly as a formula. While G is usually taken to be $SE(3)$, for spherical manipulators in which there is a fixed point for the end-effector $G = SO(3)$. For a robot hand or multi-legged walking robot, G may be a product of several copies of $SE(3)$. For a positional manipulator, for example a 3R arm assembly that determines the wrist-centre for a wrist-partitioned serial manipulator, the range is simply \mathbb{R}^3 .

4.2 Infinitesimal kinematics

Associated with a given point in either of these spaces $q \in Q$, $g \in G$, is its *tangent space*, denoted $T_q Q$, $T_g G$, consisting of tangent vectors of paths through that point. The tangent spaces are

vector spaces of the same dimension as the corresponding manifold. In terms of a choice of local coordinates \mathbf{q} on a neighbourhood of $q \in Q$ and \mathbf{x} near $g \in G$, these tangent vectors will correspond to the vectors $\dot{\mathbf{q}}, \dot{\mathbf{x}}$. If $g = f(q)$ then there is a linear transformation $T_q f : T_q Q \rightarrow T_g G$ whose matrix representation is simply the Jacobian matrix $Jf(\mathbf{q})$.

Working locally, we are free to choose coordinates so that the given configuration is the identity $e \in G$ and so we are interested in $T_e G$ especially. This vector space represents infinitesimal displacements of the end-effector. It has additional structure, namely that of a Lie algebra, characteristically denoted \mathfrak{g} : it has a bilinear, skew-symmetric "bracket" product $[u_1, u_2] \in \mathfrak{g}$ for $u_1, u_2 \in \mathfrak{g}$, that satisfies an additional property, the Jacobi identity. See, for example, Murray et al. (1994); Selig (2005) for further details in the context of robot kinematics.

For the Euclidean group of displacements of a rigid body in \mathbb{R}^3 , its Lie algebra $\mathfrak{se}(3)$ inherits the semi-direct product structure of $SE(3)$ and is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{t}(3)$; the infinitesimal rotations $\mathfrak{so}(3)$ can be represented by 3×3 skew-symmetric matrices

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad (4)$$

or equivalently the corresponding 3-vector $\boldsymbol{\omega}$ that spans the kernel of the matrix ($\mathbf{0}$ for the zero matrix); the infinitesimal translations are also 3-vectors \mathbf{v} . Hence elements $X \in \mathfrak{se}(3)$, termed *twists* are represented by pairs of 3-vectors $(\boldsymbol{\omega}, \mathbf{v})$. For $X \neq 0$, the line in $\mathfrak{se}(3)$ spanned by X is called a *screw*.

The Lie bracket on $\mathfrak{se}(3)$ can be defined in terms of the standard vector product \times on \mathbb{R}^3 by

$$[(\boldsymbol{\omega}_1, \mathbf{v}_1), (\boldsymbol{\omega}_2, \mathbf{v}_2)] = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2) \quad (5)$$

Thought of as a 6-vector, the components of this representation are generally known as *screw coordinates* though more properly they are twist coordinates. The *pitch* of a twist $(\boldsymbol{\omega}, \mathbf{v})$ is the ratio of the two fundamental invariants, the Klein and Killing forms, expressed as scalar products of the component 3-vectors of the twists as:

$$h = \frac{\boldsymbol{\omega} \cdot \mathbf{v}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} \quad (6)$$

A twist X of pitch 0 corresponds to infinitesimal rotation about an axis and the corresponding screw can be identified with the line in \mathbb{R}^3 that is its axis; in that case, screw coordinates are identical with classical Plücker line coordinates. When $\boldsymbol{\omega} = \mathbf{0}$, the pitch is not well-defined by this formula but is conventionally said to be ∞ . Note that the pitch is in fact an invariant of screws, not just twists.

The Klein form arises as the quadratic form associated with the non-degenerate bilinear form

$$Q_0((\boldsymbol{\omega}_1, \mathbf{v}_1), (\boldsymbol{\omega}_2, \mathbf{v}_2)) = \frac{1}{2}(\boldsymbol{\omega}_1 \cdot \mathbf{v}_2 + \boldsymbol{\omega}_2 \cdot \mathbf{v}_1) \quad (7)$$

The form Q_0 gives rise to a natural pairing between the Lie algebra and its dual space of wrenches (force plus torque) so it is possible to identify wrenches and twists. As Q_0 is indefinite, there exist non-zero *reciprocal twists* X_1, X_2 satisfying $Q_0(X_1, X_2) = 0$. In statics, a wrench X_2 is reciprocal to a twist X_1 if it performs no work on a body free to move along X_1 .

In order to describe the infinitesimal capabilities of a rigid body with several degrees of freedom, define a *screw system* to be any subspace $S \subseteq \mathfrak{se}(3)$ (Gibson & Hunt, 1990; Davidson & Hunt, 2004). If X_1, \dots, X_k form a set of independent twists, then they span a k -(screw) system. Associated with S is a reciprocal $(6 - k)$ -system S^\perp consisting of the constraints or wrenches that perform no work when acting on the end-effector. (These are not necessarily distinct from S .)

4.3 Product of exponentials

It is valuable to have a standard form in which to express the kinematic mapping. In particular, the *product of exponentials*, allows us to make use of the rich theory of Lie groups. For any Lie group G there is a natural *exponential mapping*, \exp , from the Lie algebra \mathfrak{g} to the group G itself. When the elements of G , and hence \mathfrak{g} , are written as matrices then for any matrix $U \in \mathfrak{g}$ and $q \in \mathbb{R}$,

$$\exp(qU) = \sum_{n=0}^{\infty} \frac{U^n}{n!} q^n \in G \quad (8)$$

The *one-parameter subgroups* (i.e. 1-dimensional) of a Lie group G always have the form the form $\exp(qX)$, where $X \neq 0$ is an element of the Lie algebra \mathfrak{g} and $q \in \mathbb{R}$. As noted previously, these correspond to the motions generated by R, P and H joints. Note that any non-zero multiple of X may be used to represent the same joint: in effect, the joint is uniquely represented by a *screw*. For an R joint the pitch $h = 0$, while for a P joint $h = \infty$.

Brockett (1984) adapted an approach for representing kinematic mappings originally due to Denavit & Hartenberg (1955) and demonstrated that the motion of the end-effector of a serial manipulator can be written as a product of exponentials:

$$f(q_1, \dots, q_k) = \exp(q_1 X_1) \cdot \exp(q_2 X_2) \cdots \exp(q_k X_k) \quad (9)$$

where q_i denotes the joint variable of the i th joint and X_i its twist relative to a fixed set of base link coordinates, for $i = 1, \dots, k$. An alternative approach uses coordinates in each link and expresses the invariant relation between successive links by a standard form of matrix in terms of its (Denavit–Hartenberg) design parameters. However the pure product of exponentials formulation permits the use of a classical formula from Lie theory, the Baker–Campbell–Hausdorff (BCH) formula (Selig, 2005; Donelan, 2007b). Given f in the form (9), since $\left. \frac{d}{dq} \exp(qX) \right|_{q=0} = X$ then at $q_1 = \dots = q_k = 0$, the twists X_1, \dots, X_k span a screw system S of dimension $\delta \leq k$, that describes the infinitesimal capabilities of the end-effector. Here, δ is the *infinitesimal mobility*.

The product-of-exponentials formalism (9) can be extended to chains that include spherical joints. For any point $\mathbf{a} \in \mathbb{R}^3$ there is a 3-dimensional subgroup $G_{\mathbf{a}} \in SE(3)$ of rotations about \mathbf{a} , leaving \mathbf{a} fixed. If $\mathfrak{g}_{\mathbf{a}} \subset \mathfrak{se}(3)$ is the corresponding Lie subalgebra then the restriction of the exponential on $\mathfrak{se}(3)$ to $\mathfrak{g}_{\mathbf{a}}$ is surjective so that every element of $G_{\mathbf{a}}$ can be written (not uniquely) in the form $\exp(q_1 X_1 + q_2 X_2 + q_3 X_3)$ where X_i , $i = 1, 2, 3$ form a basis for $\mathfrak{g}_{\mathbf{a}}$; for example they can be taken to be infinitesimal rotations about three orthogonal lines through \mathbf{a} . The product then includes an exponential of this extended form.

5. Serial Manipulator Singularities

The most important characteristic of a linear transformation, invariant under linear change of coordinates, is its *rank*, the dimension of its image. Since a linear transformation cannot increase dimension and the image is itself a subspace of the range, the rank can be no greater than either the dimension of the domain or of the range of the transformation. This provides the basis for the precise definition of a singularity:

Definition 1 *A serial manipulator with kinematic mapping $f : Q \rightarrow G$, where the joint space Q has dimension k and the displacement group G has dimension n , has a **kinematic singularity** at $q \in Q$ if $\text{rank } T_q f$ has rank less than both k and n .*

In particular, if $k = n$ then there is a kinematics singularity when $\text{rank } T_q f < n$, or equivalently $\det Jf(q) = 0$. In terms of mobility, the definition is equivalent to $\delta < \mu$.

Using topological methods, it can be shown that for standard architectures such as 6R serial manipulators, where the joint space is the 6-dimensional torus $Q = T^6$ and the end-effector space is $SE(3)$, the kinematic mapping must have singularities (Gottlieb, 1986). These can only be excluded from the joint space by imposing restrictions on the joint variables. In particular, if the joint space is compact, such as T^6 , then so its image, the *work space*, and is hence bounded in $SE(3)$. The kinematic mapping will have singularities at the boundary configurations. However it may also have singularities interior to the workspace.

A fundamental problem is to determine the locus of kinematic singularities in the joint space of a manipulator and, if possible, to stratify it in a natural way. As has been mentioned before, actually determining the locus remains a difficult problem for most manipulators. However one can address the question of whether the locus is itself a submanifold of the joint space using singularity theory methods. The singularities of rank deficiency 1 (corank 1) can be recognized by whether, at a given configuration, the first-order Taylor polynomial of the kinematic mapping $f : Q \rightarrow SE(3)$ lies in a certain manifold. Transversality to this manifold, a linear-algebraic condition that holds when a certain set of twists span the Lie algebra $\mathfrak{se}(3)$, is enough to guarantee that the corank 1 part of the singularity locus is a manifold of dimension $|6 - k| + 1$. The resulting condition involves the joint twists and certain Lie brackets involving them (Donelan, 2008).

The approach can also, in principle, be extended to a manipulator architecture by including the design parameters. In this case, one may expect to encounter more degenerate singularities for specific values of the design parameter. An important problem is to determine the bifurcation set that separates classes within the architecture of manipulators with distinct topological type of singularity locus.

6. Parallel Manipulator Singularities

6.1 Infinitesimal analysis of closure

In contrast to serial manipulators, in which all the joints are actuated, for a parallel manipulator only some of the joints will be actuated, the remainder being passive. The approach to parallel manipulator singularities pioneered by Gosselin & Angeles (1990) makes use of the actuated joint variables \mathbf{q} and output (end-effector) variables \mathbf{x} , constrained by an equation of the form:

$$F(\mathbf{q}, \mathbf{x}) = 0 \quad (10)$$

Differentiating (10), the infinitesimal kinematics can be written in the form:

$$\begin{pmatrix} \frac{\partial F}{\partial \mathbf{q}} & \frac{\partial F}{\partial \mathbf{x}} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{x}} \end{pmatrix} = J_q \dot{\mathbf{q}} + J_x \dot{\mathbf{x}} = \mathbf{0} \quad (11)$$

where the two Jacobian matrices $J_q = \partial F / \partial \mathbf{q}$, $J_x = \partial F / \partial \mathbf{x}$ depend on both \mathbf{q} and \mathbf{x} . If one assumes no redundancy, then the number of actuator variables equals the number of output variables, i.e. the mobility μ of the manipulator. The number of constraints, the dimension of the range of F , should then be the difference between the number of variables 2μ and the mobility μ , hence also μ . Thus the Jacobians are both $\mu \times \mu$ matrices.

This leads to three types of singularity, depending on loss of rank of J_q (type I), or J_x (type II), or both (type III). A type I singularity is comparable to the kinematic singularity of a serial manipulator in Definition 1, where there exist theoretical end-effector velocities that

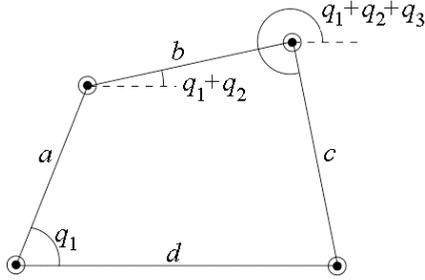


Fig. 1. Planar 4R manipulator

are not achievable by means of any input joint variable velocity. On the other hand, in a type II singularity, there is a non-zero vector $\dot{\mathbf{x}}$ in the kernel of J_x which therefore gives a solution to (11) with $\dot{\mathbf{q}} = \mathbf{0}$, in other words when the actuated joints are static or locked. Such a solution may be isolated but may also correspond to a finite branch of motion, referred to as an *architecture singular motion* (Ma & Angeles, 1992) or *self-motion* (Karger & Husty, 1998). (Note that this term is also used for motions of a serial redundant manipulator in which the end-effector remains static.)

In practice, as noted by Gosselin & Angeles (1990), the constraint equations one actually formulates do not necessarily have the form (10). Denavit & Hartenberg (1955) observed that for a closed chain the matrix product must be the identity and, likewise, for the product-of-exponentials formulation. For a simple closed chain, the end-effector is identified with the base, so gives rise to a *closure equation* for the kinematic mapping of the form:

$$\phi(q_1, \dots, q_k) = \exp(q_1 X_1) \cdot \exp(q_2 X_2) \cdots \exp(q_k X_k) = I \quad (12)$$

where I is the identity transformation that leaves the end-effector fixed with respect to the base link.

By way of a relatively simple example, consider a planar 4R closed chain (Gibson & Newstead, 1986) where the closure equation involves the design parameters (link lengths) a, b, c, d and four joint variables $q_i, i = 1, 2, 3, 4$ as in Figure 1. Since the kinematics concern $SE(2)$, which is 3-dimensional, there are three scalar closure equations, two for translation and one for rotation:

$$a \cos q_1 + b \cos(q_1 + q_2) + c \cos(q_1 + q_2 + q_3) + d \cos(q_1 + q_2 + q_3 + q_4) = 0 \quad (13)$$

$$a \sin q_1 + b \sin(q_1 + q_2) + c \sin(q_1 + q_2 + q_3) + d \sin(q_1 + q_2 + q_3 + q_4) = 0 \quad (14)$$

$$q_1 + q_2 + q_3 + q_4 = 0 \pmod{2\pi} \quad (15)$$

Usually one joint variable is eliminated via (15), which is then omitted, and (13,14) simplified. One of q_1, q_3 is taken as actuator variable. Hence, to obtain a constraint of the form (10), it is necessary first to eliminate the remaining passive joint variables. This can be done by using $\cos^2 \theta_i + \sin^2 \theta_i = 1, i = 1, \dots, 4$, at the cost of obtaining a formula involving the two branches of a quadratic. In particular, one loses differentiability where the discriminant vanishes. Alternatively, the equations can be rewritten as algebraic equations either using tan half-angle as variable, or replacing both $\cos \theta_i$ and $\sin \theta_i$ by new variables, together with corresponding Pythagorean constraint equations. In either case it is, in theory, possible to use Gröbner basis techniques to eliminate variables (Cox et al., 2004).

The situation can, however, become much more complex. For the Gough–Stewart platform in its most general architecture, for a given set of actuator variables \mathbf{q} there may be up to 40 possible configurations \mathbf{x} for the platform (Lazard, 1993; Mourrain, 1993; Dietmaier, 1998).

The omission of passive joint variables has additional drawbacks. Firstly, the choice of actuated variable may be arbitrary, whereas it is sometimes preferable to allow freedom in this choice. Secondly, a manipulator may gain finite or infinitesimal freedom with respect to only its passive joints in certain configurations. This phenomenon was observed by Di Gregorio & Parenti-Castelli (2002) who showed that a 3-UPU manipulator designed for translational motion could undergo rotation in some configurations. Such solutions may not be apparent when solving the restricted constraint equations (10).

The need to include passive joints for the instantaneous kinematics of a general parallel manipulator was recognized by Zlatanov et al. (1994a;b). They distinguished six types of singularity in total. In addition to those arising from inclusion of the passive joints, they also allowed that the instantaneous mobility could exceed the full-cycle mobility μ in (1). The resulting singularity type was denoted *increased instantaneous mobility* (IIM). The phenomenon had already been identified for simple closed chains by Hunt (1978), who termed it *uncertainty configuration*, as it corresponds to an intersection of branches of the configuration space, allowing the end-effector motion in at least two different modes. This has also been termed a *configuration space singularity* (Zlatanov et al., 2001) or *topological singularity* (Shvalb et al., 2009). Indeed, even in the relatively simple case of the planar 4R closed chain, if the link lengths $e_1 \leq e_2 \leq e_3 \leq e_4$ (in increasing order) satisfy the non-Grashof condition $e_1 + e_4 = e_2 + e_3$ then the 4-bar can fold flat, with all joints collinear, and the configuration space has a singularity (Gibson & Newstead, 1986).

6.2 A unified approach

Consider a general parallel manipulator with l links and k joints, the i th joint having δ_i degrees of freedom, $i = 1, \dots, k$. Suppose Q is the product of the individual joints spaces, so that Q is a manifold of dimension $d = \sum_{i=1}^k \delta_i$. The *configuration space* $C \subseteq Q$ is the set of joint variable vectors that satisfy the constraints imposed by the manipulator's structure. As noted in Section 3, the number of independent constraints is $k - l + 1$, the dimension of the cycle space of the graph. Let q_1, \dots, q_d denote the joint variables associated with twists X_1, \dots, X_d . Each constraint can be written in the form $\phi_j(\mathbf{q}) = I$, $j = 1, \dots, k - l + 1$ with ϕ_j as in (12). The map

$$\Phi: Q \rightarrow SE(3)^{k-l+1}, \quad \Phi(\mathbf{q}) = (\phi_1(\mathbf{q}), \dots, \phi_{k-l+1}(\mathbf{q})) \quad (16)$$

defines the configuration space as $C = \{\mathbf{q} \in Q : \Phi(\mathbf{q}) = (I, \dots, I)\}$. The Pre-Image Theorem of differential topology asserts that C is a manifold of dimension $\mu = \dim Q - \dim SE(3)^{k-l+1}$ provided that $T_{\mathbf{q}}\Phi$ has maximum rank for all $\mathbf{q} \in C$. Consider, for example, a Gough–Stewart platform having 6 UPS legs connecting the base to the platform, as in Figure 2. Each leg has 6 joint variables, hence $d = 36$, while the number of links, including the base is $l = 14$ and the number of joints $k = 18$. Given $\dim SE(3) = 6$, it follows that the Jacobian is 30×36 . Although this is rather large, we can determine its structure quite simply. Suppose the legs are numbered $r = 1, \dots, 6$ and the twists in each leg denoted X_{rs} , $r, s = 1, \dots, 6$. Five independent closure equations arise from the closed chains consisting of leg 1, leg r , the base and the platform, for $r = 2, \dots, 6$ as follows:

$$\begin{aligned} & \exp(q_{11}X_{11}) \exp(q_{12}X_{12}) \exp(q_{13}X_{13}) \exp(q_{14}X_{14} + q_{15}X_{15} + q_{16}X_{16}) \\ & \cdot \exp(-q_{r4}X_{r4} - q_{r5}X_{r5} - q_{r6}X_{r6}) \exp(-q_{r3}X_{r3}) \exp(-q_{r2}X_{r2}) \exp(-q_{r1}X_{r1}) = I \end{aligned} \quad (17)$$

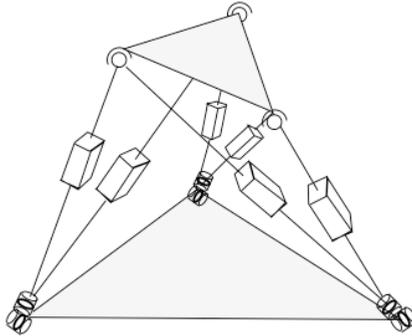


Fig. 2. 6 UPS parallel manipulator

The Jacobian matrix consists of 5 rows of 36 twists (6-vectors), the j th row having in the column of the i th joint variable either the corresponding twist X_i , if that joint is involved in the loop described by the closure equation for ϕ_j , or else $\mathbf{0}$; for example the first row is

$$(X_{11} \quad X_{12} \quad \cdots \quad X_{16} \quad X_{21} \quad X_{22} \quad \cdots \quad X_{26} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}) \quad (18)$$

It follows that a necessary and sufficient condition for maximum rank is that the pairwise vector sum of the screw systems of legs span $\mathfrak{se}(3)$. Alternatively, there is a singularity if there exists a common reciprocal wrench. This is a *configuration space singularity*.

Suppose now that the the Jacobian is of full rank, so that the configuration space is a manifold whose dimension is the full-cycle mobility μ in the CGK formula (1), or simply that this is true in a neighbourhood of a configuration. Can the full mobility of the platform be achieved using a given set of μ actuated joints? This can be answered using the Implicit Function Theorem. One requires the corresponding joint variables to give local coordinates for C . The theorem asserts that this is the case if the square matrix, obtained by deleting the μ columns of the Jacobian corresponding to the actuator variables, is non-singular. If that fails to be the case at some configuration $\mathbf{q} \in C$, then there is a direction in the tangent space $T_{\mathbf{q}}C$ that projects to $\mathbf{0}$ in the actuator velocity space but which corresponds to a non-zero velocity of the platform. This is therefore a type II singularity in the nomenclature above, or what might be termed an *actuator singularity*.

Finally, the type I or *kinematic singularities* for a parallel manipulator at a non-singular point of the configuration space are simply the singularities of the forward kinematics from C to $SE(3)$.

Even though this model gives a reasonably complete picture of the singularity types possible for a parallel manipulator, the paradoxical mechanisms such as Bennett, Bricard, Goldberg etc., are not explained; they correspond here to manipulators for which every configuration is a configuration space singularity, as the constraint Jacobian rank is non-maximal. Nevertheless, their configuration spaces are manifolds, but of the “wrong” dimension.

7. Outlook

The analysis of manipulator singularities is a burgeoning area of research. This chapter has touched briefly on some key themes. There are significant open problems, including understanding over-constrained mechanisms, genericity theorems for manipulator architectures,

higher-order analysis and topology of the singularity loci, singularities of compliant mechanisms, bifurcation analysis and many more. The practical problems of manipulator design and control remain central goals, but their resolution involves many branches of mathematics including algebraic geometry, differential topology, Lie group theory, geometric algebra, analysis, numerical methods and combinatorics.

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