The Geometry of the Point–Paths Generated by Rigid–Body Motion in Two and Three Dimensions

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Abstract

The geometry of the point–paths of planar and spatial rigid–body motions are studied using techniques of differential geometry. In particular, we look at sets of points which, at an instant, display some particular behaviour. In the planar case this involves looking at sets of points of zero curvature (inflection points) and of stationary curvature (vertices). For spatial motion we look at sets of points of zero curvature and of zero torsion. We then examine how these sets degenerate when the motion itself has special behaviour.

For the spatial motions we detail a method of putting the motion into a normal form which enables calculations to be made more easily and classify the motions according to the type of normal form that results. We then use these classifications to describe the degeneracies of the zero–curvature curve and the zero–torsion surface.

We also give a brief description of some of the other methods which can be used to represent spatial motion.

1 Rigid–Body Kinematics

1.1 Notation

Throughout, the derivative of a general curve will be denoted by a dash after the symbol representing the curve and the derivative of a quantity with time parameter will be denoted by a dot above the symbol representing the quantity. The number of dashes or dots represents the order of the derivative.

1.2 n–Dimensional Rigid–Body Motion

A *rigid–body displacement* in n dimensions is a transformation of Euclidean n–space, E^n, which preserves length and orientation. E^n is defined as the vector space $\mathbb{R}^n$, together with the standard inner product $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i y_i$ and length is defined in the usual way: $\|\mathbf{x}\| = \sqrt{< \mathbf{x}, \mathbf{x} >}$. The set of rigid–body displacements of E^n forms a group with the group operation being that of composition. It is a fundamental theorem that any displacement of E^n can be shown to be the combination of a rotation about a fixed point, usually taken to be the origin, together with a translation, though either rotation or translation may be zero. A proof of this fact may be found in [2] for example.

Rotations of E^n about $\mathbf{0}$ are represented by orthogonal $n \times n$ matrices of determinant 1 and so the set of rotations of E^n form the group $SO(n)$ with group operation matrix multiplication. Translations of E^n are represented by n–vectors and form the group $\mathbb{R}^n$ with group operation vector addition. The full displacement group is the *Euclidean group* $SE(n)$, which is a semi–direct product of the rotation group $SO(n)$ and the translation group $\mathbb{R}^n$. Elements of $SE(n)$ can be denoted in the form $(A, \mathbf{a})$ where $A \in SO(n)$ and $\mathbf{a} \in \mathbb{R}^n$, while the group operation is defined as follows:

$$(A_2, \mathbf{a}_2)(A_1, \mathbf{a}_1) = (A_2 A_1, A_2 \mathbf{a}_1 + \mathbf{a}_2)$$

$SE(n)$ is a Lie group, that is to say a group with the structure of a C^∞ manifold, with respect to which the group operation and inverse operation are differentiable, see, for example, [8] or [14]. The full strength of this is not relevant here, but we do need to know that it is possible to differentiate along paths in $SE(n)$.

The Euclidean group acts on E^n to describe the displacement in the following way:

$$SE(n) \times E^n \to E^n, \quad ((A, \mathbf{a}), \mathbf{x}) \mapsto A\mathbf{x} + \mathbf{a}$$

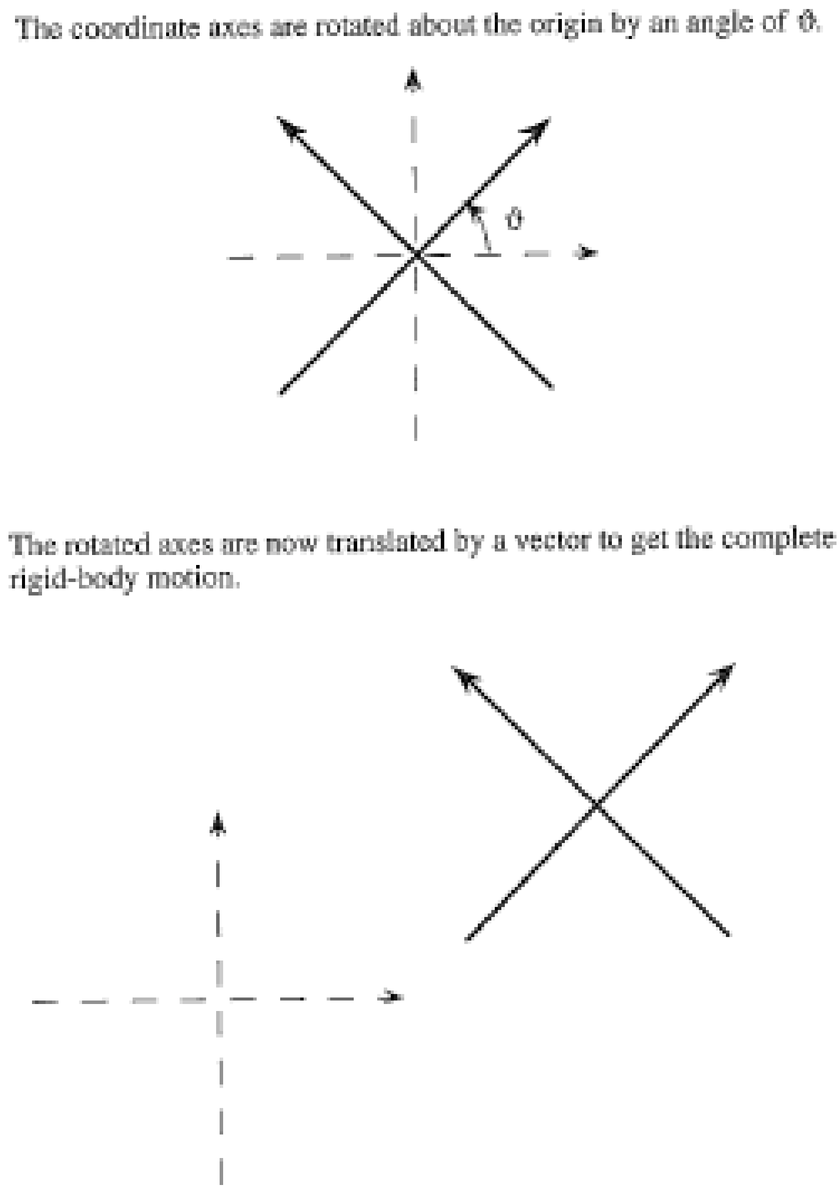

Figure 1.1: A rigid-body motion consisting of a rotation about the origin followed by a translation

A *rigid–body motion* is a continuous family of displacements with (time) parameter, $t \in \mathbb{R}$;

that is a continuous map $\mu : \mathbb{R} \to SE(n)$. We shall, in fact, only be interested in smooth rigid–body motions, that is, ones for which derivatives of all orders exist. Associated to the motion is its action on the rigid body represented by a 1–parameter family of displacements via the map $\Phi : \mathbb{R} \times E^n \to E^n$ given by

$$\Phi(t, \mathbf{x}) = A(t)\mathbf{x} + \mathbf{a}(t).$$

For any point $\mathbf{x}$ on the rigid body, the point–path of the motion is given by $\Phi_{\mathbf{x}} : \mathbb{R} \to E^n$ where $\Phi_{\mathbf{x}}(t) = \Phi(t, \mathbf{x})$.

For the purposes of further analysis, we are only interested in the cases when $n = 2, 3$.

1.3 Representation of Motions for $\mathbf{n = 2}$

For $n = 2$, every orthogonal 2×2 matrix of determinant 1 has the special form

$$A = \exp \theta J = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\theta \in \mathbb{R}$. This allows us to identify $SO(2)$ topologically with the unit circle S^1. In particular, a rigid–body motion can be written

$$(A(t), \mathbf{a}(t)) = (\exp \theta(t) J, \mathbf{a}(t))$$

where $\theta : \mathbb{R} \to \mathbb{R}$ and $\mathbf{a}$ are continuous.

1.4 Representation of Motions for $\mathbf{n = 3}$

For $n = 3$ it can easily be shown that every $A \in SO(3)$ except $A = I$ has a unique axis of rotation corresponding to the eigenvector with eigenvalue 1, see for example [7] or [9]. There are various ways of representing elements of $SO(3)$ and hence a motion.

Euler Angles

One of these ways is via Euler angles, whereby the rotation matrix is written as a product of 3 matrices, each involving a single rotation about one of the axes. Traditionally, one decomposes $A \in SO(3)$ into a rotation of θ_1 about the z-axis, followed by a rotation of θ_2 about the x-axis, followed by a rotation of θ_3 about the z-axis (although other choices of axis are possible), with some restriction of angles. So a continuous rotation $A(t) = A_3(t)A_2(t)A_1(t)$ where

$$A_i(t) = \begin{pmatrix} \cos\theta_i(t) & -\sin\theta_i(t) & 0 \\ \sin\theta_i(t) & \cos\theta_i(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 1, 3$$

$$A_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2(t) & -\sin\theta_2(t) \\ 0 & \sin\theta_2(t) & \cos\theta_2(t) \end{pmatrix}$$

Even for simple motions, calculating the Euler angles can be difficult and movement between them and the axis of rotation is not easily achieved. Also some specific rotations can be represented by infinitely many choices of Euler angles. Nevertheless, they sometimes turn out to be the most useful description of a motion.

Dual Quaternions

Another method for representing the motion in 3 dimensions is by the use of quaternions and dual quaternions, cf. [2] and [10]. A *quaternion*, Q, is a number depending on four units 1, i, j, k:

$$Q = a + bi + cj + dk, \quad a, b, c, d \in \mathbb{R}$$

where the units satisfy

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j$$

These relations allow two quaternions to be multiplied as follows:

$$\begin{aligned} Q_1Q_2 &= (a_1 + b_1i + c_1j + d_1k)(a_2 + b_2i + c_2j + d_2k) \\ &= a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ &\quad + (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)j + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)k \end{aligned}$$

Thus the quaternions form a 4–dimensional real algebra and any quaternion $Q = a + bi + cj + dk$ can be written as the 4–vector (a, b, c, d).

The *conjugate* of a quaternion $Q = a + bi + cj + dk$, is defined to be $\tilde{Q} = a - bi - cj - dk$, giving $Q\tilde{Q} = \tilde{Q}Q = a^2 + b^2 + c^2 + d^2$, which is called the *norm* of Q, $N(Q)$. If $N(Q) = 1$, Q is a *unit* quaternion.

If $a = 0$ then Q is a *pure* quaternion and the components (b, c, d) may be considered as those of a Euclidean vector in $\mathbb{R}^3$.

Let Q be the unit quaternion with components (a, b, c, d) and $\mathbf{x} = (x_1, x_2, x_3)$ be a vector in $\mathbb{R}^3$. Let x represent the pure quaternion $(0, x_1, x_2, x_3)$. Then

$$\begin{aligned}
Qx\tilde{Q} &= (a + bi + cj + dk)(x_1 i + x_2 j + x_3 k)(a - bi - cj - dk) \\
&= (a + bi + cj + dk)((bx_1 + cx_2 + dx_3) + (ax_1 - dx_2 + cx_3)i \\
&\qquad + (ax_2 - bx_3 + dx_1)j + (ax_3 - cx_1 + bx_2)k) \\
&= ((a^2 + b^2 - c^2 - d^2)x_1 + 2(bc - ad)x_2 + 2(ac + bd)x_3)i \\
&\qquad + (2(ad + bc)x_1 + (a^2 - b^2 + c^2 - d^2)x_2 + 2(cd - ab)x_3)j \\
&\qquad + (2(bd - ac)x_1 + 2(ab + cd)x_2 + (a^2 - b^2 - c^2 + d^2)x_3)k \\
&= X_1 i + X_2 j + X_3 k \\
&= X
\end{aligned}$$

So the quaternion 'conjugation' operation on a pure quaternion x produces another pure quaternion X. Moreover the operation preserves the norm.

Any unit quaternion can be written (as a 4–vector) in the form

$$Q = (\cos \tfrac{1}{2}\alpha, a_1 \sin \tfrac{1}{2}\alpha, a_2 \sin \tfrac{1}{2}\alpha, a_3 \sin \tfrac{1}{2}\alpha)$$

where $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ is a unit vector. With Q in this form, the multiplication $Qx\tilde{Q}$ represents a rotation of the 3–dimensional space of pure quaternions by angle α about axis (a_1, a_2, a_3).

Thus members of $SO(3)$ can be represented by quaternions. As with Euler angles, the representation is not unique, but there are always exactly two quaternions for any rotation. The other advantage quaternions have over Euler angles is that the composition of rotations is simply given by quaternion multiplication.

The proper description of rotations via quaternions was first given by Rodrigues in 1840 (see [1]).

For the full displacement group $SE(3)$, we make use of dual quaternions.

A *dual quaternion* is of the form $\hat{Q} = q + r\epsilon$ where q and r are quaternions and ϵ has the property that $\epsilon^2 = 0$. Two dual quaternions are then multiplied as follows:

$$\begin{aligned}
\hat{Q}_1\hat{Q}_2 &= (q_1 + r_1\epsilon)(q_2 + r_2\epsilon) \\
&= q_1 q_2 + (q_1 r_2 + r_1 q_2)\epsilon + r_1 r_2 \epsilon^2 \\
&= q_1 q_2 + (q_1 r_2 + r_1 q_2)\epsilon
\end{aligned}$$

The *conjugate* of a dual quaternion is as expected: $(q+r\epsilon)\tilde{} = \tilde{q} + \tilde{r}\epsilon$. $\hat{Q}$ is a unit dual quaternion if $\hat{Q}\tilde{\hat{Q}} = (q+r\epsilon)(q+r\epsilon)\tilde{} = q\tilde{q} + (q\tilde{r} + r\tilde{q})\epsilon = 1$. This requires q to be a unit quaternion and $q\tilde{r} + r\tilde{q} = 0$. If $q = a + bi + cj + dk$ and $r = e + fi + gj + hk$,

$$\begin{aligned}
q\tilde{r} &= (a+bi+cj+dk)(e-fi-gj-hk) \\
&= ae+bf+cg+dh+(-af+be-ch+dg)i \\
&\quad +(-ag+bh+ce-df)j+(-ah-bg+cf+de)k \\
r\tilde{q} &= (e+fi+gj+hk)(a-bi-cj-dk) \\
&= ae+bf+cg+dh+(af-be+ch-dg)i \\
&\quad +(ag-bh-ce+df)j+(ah+bg-cf-de)k \\
\Rightarrow q\tilde{r}+r\tilde{q} &= 2(ae+bf+cg+dh)
\end{aligned}$$

As a consequence of $q\tilde{r}+r\tilde{q} = 0$ we have $ae+bf+cg+dh = 0$, so $q\tilde{r}$ and $r\tilde{q}$ are both pure quaternions and $r\tilde{q} = -q\tilde{r}$. $\hat{Q}$ is a dual pure quaternion if q and r are pure quaternions.

If x is a pure quaternion representing the vector $\mathbf{x} \in \mathbb{R}^3$ and $\hat{Q}$ is a unit dual quaternion, we introduce the dual quaternion $\hat{x} = 1 + x\epsilon$ and the modified 'conjugation' map $\hat{x} \mapsto \hat{Q}\hat{x}\tilde{\hat{Q}}^*$, where $(q+r\epsilon)^* = q - r\epsilon$, and observe that

$$\begin{aligned}
\hat{Q}\hat{x}\tilde{\hat{Q}}^* &= (q+r\epsilon)(1+x\epsilon)(\tilde{q}-\tilde{r}\epsilon) \\
&= q\tilde{q} + (qx\tilde{q} - q\tilde{r} + r\tilde{q})\epsilon \\
&= 1 + (qx\tilde{q} - 2q\tilde{r})\epsilon
\end{aligned}$$

Now $q\tilde{r}$ is a pure quaternion, so it can be identified with a vector in $\mathbb{R}^3$ and, as before, $qx\tilde{q}$ represents a rotation. Thus the transformation $x \mapsto qx\tilde{q} - 2q\tilde{r}$ represents a rotation about the origin followed by a translation.

So any element of $SE(3)$ can be represented by the dual quaternion $q + r\epsilon$ for appropriate choices of q and r.

Although this gives a nice algebraic structure to the elements of $SE(3)$, in practice the calculations relating to spatial motions are no simpler than those for the standard matrix representation. The main advantage of this representation is when we have consecutive motions which can be achieved simply by multiplying the relevant quaternions. It is mostly of use when the motion concerned is a finite displacement rather than the more general continuous motion we are considering.

1.5 Examples

To illustrate these two methods of representation, consider the motion consisting of a rotation by angle t about the line $y = x$ followed by the translation (t, t^2, t^3).

In this case the axis of rotation is perpendicular to the z–axis, so it can be easily seen that we can perform the required rotation by first rotating about the z–axis by an angle of $-\frac{\pi}{4}$ to bring the axis of rotation in line with the x–axis, then rotating about the x–axis by the required angle t, then rotating about the z–axis by an angle of $\frac{\pi}{4}$ to bring the axis of rotation back into it's correct position. From this the Euler angles can clearly be identified: $\theta_1(t) = -\frac{\pi}{4}$, $\theta_2(t) = t$, $\theta_3(t) = \frac{\pi}{4}$.

In quaternion representation, where the motion is represented by the dual quaternion $q + r\epsilon$, with $qx\tilde{q}$ being the rotational component and $-2q\tilde{r}$ being the translational component, the relevant quaternions for this first example are

$$\begin{aligned} q_1 &= \left(\cos(\tfrac{1}{2}t), \tfrac{1}{\sqrt{2}}\sin(\tfrac{1}{2}t), \tfrac{1}{\sqrt{2}}\sin(\tfrac{1}{2}t), 0\right) \\ r_1 &= \left(-\tfrac{1}{2\sqrt{2}}t(1+t)\sin(\tfrac{1}{2}t), -\tfrac{1}{4}t\left(-2\cos(\tfrac{1}{2}t) + \sqrt{2}\,t^2\sin(\tfrac{1}{2}t)\right),\right. \\ &\qquad \left.\tfrac{1}{4}t^2\left(2\cos(\tfrac{1}{2}t) + \sqrt{2}\,t\sin(\tfrac{1}{2}t)\right), \tfrac{1}{4}t\left(\sqrt{2}\,(1-t)\sin(\tfrac{1}{2}t) + 2t^2\cos(\tfrac{1}{2}t)\right)\right) \end{aligned}$$

This example is well–suited to Euler angles, but the quaternion representation is not so nice. However, if we follow this motion by a second motion involving a rotation about the line $y = -x$ by an angle of t^2, the quaternion representation for the complete motion is easily calculated by multiplying the quaternions representing the two separate motions. Motions of this sort arise in robotics where the rigid body is the end–effector attached to an arm with joints connected in series.

This second motion has quaternion representation

$$\begin{aligned} q_2 &= \left(\cos(\tfrac{1}{2}t^2), \tfrac{1}{\sqrt{2}}\sin(\tfrac{1}{2}t^2), -\tfrac{1}{\sqrt{2}}\sin(\tfrac{1}{2}t^2), 0\right) \\ r_2 &= (0, 0, 0, 0) \end{aligned}$$

So the complete motion has quaternion representation

$$\begin{aligned}
q &= q_2 q_1 \\
&= \Big(\cos(\tfrac{1}{2}t)\cos(\tfrac{1}{2}t^2), \tfrac{1}{\sqrt{2}}\sin(\tfrac{1}{2}(t+t^2)), \tfrac{1}{\sqrt{2}}\sin(\tfrac{1}{2}(t-t^2)), 0\Big) \\
r &= q_2 r_1 + r_2 q_1 \\
&= q_2 r_1 \\
&= \frac{1}{2}\begin{pmatrix} t^3\sin(\tfrac{1}{2}t)\sin(\tfrac{1}{2}t^2) + \tfrac{1}{\sqrt{2}}t^2\sin(\tfrac{1}{2}(t^2-t)) - \tfrac{1}{\sqrt{2}}t\sin(\tfrac{1}{2}(t^2+t)) \\ t\cos(\tfrac{1}{2}(t^2+t)) - \tfrac{1}{\sqrt{2}}t^3\sin(\tfrac{1}{2}(t^2+t)) \\ t^2\cos(\tfrac{1}{2}(t-t^2)) + \tfrac{1}{\sqrt{2}}t^3\sin(\tfrac{1}{2}(t-t^2)) \\ t^3\cos(\tfrac{1}{2}t)\cos(\tfrac{1}{2}t^2) + \tfrac{1}{\sqrt{2}}t^2\sin(\tfrac{1}{2}(t^2-t)) + \tfrac{1}{\sqrt{2}}t\sin(\tfrac{1}{2}(t^2+t)) \end{pmatrix}^T
\end{aligned}$$

Although this is messy, it is relatively easy to calculate.

If we wanted to represent the complete motion via Euler angles, we would need to break down the matrix representing the complete rotation into the three Euler matrices. The Euler angles for the first motion are $\theta_1 = -\frac{\pi}{4}$, $\theta_2 = t$ and $\theta_3 = \frac{\pi}{4}$. The Euler angles for the second motion are $\theta_1 = \frac{\pi}{4}$, $\theta_2 = t^2$ and $\theta_3 = -\frac{\pi}{4}$. This gives matrix representation of the rotation as

$$A(t) = A_2(t)A_1(t)$$

$$\text{where} \quad A_1(t) = \begin{pmatrix} \frac{1}{2}(1+\cos t) & \frac{1}{2}(1-\cos t) & \frac{1}{\sqrt{2}}\sin t \\ \frac{1}{2}(1-\cos t) & \frac{1}{2}(1+\cos t) & -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\sin t & \cos t \end{pmatrix}$$

$$\text{and} \quad A_2(t) = \begin{pmatrix} \frac{1}{2}(1+\cos t^2) & -\frac{1}{2}(1-\cos t^2) & -\frac{1}{\sqrt{2}}\sin t^2 \\ -\frac{1}{2}(1-\cos t^2) & \frac{1}{2}(1+\cos t^2) & -\frac{1}{\sqrt{2}}\sin t^2 \\ \frac{1}{\sqrt{2}}\sin t^2 & \frac{1}{\sqrt{2}}\sin t^2 & \cos t^2 \end{pmatrix}$$

$A(t)$ cannot be explicitly solved for Euler angles as defined.

1.6 Change of Coordinates

It is frequently useful to make a change of coordinates in $E(n)$ to get the rotation matrix $A(t)$ and the translation vector $\mathbf{a}(t)$ into certain standard forms which make calculations easier.

When studying a rigid–body motion locally, say in a neighbourhood of $t = 0$, we can choose coordinates in the ambient space and in the rigid body itself which match, so that $\mu(0) = (A(0), \mathbf{a}(0)) = (I, \mathbf{0}) \in SE(n)$.

A simultaneous change of coordinates on the ambient space and on the rigid body is given by $(G, \mathbf{g}) \in SE(n)$. It's effect on a motion $(A(t), \mathbf{a}(t))$ is represented by conjugation in $SE(n)$:

$$(G, \mathbf{g})(A, \mathbf{a})(G, \mathbf{g})^{-1} = (GAG^{-1}, G\mathbf{a} - GAG^{-1}\mathbf{g} + \mathbf{g}) \tag{1.1}$$

Let us first of all consider the rotational part of the motion. We may determine the local behaviour of the motion by considering its Taylor expansion at some point $t_0 \in \mathbb{R}$. Let us take $t_0 = 0$ as above. Consider a smooth rotation of E^n described by the family of $n \times n$ orthogonal matrices $A(t)$ with $A(0) = I$. Using the condition $A^T A = I$ one can show that the Taylor Series in the neighbourhood of $t = 0$ has the form

$$A(t) = I + (B_1 + C_1)t + \tfrac{1}{2!}(B_2 + C_2)t^2 + \tfrac{1}{3!}(B_3 + C_3)t^3 + \dots$$

where B_k ($k = 1, 2, \dots$) is an arbitrary skew–symmetric matrix and C_k ($k = 1, 2, \dots$) is a symmetric matrix which depends on $B_1, B_2, \dots, B_{k-1}$ and is given by the recursion formula

$$C_k = -\tfrac{1}{2} \sum_{i=1}^{k-1} \binom{k}{i} (C_{k-i} + B_{k-i})(C_i - B_i)$$

with $C_1 = 0$ (cf [2]).

Explicitly, the first few terms are

$$A(t) = I + B_1 t + \tfrac{1}{2}(B_1^2 + B_2)t^2 + \tfrac{1}{6}(\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)t^3 + \dots$$

Now the transformation of $A(t)$ under a change of coordinates determines a transformation of the B_i's. From equation (1.1) we see that $B_i \mapsto GB_iG^{-1}$.

The Taylor expansion of the translation vector $\mathbf{a}(t)$ is simply

$$\mathbf{a}(t) = \mathbf{a}(0) + \dot{\mathbf{a}}(0)t + \tfrac{1}{2}\ddot{\mathbf{a}}(0)t^2 + \tfrac{1}{6}\dddot{\mathbf{a}}(0)t^3 + \dots$$

Let

$$\begin{aligned}\mathbf{b}_1 &= \dot{\mathbf{a}}(0)\\ \mathbf{b}_2 &= \ddot{\mathbf{a}}(0)\\ \mathbf{b}_3 &= \dddot{\mathbf{a}}(0)\end{aligned}$$

Again, from equation (1.1), we see that $\mathbf{a}(t) \mapsto G\mathbf{a}(t) - GA(t)G^{-1}\mathbf{g} + \mathbf{g}$. Differentiating this, we get that $\dot{\mathbf{a}}(t) \mapsto G\dot{\mathbf{a}}(t) - G\dot{A}(t)G^{-1}\mathbf{g}$ and

$$\begin{aligned}\mathbf{b}_1 &\mapsto G\mathbf{b}_1 - GB_1G^{-1}\mathbf{g}\\ \mathbf{b}_2 &\mapsto G\mathbf{b}_2 - G(B_1^2 + B_2)G^{-1}\mathbf{g}\\ \mathbf{b}_3 &\mapsto G\mathbf{b}_3 - G(\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)G^{-1}\mathbf{g}\end{aligned}$$

Thus we shall be interested in finding a change of coordinates that simultaneously reduces the skew–symmetric B_i's and the vector $\mathbf{b}_i$'s to simpler forms.

This is the method we will be using for analysing the case $n = 3$ and will be explored further in Section 4.

2 Differential Geometry

2.1 Introduction

When $n = 2$, the point–paths of the motion are a 2–parameter family of plane curves, $\Phi_{\mathbf{x}} : \mathbb{R} \to E^2$, when $n = 3$, they are a 3–parameter family of space curves, $\Phi_{\mathbf{x}} : \mathbb{R} \to E^3$; and their geometry can be studied using the techniques of differential geometry for general plane and space curves.

We will be looking for connections between the geometry of the plane and space curves and the motion to which they correspond.

The approach we are presenting is that described by Bruce and Gibson in [3].

2.2 Plane Curves

A general smooth *plane curve* is a map $\gamma : I \to \mathbb{R}^2$ where I is an open interval in $\mathbb{R}$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and each function γ_1, γ_2 has derivatives of all orders, for all $t \in I$. We say γ is *regular* provided there is no $t \in I$ with $\gamma_1'(t) = \gamma_2'(t) = 0$.

A plane curve, $\alpha(t)$, is *unit speed* if the derivative vector has length $\|\alpha'(t)\| = 1$. We define the (unit) *tangent vector* to α at t by $T(t) = \alpha'(t)$ and the (unit) *normal vector*, $N(t)$, to be obtained from the tangent vector by rotating anticlockwise through an angle of $\frac{\pi}{2}$. So $N(t) \perp T(t)$ and we have the picture in Fig. 2.1.

As $T(t)$ is a unit vector, we have $T \cdot T = 1$ and hence, after differentiating:

$$T \cdot T' = 0$$

so the derivative of the tangent vector is a (possibly zero) vector perpendicular to the tangent vector. This means that there is some real number $\kappa(t)$, called the *curvature* of α at t, such that

$$T'(t) = \kappa(t)N(t)$$

A regular plane curve that is not unit speed can be made so by a reparametrisation using arclength ($l(t) = \int_{t_0}^{t} \|\gamma'(u)\|\, du$) from a fixed $t = t_0$ as the new parameter. The curve $\alpha = \gamma \circ l^{-1}$ is then unit speed and the curvature of γ is defined as the curvature of its unit–speed reparametrisation α. It can be shown, after such a reparametrisation, that for

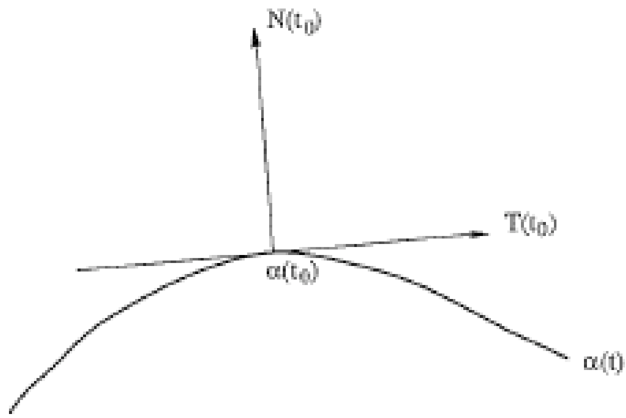

Figure 2.1: The tangent and normal for a plane curve

a plane curve $\gamma(t) = (X(t), Y(t))$,

$$T(t) = \frac{(X'(t), Y'(t))}{(X'(t)^2 + Y'(t)^2)^{\frac{1}{2}}} = \frac{\gamma'(t)}{\|\gamma'(t)\|} \tag{2.1}$$

$$N(t) = \frac{(-Y'(t), X'(t))}{(X'(t)^2 + Y'(t)^2)^{\frac{1}{2}}} = J\frac{\gamma'(t)}{\|\gamma'(t)\|} \tag{2.2}$$

$$\kappa(t) = \frac{X'(t)Y''(t) - X''(t)Y'(t)}{(X'(t)^2 + Y'(t)^2)^{\frac{3}{2}}} = \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t)\|^3} \tag{2.3}$$

where for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, we define $\mathbf{u} \times \mathbf{v} = u_1 v_2 - u_2 v_1$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as before. (See [3]).

Now $N(t) \perp N'(t)$, so $N'(t) = \lambda T(t)$ for some $\lambda \in \mathbb{R}$. Thus we have

$$\begin{aligned} T \cdot N &= 0 \\ \Rightarrow T' \cdot N + T \cdot N' &= 0 \\ \Rightarrow \kappa N \cdot N + T \cdot \lambda T &= 0 \\ \Rightarrow \kappa + \lambda &= 0 \\ \Rightarrow \lambda &= -\kappa \end{aligned}$$

This gives the Serret–Frenet formulae for plane curves: for α unit speed, we have

$$T'(t) = \kappa(t)N(t)$$
$$N'(t) = -\kappa(t)T(t)$$

and for an arbitrary plane curve, γ,

$$T'(t) = \kappa(t)N(t)\|\gamma'(t)\|$$
$$N'(t) = -\kappa(t)T(t)\|\gamma'(t)\|$$

2.3 Contact

We can reinterpret κ in terms of best–fitting circles and lines as measured by the idea of "contact" of curves. Important geometric properties of a curve can then be characterised by, for example, $\kappa = 0$ or $\kappa' = 0$.

We say a parametrised curve $\gamma(t)$ has *order of contact* k or k*–point contact* at $t = t_0$ with a curve defined by $F^{-1}(0) = \{\, \mathbf{x} \mid F(\mathbf{x}) = 0 \,\}$, where $F : \mathbb{R}^2 \to \mathbb{R}$, if $(F \circ \gamma)(t_0) = (F \circ \gamma)'(t_0) = \cdots = (F \circ \gamma)^{(k-1)}(t_0) = 0$ and $(F \circ \gamma)^{(k)}(t_0) \neq 0$. Intuitively, small perturbations of γ may intersect $F^{-1}(0)$ in up to k–points. The number of intersection points for a particular curve tells us the order of contact with $F^{-1}(0)$ for that particular curve.

2.4 Inflection Points

A unit speed plane curve, $\alpha(t)$, has an *ordinary inflection point* at $t = t_0$ if the tangent line at t_0, which has equation $F(\mathbf{x}) = (\mathbf{x} - \alpha(t_0)) \cdot N(t_0)$, has 3–point contact with the curve at $t = t_0$. In general, the tangent line to a curve will have 2–point contact with that curve: if you change the curve slightly it will be intersected by the tangent line at 2

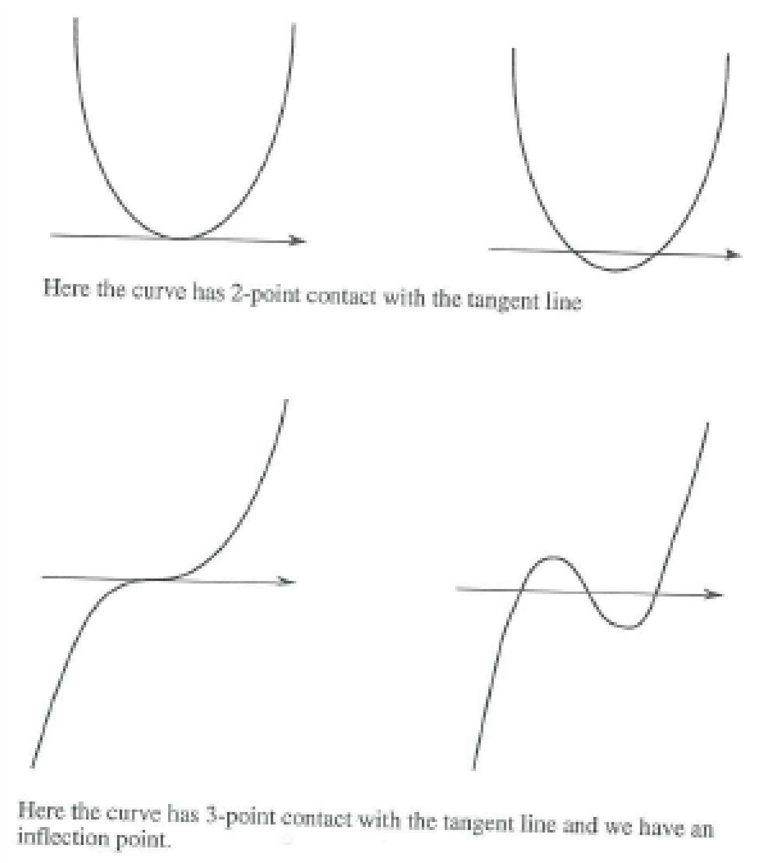

Figure 2.2: 2– and 3–pont contact with the tangent line

points. At an inflection point, changing the curve slightly will cause the tangent line to intersect it at 3 points.

If the tangent line has k–point ($k \geq 3$) contact with the curve at $t = t_0$, the curve has a *higher inflection* there.

We can determine the conditions for the tangent line to have k–point ($k \geq 3$) contact as follows:

$$
\begin{aligned}
F(\alpha(t_0)) &= (\alpha(t_0) - \alpha(t_0)) \cdot N(t_0) = 0 \\
F'(\alpha(t_0)) &= \alpha'(t_0) \cdot N(t_0) \\
&= T(t_0) \cdot N(t_0) = 0 \\
F''(\alpha(t_0)) &= T'(t_0) \cdot N(t_0) \\
&= \kappa(t_0) N(t_0) \cdot N(t_0) \\
&= \kappa(t_0) \\
F'''(\alpha(t_0)) &= \kappa'(t_0) + \kappa(t_0) N'(t_0) \cdot N(t_0)
\end{aligned}
$$

Thus $\alpha(t)$ has an inflection point at $t = t_0$ if and only if $\kappa(t_0) = 0$. If, in addition, $\kappa'(t_0) \neq 0$, $\alpha(t)$ has an ordinary inflection point at $t = t_0$. Typically one would expect κ to vanish only at isolated points on a curve.

2.5 Vertices

The curve $\alpha(t)$ has an *ordinary vertex* at $t = t_0$ if there is a circle passing through $\alpha(t_0)$ which has 4–point contact with the curve at $t = t_0$. If the circle has k–point ($k > 4$) contact, the curve has a *higher vertex* there. The existence of a vertex is a measure of the symmetry of α about the normal $N(t_0)$.

The circle, centre u, passing through $\alpha(t_0)$ has equation $F(\mathbf{x}) = 0$ where

$$F(\mathbf{x}) = \|\mathbf{x} - u\|^2 - \|\alpha(t_0) - u\|^2$$

To determine its contact with the curve, we calculate:

$$\begin{aligned}
F(\alpha(t_0)) &= \|\alpha(t_0) - u\|^2 - \|\alpha(t_0) - u\|^2 = 0 \\
\tfrac{1}{2}F'(\alpha(t_0)) &= \alpha'(t_0) \cdot (\alpha(t_0) - u) \\
&= T(t_0) \cdot (\alpha(t_0) - u) \\
\tfrac{1}{2}F''(\alpha(t_0)) &= T'(t_0) \cdot (\alpha(t_0) - u) + T(t_0) \cdot \alpha'(t_0) \\
&= \kappa(t_0)N(t_0) \cdot (\alpha(t_0) - u) + T(t_0) \cdot T(t_0) \\
&= \kappa(t_0)N(t_0) \cdot (\alpha(t_0) - u) + 1 \\
\tfrac{1}{2}F'''(\alpha(t_0)) &= (\kappa'(t_0)N(t_0) + \kappa(t_0)N'(t_0)) \cdot (\alpha(t_0) - u) \\
&= (\kappa'(t_0)N(t_0) - \kappa(t_0)^2T(t_0)) \cdot (\alpha(t_0) - u) \\
\tfrac{1}{2}F^{iv}(\alpha(t_0)) &= (\kappa''(t_0)N(t_0) - 3\kappa(t_0)\kappa'(t_0)T(t_0) - \kappa(t_0)^3N(t_0)) \cdot (\alpha(t_0) - u) \\
&\quad + (\kappa'(t_0)N(t_0) - \kappa(t_0)^2T(t_0)) \cdot T(t_0) \\
&= ((\kappa''(t_0) - \kappa(t_0)^3)N(t_0) - 3\kappa(t_0)\kappa'(t_0)T(t_0)) \cdot (\alpha(t_0) - u) \\
&\quad - \kappa(t_0)^2
\end{aligned}$$

Thus we have the following (suppressing parameter t_0):

$$\begin{aligned}
F' = 0 &\iff \alpha - u = \lambda N \\
F' = F'' = 0 &\iff \lambda\kappa + 1 = 0 \\
&\iff \lambda = \tfrac{-1}{\kappa} && (2.4) \\
F' = F'' = F''' = 0 &\iff (\kappa' N - \kappa^2 T) \cdot \tfrac{-N}{\kappa} = 0 \\
&\iff \tfrac{-\kappa'}{\kappa} = 0 \\
&\iff \kappa' = 0 && (2.5) \\
F' = F'' = F''' = F^{iv} = 0 &\iff ((\kappa'' - \kappa^3)N - 3\kappa\kappa' T) \cdot \tfrac{-N}{\kappa} - \kappa^2 = 0 \\
&\iff \tfrac{-(\kappa'' - \kappa^3)}{\kappa} - \kappa^2 = 0 \\
&\iff \tfrac{\kappa^3 - \kappa''}{\kappa} = \kappa^2 \\
&\iff \kappa^3 - \kappa'' = \kappa^3 \\
&\iff \kappa'' = 0 && (2.6)
\end{aligned}$$

From (2.4), provided $\kappa(t_0) \neq 0$ there is a unique circle having at least 3–point contact with α. It is called the *osculating circle* or *circle of curvature* at t_0. Its centre, $\alpha(t_0) + N(t_0)/\kappa(t_0)$ is called the *centre of curvature* of α at t_0; its radius, $\rho(t_0) = 1/\kappa(t_0)$, is called the *radius of curvature*.

By (2.5), this unique circle has at least 4–point contact if and only if $\kappa'(t_0) = 0$ so α has a vertex at $t = t_0$ if $\kappa(t_0) \neq 0$ and $\kappa'(t_0) = 0$. If, in addition, $\kappa''(t_0) \neq 0$, by (2.6), α has an ordinary vertex at $t = t_0$. So an ordinary vertex occurs at a simple maximum or minimum of curvature (provided $\kappa(t) \neq 0$).

2.6 Space Curves

General and unit speed *space curves* are defined in the same way as plane curves. For a unit speed space curve, $\alpha(t)$, $\alpha : I \to \mathbb{R}^3$, the (unit) *tangent vector* is given by $T(t) = \alpha'(t)$ and the *curvature* is defined to be $\kappa(t) = \|T'(t)\|$. This is in direct contrast to the planar case: here the definition of κ has no relation to the normal, moreover $\kappa \geq 0$ always. Again $T(t)$ is a unit vector, so $T \cdot T' = 0$. Hence, provided $\kappa(t) \neq 0$, there is a (unit) *principal normal*, $N(t)$, parallel to T' (and perpendicular to $T(t)$) with $T'(t) = \kappa(t)N(t)$. Since $T(t)$ and $N(t)$ are perpendicular unit vectors there is a unique third unit vector, $B(t)$, called the *binormal vector*, which is perpendicular to both and such that $\langle T(t), N(t), B(t)\rangle$ is a right-handed system (i.e. $B(t) = T(t) \times N(t)$).

We will proceed for now on the assumption that $\kappa \neq 0$.

The plane through $\alpha(t)$ spanned by $N(t)$ and $B(t)$ is called the *normal* plane at t and the plane spanned by $T(t)$ and $N(t)$ is called the *osculating*, meaning 'touching', plane at t.

As before, since N is a unit vector, N' is perpendicular to N, so $N' = \lambda T + \tau B$ for some λ, τ in $\mathbb{R}$. Now

$$\begin{aligned} T \cdot N &= 0 \\ \Rightarrow T' \cdot N + T \cdot N' &= 0 \\ \Rightarrow \kappa N \cdot N + \lambda T \cdot T + \tau T \cdot B &= 0 \\ \Rightarrow \kappa + \lambda &= 0 \\ \Rightarrow \lambda &= -\kappa \end{aligned}$$

Hence $N'(t) = -\kappa(t)T(t) + \tau(t)B(t)$ for some real number $\tau(t)$, called the *torsion* of α at t. The torsion measures the rate of turning of the osculating plane. For a plane curve, the osculating plane is the fixed plane of the curve, so $\tau = 0$. The converse is also true: if a space curve has identically vanishing torsion, then it is planar (see [3]).

Also, B' is perpendicular to B, so $B' = \mu T + \nu N$ for some μ, ν in $\mathbb{R}$.

$$\begin{array}{rcr@{\qquad}rcr} B \cdot T = 0 & \Rightarrow & B' \cdot T + B \cdot T' = 0 & B \cdot N = 0 & \Rightarrow & B' \cdot N + B \cdot N' = 0 \\ & \Rightarrow & \mu + 0 = 0 & & \Rightarrow & \nu + \tau = 0 \\ & \Rightarrow & \mu = 0 & & \Rightarrow & \nu = -\tau \end{array}$$

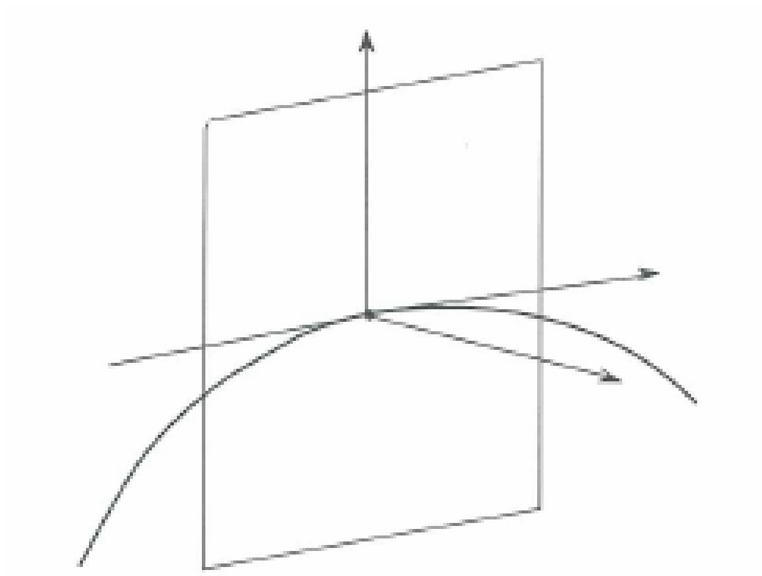

Figure 2.3: The osculating plane of a space curve

So the Serret–Frenet formulae for unit speed space curves are:

$$\begin{aligned} T'(t) &= \kappa(t)N(t) \\ N'(t) &= -\kappa(t)T(t) + \tau(t)B(t) \\ B'(t) &= -\tau(t)N(t) \end{aligned}$$

Non unit–speed space curves are discussed in Section 2.8.

2.7 Contact With Osculating Plane

Order of contact for a space curve with a surface $F^{-1}(0)$, $F : \mathbb{R}^3 \to \mathbb{R}$, is defined in the same way as for a plane curve. For a space curve, $\alpha(t)$, we wish to measure its contact at $t = t_0$ with the osculating plane and the sphere through $\alpha(t_0)$ centred at some point $u \in \mathbb{R}^3$.

The osculating plane can be given by equation $F(\mathbf{x}) = 0$ where

$$F(\mathbf{x}) = (\mathbf{x} - \alpha(t_0)) \cdot B(t_0)$$

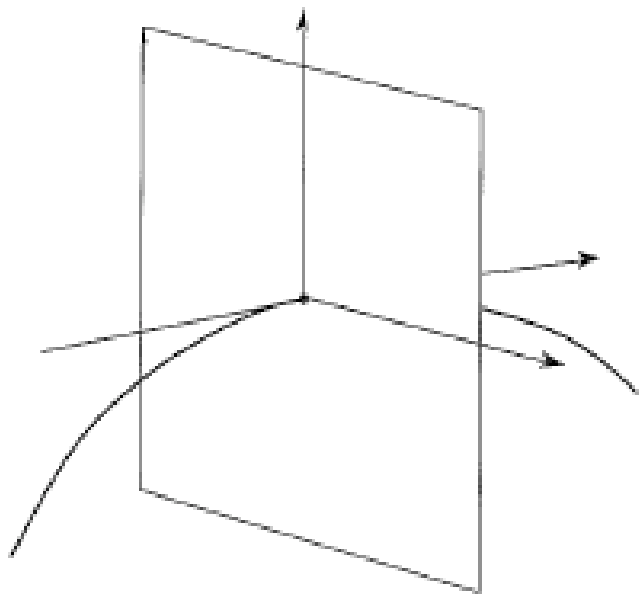

Figure 2.4: The normal plane of a space curve

Differentiating and evaluating at $\alpha(t_0)$ we get:

$$\begin{aligned}
F(\alpha(t_0)) &= (\alpha(t_0) - \alpha(t_0)) \cdot B(t_0) = 0 \\
F'(\alpha(t_0)) &= T(t_0) \cdot B(t_0) = 0 \\
F''(\alpha(t_0)) &= T'(t_0) \cdot B(t_0) \\
&= \kappa(t_0) N(t_0) \cdot B(t_0) = 0 \\
F'''(\alpha(t_0)) &= (\kappa'(t_0) N(t_0) + \kappa(t_0) N'(t_0)) \cdot B(t_0) \\
&= (\kappa'(t_0) N(t_0) - \kappa(t_0)^2 T(t_0) + \kappa(t_0)\tau(t_0) B(t_0)) \cdot B(t_0) \\
&= \kappa(t_0)\tau(t_0) \\
F^{iv}(\alpha(t_0)) &= \big(-3\kappa'(t_0)\kappa(t_0) T(t_0) + (\kappa''(t_0) - \kappa(t_0)^3 - \kappa(t_0)\tau(t_0)^2) N(t_0) \\
&\qquad + (2\kappa'(t_0)\tau(t_0) + \kappa(t_0)\tau'(t_0)) B(t_0)\big) \cdot B(t_0) \\
&= 2\kappa'(t_0)\tau(t_0) + \kappa(t_0)\tau'(t_0)
\end{aligned}$$

From this it follows that the osculating plane always has at least 3–point contact with α and it has 4–point contact if $\tau(t_0) = 0$, higher contact if, in addition $\tau'(t_0) = 0$.

2.8 Sphere Of Curvature

The sphere, passing through $\alpha(t_0)$ with centre $u \in \mathbb{R}^3$, has equation $F(\mathbf{x}) = 0$ where $F(\mathbf{x}) = \|\mathbf{x} - u\|^2 - \|\alpha(t_0) - u\|^2$. To establish the order of contact with α at t_0, we calculate:

$$\begin{aligned}
F(\alpha(t_0)) &= \|\alpha(t_0) - u\|^2 - \|\alpha(t_0) - u\|^2 = 0 \\
\tfrac{1}{2}F'(\alpha(t_0)) &= T(t_0) \cdot (\alpha(t_0) - u) \\
\tfrac{1}{2}F''(\alpha(t_0)) &= T(t_0) \cdot T(t_0) + T'(t_0) \cdot (\alpha(t_0) - u) \\
&= 1 + \kappa(t_0)N(t_0) \cdot (\alpha(t_0) - u) \\
\tfrac{1}{2}F'''(\alpha(t_0)) &= \kappa(t_0)N(t_0) \cdot T(t_0) + (\kappa'(t_0)N(t_0) + \kappa(t_0)N'(t_0)) \cdot (\alpha(t_0) - u) \\
&= (\kappa'(t_0)N(t_0) - \kappa(t_0)^2T(t_0) + \kappa(t_0)\tau(t_0)B(t_0)) \cdot (\alpha(t_0) - u)
\end{aligned}$$

Hence, we have (suppressing parameter t_0)

$$\begin{aligned}
F' = 0 &\iff u \text{ is in the normal plane at } t_0 \\
F' = F'' = 0 &\iff \alpha - u = -\tfrac{1}{\kappa}N + \mu B \ \text{ for some } \mu \in \mathbb{R} \\
F' = F'' = F''' = 0 &\iff \alpha - u = -\tfrac{1}{\kappa}N + \tfrac{\kappa'}{\kappa^2\tau}B
\end{aligned}$$

Thus we can see that, provided $\kappa(t_0) \neq 0$ and $\tau(t_0) \neq 0$, there is a unique sphere called the *sphere of curvature* having at least 4–point contact with α at t_0. The centre of this sphere, called the *centre of spherical curvature* is $\alpha + N/\kappa - (\kappa'/\kappa^2\tau)B$.

2.9 Zero Curvature and Torsion

Using the principle that increasing order of contact measures increasing geometric degeneracy, for a space curve, we are interested in points where the curvature vanishes – here the principal normal and the binormal are not well–defined – and points where the torsion vanishes – here the osculating plane has greater than 3–point contact with the curve.

For a regular space curve, $\gamma(t)$, not necessarily unit speed, we define curvature and torsion to be those for the reparametrisation as a unit speed curve. It can be shown (see [2]) that

$$\begin{aligned}
\kappa &= \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} \\
\tau &= \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}
\end{aligned} \tag{2.7}$$

Thus

$$
\begin{aligned}
\kappa = 0 &\iff \|\gamma' \times \gamma''\| = 0 \\
&\iff \gamma' \times \gamma'' = 0 \\
&\iff \gamma'' \parallel \gamma' \quad \text{or} \quad \gamma'' = 0 && (2.8) \\
\kappa \neq 0 \text{ and } \tau = 0 &\iff (\gamma' \times \gamma'') \cdot \gamma''' = 0 \\
&\iff \gamma' \times \gamma'' \perp \gamma''' \\
&\iff \gamma',\ \gamma'' \text{ and } \gamma''' \text{ are coplanar} && (2.9)
\end{aligned}
$$

So vanishing curvature imposes two constraints on γ at $t = t_0$ and vanishing torsion imposes one constraint. Hence for a 'typical' curve we'd expect isolated points where $\tau = 0$ but no points where $\kappa = 0$. However, later on when we are considering families of curves, we will not be able to avoid the situation when $\kappa = 0$.

3 Planar Motion

3.1 Introduction

In this section we will be looking at 2–dimensional rigid–body motions and the geometry of their trajectories. In particular we will study the inflection curve and the vertex curve for a general 2–dimensional rigid–body motion and examine how these degenerate when the motion has certain characteristics.

Much of this is covered in [12] by Veldkamp, among others, but, in particular, Veldkamp does not consider the case when $\dot{\theta}(0)$ and $\ddot{\theta}(0)$ both vanish.

As in section 1 the trajectories of a 2–dimensional rigid–body motion form a 2–parameter family of functions $\Phi_{\mathbf{x}} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\Phi_{\mathbf{x}}(t) = A(t)\mathbf{x} + \mathbf{a}(t)$$

with

$$A(t) = \exp\theta(t)J = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}$$

and $\mathbf{a}(t)$ a vector in $\mathbb{R}^2$.

3.2 The Moving Centrode

An *instantaneous stationary point* for the motion at $t = t_0$ is the value of $\mathbf{x}$ for which $\dot{\Phi}_{\mathbf{x}}(t_0) = 0$. This is the locus of points $\mathbf{x} \in \mathbb{R}^2$ satisfying

$$\dot{A}(t)\mathbf{x} + \dot{\mathbf{a}}(t) = 0 \tag{3.1}$$

If $\det \dot{A}(t_0) \neq 0$, or equivalently $\dot{\theta}(t_0) \neq 0$, this defines a unique stationary point at $t = t_0$, called an *instantaneous centre of rotation* for the motion. The locus of instantaneous centres of rotation of the motion constructed in the coordinates of the body is called the *moving centrode*, c_m, and is given by

$$\begin{aligned} c_m(t) &= -\dot{A}(t)^{-1}\dot{\mathbf{a}}(t) \\ &= \frac{1}{\dot{\theta}(t)} J \exp(-\theta(t)J)\mathbf{a}(t) \end{aligned} \tag{3.2}$$

If $\det \dot{A}(t_0) = 0$ and $\dot{\mathbf{a}}(t_0) \neq 0$, then c_m is undefined at $t = t_0$ and if $\det \dot{A}(t_0) = 0$ and $\dot{\mathbf{a}}(t_0) = 0$, then c_m at $t = t_0$ is the whole of $\mathbb{R}^2$.

Provided neither of these two conditions hold, $c_m(t)$ is a smooth curve.

3.3 Inflection Curve

Since the existence of an inflection imposes one condition, we would expect a curve of points $\mathbf{x}$ in the plane whose trajectories have inflections at $t = t_0$. Now, the points whose trajectories have inflections at $t = 0$ are those whose trajectories have zero curvature (as in Section 2.3). By equation (2.3) the points of zero curvature must satisfy

$$\dot{\Phi}_{\mathbf{x}}(0) \times \ddot{\Phi}_{\mathbf{x}}(0) = 0 \tag{3.3}$$

Note that even though $c_m(0)$ satisfies equation (3.3) it has a singular trajectory so is not an inflection point. In the general case that c_m is uniquely defined (i.e $\dot{\theta} \neq 0$) we can make a suitable choice of coordinates so that $c_m(0) = 0$ and, in particular $\dot{\mathbf{a}}(0) = 0$; then equation (3.3) becomes

$$x_1^2 + x_2^2 - \frac{1}{\dot{\theta}(0)^2}(x_1\ddot{a}_1(0) + x_2\ddot{a}_2(0)) = 0 \tag{3.4}$$

which is a circle passing through $c_m(0)$ and tangent to the centrode, $c_m(t)$. This is generally known as the *inflection curve* (or *inflection circle*), (see [12]).

If, on the other hand, $\dot{\theta}(0) = 0$ and $\dot{\mathbf{a}}(0) \neq 0$, then, from equation (3.2), $c_m(t) \to \infty$ as $t \to 0$, and the motion is instantaneously translatory (the vector field $\dot{\Phi}_{\mathbf{x}}(0)$ is constant). In this case it is instructive to change to homogeneous coordinates (x_1, x_2, x_3) in the real projective plane, $\mathbb{P}^2$, where $x_3 = 1$ defines the Euclidean plane and $x_3 = 0$ defines the *line at infinity*. In this setting it is possible to regard $c_m(0)$ as a point at infinity: from equation (3.1) $c_m(t)$ in cartesian coordinates is the locus of points satisfying

$$\begin{aligned}
-\dot{\theta}(t)\,(\sin\theta(t)x_1 + \cos\theta(t)x_2) + \dot{a}_1(t) &= 0 \\
\dot{\theta}(t)\,(\cos\theta(t)x_1 - \sin\theta(t)x_2) + \dot{a}_2(t) &= 0
\end{aligned}$$

Changing to homogeneous coordinates and evaluating at $t = 0$ we have

$$\begin{aligned}
\dot{\theta}(0)x_1 &= -\dot{a}_2(0)x_3 \\
\dot{\theta}(0)x_2 &= \dot{a}_1(0)x_3
\end{aligned}$$

which gives the point $(-\dot{a}_2(0), \dot{a}_1(0), \dot{\theta}(0))$. Thus, when $\dot{\theta}(0) = 0$, $c_m(0)$ is the point at infinity $(-\dot{a}_2(0), \dot{a}_1(0), 0)$.

In homogeneous coordinates, the inflection curve becomes

$$\begin{aligned}
\dot{\theta}(0)^3(x_1^2 + x_2^2) + \dot{\theta}(0)^2(\dot{a}_2(0)x_1 - \dot{a}_1(0)x_2)x_3 - \dot{\theta}(0)(\ddot{a}_1(0)x_1 + \ddot{a}_2(0)x_2)x_3 \\
+ \ddot{\theta}(0)\,(\dot{a}_1(0)x_1 + \dot{a}_2(0)x_2)\,x_3 + (\dot{a}_1(0)\ddot{a}_2(0) - \dot{a}_2(0)\ddot{a}_1(0))x_3^2 = 0
\end{aligned} \tag{3.5}$$

Setting $\dot{\theta}(0) = 0$ we have

$$x_3 \left(\ddot{\theta}(0) \left(\dot{a}_1(0) x_1 + \dot{a}_2(0) x_2 \right) + \left(\dot{a}_1(0)\ddot{a}_2(0) - \dot{a}_2(0)\ddot{a}_1(0) \right) x_3 \right) = 0 \tag{3.6}$$

So the inflection curve degenerates to the line at infinity $x_3 = 0$ and the line (in cartesian coordinates)

$$x_1 \dot{a}_1(0) + x_2 \dot{a}_2(0) = -\frac{1}{\ddot{\theta}(0)} \left(\dot{a}_1(0)\ddot{a}_2(0) - \dot{a}_2(0)\ddot{a}_1(0) \right) \tag{3.7}$$

when $\dot{\theta}(0) = 0$ (c.f. [11]). The line (3.7) is in fact the asymptote to c_m so we can see that the line of inflections given by equation (3.6) is, once again, tangent in $\mathbb{P}^2$ to the centrode, $c_m(t)$, at $t = 0$.

When $\dot{\theta}(0) = \ddot{\theta}(0) = 0$, the inflection curve further degenerates. Setting $\ddot{\theta}(0) = 0$ in equation (3.6) we have

$$\left(\dot{\mathbf{a}}(0) \times \ddot{\mathbf{a}}(0) \right) x_3^2 = 0 \tag{3.8}$$

If $\dot{\mathbf{a}}(0) \times \ddot{\mathbf{a}}(0) = 0$, the inflection curve is the whole of the projective plane; if $\dot{\mathbf{a}}(0) \times \ddot{\mathbf{a}}(0) \neq 0$ the inflection curve degenerates to the line at infinity, $x_3 = 0$, repeated.

3.4 Vertex Curve

Also of interest are the points whose trajectories have vertices at $t = 0$. From Section 2.4, this happens when the trajectory has stationary curvature, i.e. $\kappa' = 0$. Differentiating equation (2.3), we see that such points, $\mathbf{x}$, are given by

$$\|\dot{\Phi}_{\mathbf{x}}\|^2 \left(\dot{\Phi}_{\mathbf{x}} \times \dddot{\Phi}_{\mathbf{x}} \right) = 3 \left(\dot{\Phi}_{\mathbf{x}} \times \ddot{\Phi}_{\mathbf{x}} \right) \left(\dot{\Phi}_{\mathbf{x}} \cdot \ddot{\Phi}_{\mathbf{x}} \right) \tag{3.9}$$

(all derivatives at $t = 0$). When $c_m(0)$ is uniquely defined this reduces to a cubic, called the *vertex curve* (or *cubic of stationary curvature*). Choosing coordinates so that $c_m(0) = 0$ it has the form

$$\begin{aligned} &\left(x_1^2 + x_2^2 \right) \left(x_1 \ddot{a}_2 - x_2 \ddot{a}_1 \right) + \frac{1}{3\dot{\theta}} \left(x_1^2 + x_2^2 \right) \left(x_1 \dddot{a}_1 + x_2 \dddot{a}_2 \right) \\ &\qquad - \frac{\ddot{\theta}}{\dot{\theta}^2} \left(x_1^2 + x_2^2 \right) \left(x_1 \ddot{a}_1 + x_2 \ddot{a}_2 \right) - \frac{1}{\dot{\theta}^2} \left(x_1 \ddot{a}_2 - x_2 \ddot{a}_1 \right) \left(x_1 \ddot{a}_1 + x_2 \ddot{a}_2 \right) = 0 \end{aligned} \tag{3.10}$$

This curve passes through $c_m(0)$ twice, once tangent to the centrode and once normal to it. It intersects with the inflection curve at 3 points: twice at $c_m(0)$ and at one other point, called *Ball's point*, given by

$$x_1 = \frac{(\ddot{\mathbf{a}} \times \dddot{\mathbf{a}}) \left(\dot{\theta}\dddot{a}_2 - 3\ddot{\theta}\ddot{a}_2\right)}{\dot{\theta}\left(\left(\dot{\theta}\dddot{a}_1 - 3\ddot{\theta}\ddot{a}_1\right)^2 + \left(\dot{\theta}\dddot{a}_2 - 3\ddot{\theta}\ddot{a}_2\right)^2\right)}$$

$$x_2 = \frac{-(\ddot{\mathbf{a}} \times \dddot{\mathbf{a}}) \left(\dot{\theta}\dddot{a}_1 - 3\ddot{\theta}\ddot{a}_1\right)}{\dot{\theta}\left(\left(\dot{\theta}\dddot{a}_1 - 3\ddot{\theta}\ddot{a}_1\right)^2 + \left(\dot{\theta}\dddot{a}_2 - 3\ddot{\theta}\ddot{a}_2\right)^2\right)}$$

The vertex curve also degenerates when $\dot{\theta}(0) = 0$. Once again changing to homogeneous coordinates, we have

$$\begin{aligned}
\|\dot{\Phi}_{\mathbf{x}}\|^2 &= \dot{\theta}^2(x_1^2 + x_2^2) + 2\dot{\theta}(\dot{a}_2 x_1 x_3 - \dot{a}_1 x_2 x_3) + \|\dot{\mathbf{a}}\|^2 x_3^2 \\
\dot{\Phi}_{\mathbf{x}} \times \dddot{\Phi}_{\mathbf{x}} &= 3\dot{\theta}^2\ddot{\theta}(x_1^2 + x_2^2) + (\dddot{\theta}\dot{a}_1 - \dot{\theta}^3\dot{a}_1 + 3\dot{\theta}\ddot{\theta}\dot{a}_2 - \dot{\theta}\dddot{a}_1)x_1 x_3 \\
&\quad + (\dddot{\theta}\dot{a}_2 - \dot{\theta}^3\dot{a}_2 - 3\dot{\theta}\ddot{\theta}\dot{a}_1 - \dot{\theta}\dddot{a}_2)x_2 x_3 + (\dot{\mathbf{a}} \times \dddot{\mathbf{a}})x_3^2 \\
\dot{\Phi}_{\mathbf{x}} \times \ddot{\Phi}_{\mathbf{x}} &= \dot{\theta}^3(x_1^2 + x_2^2) + (\ddot{\theta}\dot{a}_1 + \dot{\theta}^2\dot{a}_2 - \dot{\theta}\ddot{a}_1)x_1 x_3 \\
&\quad + (\ddot{\theta}\dot{a}_2 - \dot{\theta}^2\dot{a}_1 - \dot{\theta}\ddot{a}_2)x_2 x_3 + (\dot{\mathbf{a}} \times \ddot{\mathbf{a}})x_3^2 \\
\dot{\Phi}_{\mathbf{x}} \cdot \ddot{\Phi}_{\mathbf{x}} &= \dot{\theta}\ddot{\theta}(x_1^2 + x_2^2) + (\ddot{\theta}\dot{a}_2 - \dot{\theta}^2\dot{a}_1 + \dot{\theta}\ddot{a}_2)x_1 x_3 \\
&\quad - (\ddot{\theta}\dot{a}_1 + \dot{\theta}^2\dot{a}_2 + \dot{\theta}\ddot{a}_1)x_2 x_3 + (\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}})x_3^2
\end{aligned}$$

Setting $\dot{\theta}(0) = 0$ equation (3.9) becomes

$$\begin{aligned}
&x_3^2\|\dot{\mathbf{a}}\|^2 \left(\dddot{\theta}\dot{a}_1 x_1 x_3 + \dddot{\theta}\dot{a}_2 x_2 x_3 + (\dot{\mathbf{a}} \times \dddot{\mathbf{a}})x_3^2\right) \\
&\quad = 3x_3^2 \left(\ddot{\theta}\dot{a}_1 x_1 + \ddot{\theta}\dot{a}_2 x_2 + (\dot{\mathbf{a}} \times \ddot{\mathbf{a}})x_3\right)\left(\ddot{\theta}\dot{a}_2 x_1 - \ddot{\theta}\dot{a}_1 x_2 + (\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}})x_3\right)
\end{aligned} \tag{3.11}$$

So the vertex curve with equation given by (3.10) degenerates to the line at infinity $x_3 = 0$ and the conic given by (in cartesian coordinates)

$$A x_1^2 + B x_1 x_2 + C x_2^2 + D x_1 + E x_2 + F = 0 \tag{3.12}$$

where

$$
\begin{aligned}
A &= \dot{a}_1 \dot{a}_2 \\
B &= \dot{a}_2^2 - \dot{a}_1^2 \\
C &= -\dot{a}_1 \dot{a}_2 \\
D &= \frac{1}{\ddot{\theta}} \left(\dot{a}_1 \left(\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}} \right) + \dot{a}_2 \left(\dot{\mathbf{a}} \times \ddot{\mathbf{a}} \right) \right) - \frac{\dddot{\theta}}{3\ddot{\theta}^2} \|\dot{\mathbf{a}}\|^2 \dot{a}_1 \\
E &= \frac{1}{\ddot{\theta}} \left(\dot{a}_2 \left(\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}} \right) - \dot{a}_1 \left(\dot{\mathbf{a}} \times \ddot{\mathbf{a}} \right) \right) - \frac{\dddot{\theta}}{3\ddot{\theta}^2} \|\dot{\mathbf{a}}\|^2 \dot{a}_2 \\
F &= \frac{1}{\ddot{\theta}^2} \left(\left(\dot{\mathbf{a}} \times \ddot{\mathbf{a}} \right) \left(\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}} \right) - \frac{1}{3} \|\dot{\mathbf{a}}\|^2 \left(\dot{\mathbf{a}} \times \dddot{\mathbf{a}} \right) \right)
\end{aligned}
$$

The discriminant of this conic is

$$
\begin{aligned}
\triangle &= B^2 - 4AC \\
&= \left(\dot{a}_2^2 - \dot{a}_1^2 \right)^2 - 4 \left(\dot{a}_1 \dot{a}_2 \right) \left(-\dot{a}_1 \dot{a}_2 \right) \\
&= \left(\dot{a}_1^2 + \dot{a}_2^2 \right)^2 = \|\dot{\mathbf{a}}\|^4 > 0
\end{aligned}
$$

Thus, when $\dot{\theta}(0) = 0$ and $\ddot{\theta}(0) \neq 0$, the vertex curve becomes the line at infinity and a hyperbola.

In this degenerate case, the (finite) line of inflections is one of the asymptotes of the vertex curve and so the vertex curve and the inflection curve intersect at the point at infinity on this line which is $c_m(0) = (-\dot{a}_2(0), \dot{a}_1(0), 0)$. If $\dddot{\theta}(\dot{\mathbf{a}} \times \ddot{\mathbf{a}}) - \ddot{\theta}(\dot{\mathbf{a}} \times \dddot{\mathbf{a}}) = 0$, they also intersect on the whole of the degenerate inflection curve and every point on the line given by equation (3.7) is a Ball's point. If $\dddot{\theta}(\dot{\mathbf{a}} \times \ddot{\mathbf{a}}) - \ddot{\theta}(\dot{\mathbf{a}} \times \dddot{\mathbf{a}}) \neq 0$, then there are no other intersection points of the vertex curve and the inflection curve and there is no Ball's point.

When $\dot{\theta}(0) = \ddot{\theta}(0) = 0$, equation (3.11) becomes

$$
x_3^3 \|\dot{\mathbf{a}}\|^2 \left(\dddot{\theta} \dot{a}_1 x_1 + \dddot{\theta} \dot{a}_2 x_2 + (\dot{\mathbf{a}} \times \dddot{\mathbf{a}}) x_3 \right) = 3 x_3^4 (\dot{\mathbf{a}} \times \ddot{\mathbf{a}})(\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}})
$$

so the vertex curve further degenerates to the line at infinity twice and the line (in cartesian coordinates)

$$
\dddot{\theta} \|\dot{\mathbf{a}}\|^2 \left(x_1 \dot{a}_1 + x_2 \dot{a}_2 \right) = 3 \left(\dot{\mathbf{a}} \times \ddot{\mathbf{a}} \right) \left(\dot{\mathbf{a}} \cdot \ddot{\mathbf{a}} \right) - \|\dot{\mathbf{a}}\|^2 \left(\dot{\mathbf{a}} \times \dddot{\mathbf{a}} \right) \tag{3.13}
$$

This line intersects the inflection curve (the line at infinity) at $c_m(0)$ and is, in fact, tangent to the moving centrode at 0 in $\mathbb{P}^2$.

3.5 Examples

Example 1
For an example of the non–degenerate case, we can take the motion which is a rotation about the origin of angle t followed by a translation by vector (t^2, t^3). Here $\dot{\theta}(t) = 1$ always, so $c_m(t)$ is uniquely defined for all $t \in \mathbb{R}$:

$$c_m(t) = (2t \sin t - 3t^2 \cos t, 2t \cos t + 3t^2 \sin t)$$

Substituting values into equation (3.6), we get the inflection curve of the motion at $t = 0$ as

$$x_1^2 + x_2^2 - 2x_1 = 0$$

which is the circle centred at $(1, 0)$ with radius 1.

Similarly from equation (3.9) the vertex curve at $t = 0$ is

$$4x_1x_2 = 0$$

which is the axes $x_1 = 0$ and $x_2 = 0$.

The vertex curve and the inflection circle intersect twice at the origin ($c_m(0)$) and once at the Ball's point $(2, 0)$. At $t = 0$ the centrode has vertical direction, so the vertex curve passes through $c_m(0)$ once tangent to the centrode (along $x_1 = 0$) and once normal to the centrode (along $x_2 = 0$) as expected, see fig. 3.1.

This example is computationally easy but exhibits the unexpected degeneracy of the vertex curve to a pair of lines.

Example 2
A second non–degenerate example is the motion given by a rotation about the origin of angle $\theta(t) = t - \frac{1}{2}t^2$ followed by translation by vector $\mathbf{a}(t) = (3t^2 - 6t^3, 4t^2 - t^3)$. In this case $\dot{\theta}(t) = 1 - t$, so $c_m(t)$ is uniquely defined for all $t \neq 1$, in particular it is uniquely defined for $t = 0$. From equation (3.5), the inflection curve of the motion at $t = 0$ is

$$x_1^2 + x_2^2 - 6x_1 - 8x_2 = 0$$

which is the circle centered at $(3, 4)$ with radius 5. From equation (3.8) the vertex curve at $t = 0$ is

$$2x_1(x_1^2 + x_2^2) - (8x_1 - 6x_2)(6x_1 + 8x_2) = 0$$

which is an irreducible cubic.

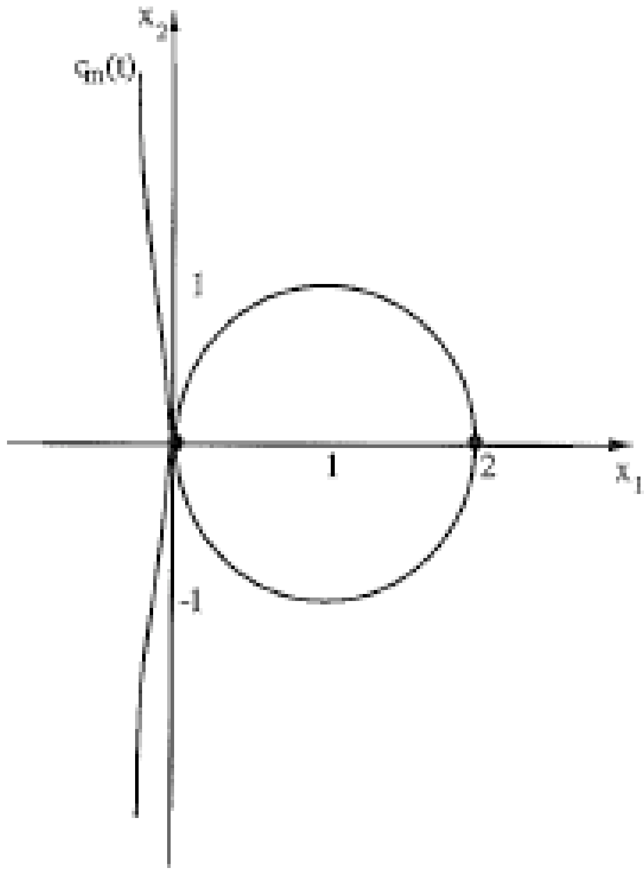

Figure 3.1: The inflection circle, vertex curve and centrode for the motion of the first example, showing the points of intersection of the vertex and inflection curves—the origin and Ball's point at $(2, 0)$

The vertex curve passes through the origin ($c_m(0)$) twice, once tangent to and once normal to the centrode and it intersects with the inflection circle there and at the Ball's point $(7, 7)$. This is illustrated in fig. 3.2.

Example 3

An example of the degeneracies that occur when $\dot{\theta}(0) = 0$ and $\ddot{\theta}(0) \neq 0$ can be seen by the motion given by a rotation about the origin of angle $\theta(t) = t^2 + t^3$ followed by a translation of $\mathbf{a}(t) = (t + t^3, t + t^2)$. Here $\dot{\theta}(t) = 2t + 3t^2$ which is zero when $t = 0$ and

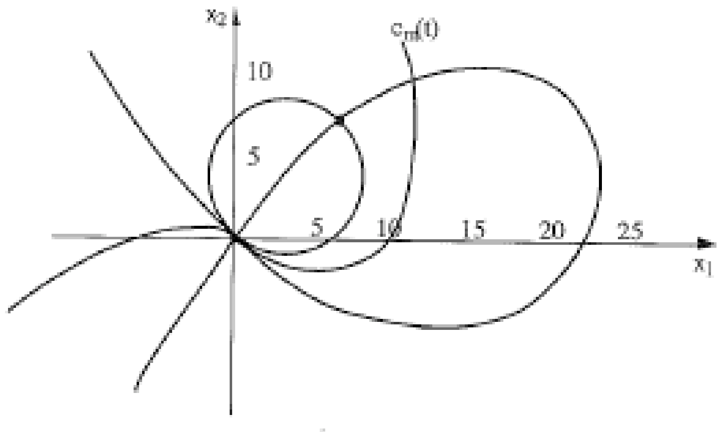

Figure 3.2: The inflection circle, vertex curve and centrode (near $c_m(0)$) for the second exampleof a planar motion

$t = -\frac{2}{3}$. Apart from these two cases, $c_m(t)$ is uniquely defined as

$$c_m(t) = \frac{1}{2t + 3t^2} \begin{pmatrix} (1+3t^2)\sin(t^2+t^3) - (1+2t)\cos(t^2+t^3) \\ (1+3t^2)\cos(t^2+t^3) + (1+2t)\sin(t^2+t^3) \end{pmatrix}$$

$c_m(0)$ is the point at infinity $(-1, 1, 0)$ and at $t = 0$ $c_m(t)$ has gradient -1.
The inflection curve at $t = 0$ is, from equation (3.6), given by

$$x_1 + x_2 = -1$$

which is tangent to the centrode at $t = 0$.
The vertex curve, from equation (3.12), is the hyperbola

$$x_1^2 - x_2^2 + x_1 - x_2 + 2 = 0$$

which has the inflection curve as one of its asymptotes. Indeed the inflection curve and the vertex curve intersect at $c_m(0)$ and as $\dddot{\theta}(\dot{\mathbf{a}} \times \ddot{\mathbf{a}}) - \ddot{\theta}(\dot{\mathbf{a}} \times \dddot{\mathbf{a}}) = 6(2) - 2(-6) = 24 \neq 0$ they do not intersect anywhere else and there is no Ball's point. This example is illustrated in fig. 3.3.

Example 4

If we take the motion given by $\theta(t) = t^3$ and $\mathbf{a}(t) = (t, t + t^2 + t^3)$ we have $\dot{\theta}(0) = \ddot{\theta}(0) = 0$ and from equation (3.7) the inflection curve is

$$2x_3 = 0$$

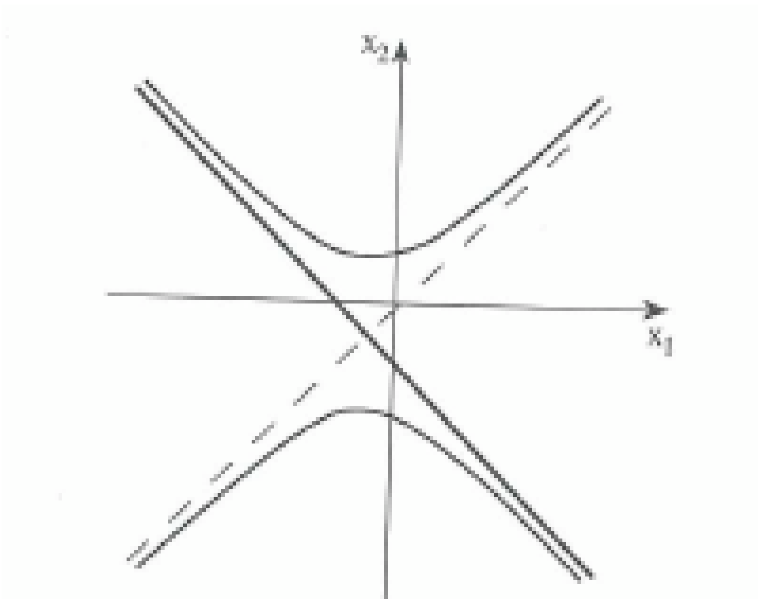

Figure 3.3: In the first degenerate case the vertex curve is a hyperbola and the inflection curve is a line which is one of the asymptotes of the vertex curve. The centrode at $t = 0$ is the point at infinity where the vertex curve and the inflection curve meet

the line at infinity.

From equation (3.13) the vertex curve at $t = 0$ is the line

$$x_1 + x_2 = 0$$

which is tangent to the centrode at $c_m(0)$ which is the point at infinity $(-1, 1, 0)$ and the vertex curve and the inflection curve intersect at $c_m(0)$.

On the other hand, if we take the motion given by $\theta(t) = t^3$ and $\mathbf{a}(t) = (t, t + t^3)$, then the inflection curve is given by

$$0x_3 = 0$$

So the inflection curve is the whole of the projective plane (excluding the point at infinity $c_m(0) = (-1, 1, 0)$).

The vertex curve is the line

$$x_1 + x_2 = -1$$

which again is tangent to the centrode at $c_m(0)$. In this case, however, the inflection curve and the vertex curve intersect on the whole of the vertex curve, so there is a line of Ball's points for this motion.

4 Spatial Motion

4.1 Introduction

This section deals with the geometry of the trajectories of 3–dimensional rigid–body motions. In particular, we will be looking at the zero–curvature curve and the zero–torsion surface and the ways in which they degenerate under certain characteristics of the motion.

By analogy with the planar case, the trajectories of a 3–dimensional rigid–body motion, $(A(t), \mathbf{a}(t))$, form a 3–parameter family of functions $\Phi_{\mathbf{x}} : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\Phi_{\mathbf{x}}(t) = A(t)\mathbf{x} + \mathbf{a}(t)$$

with $A(t) \in SO(3)$ and $\mathbf{a}(t)$ a vector in $\mathbb{R}^3$. As $A(t)$ is an orthogonal matrix, we have

$$A(t)A(t)^T = I.$$

Differentiating this gives

$$\dot{A}(t)A(t)^T + A(t)\dot{A}(t)^T = 0$$

which tells us that $\dot{A}(t)A(t)^T$ is a 3×3 skew–symmetric matrix and as such has even rank, i.e. rank 0 or 2, see [7], [9]. As $A(t)^T$ is a non–singular matrix, this means that $\dot{A}(t)$ is singular.

4.2 Geometric Invariants

In direct analogy to the planar case in which an instantaneous centre is the unique instantaneous stationary point we may wish to consider the locus of stationary points in the spatial case. However, in general we do not get stationary points as, for any $t_0 \in \mathbb{R}$, $\dot{A}(t_0)$ is a singular matrix and so $\dot{A}(t_0)\mathbf{x} + \dot{\mathbf{a}}(t_0) = 0$ does not usually have a solution. Hence the best we can do for the spatial case, following the approach of Veldkamp in [13], is look for points having a velocity vector with minimal norm. So we wish to minimise $\|\dot{\Phi}_{\mathbf{x}}(t_0)\|^2$ with respect to $\mathbf{x}$.

To do this we first require $\partial\left(\dot{\Phi}_{\mathbf{x}}(t_0)\cdot\dot{\Phi}_{\mathbf{x}}(t_0)\right)/\partial\mathbf{x}=0$. We have

$$\begin{aligned}\frac{\partial\left(\dot{\Phi}_{\mathbf{x}}(t_0)\cdot\dot{\Phi}_{\mathbf{x}}(t_0)\right)}{\partial x_i} &= \frac{\partial\left((\dot{A}(t_0)\mathbf{x}+\dot{\mathbf{a}}(t_0))\cdot(\dot{A}(t_0)\mathbf{x}+\dot{\mathbf{a}}(t_0))\right)}{\partial x_i}\\ &= 2\left(\frac{\partial\left(\dot{A}(t_0)\mathbf{x}\right)}{\partial x_i}\right)\cdot(\dot{A}(t_0)\mathbf{x}+\dot{\mathbf{a}}(t_0))\end{aligned}$$

Now $\partial\left(\dot{A}(t_0)\mathbf{x}\right)/\partial x_i$ is just the ith column of the matrix $\dot{A}(t_0)$, so the requirement is simply that $\dot{A}(t_0)^T(\dot{A}(t_0)\mathbf{x}+\dot{\mathbf{a}}(t_0))=0$. Let $\mathbf{e}^*$ be a unit vector for which $\dot{A}(t_0)^T\mathbf{e}^*=0$. Such an $\mathbf{e}^*$ exists because $\dot{A}(t_0)$ is a singular matrix as already stated. Provided $\dot{A}(t_0)\neq 0$ the rank of $\dot{A}(t_0)$ is 2 so nullity$\dot{A}(t_0)=1$ and $\mathbf{e}^*$ is unique up to sign. On the other hand, if $\dot{A}(t_0)=0$ then $\mathbf{e}^*\in S^2$ is arbitrary. Assuming $\dot{A}(t_0)\neq 0$, $\mathbf{x}$ must satisfy

$$\dot{A}(t_0)\mathbf{x}+\dot{\mathbf{a}}(t_0)=\mu\mathbf{e}^* \tag{4.1}$$

for some $\mu\in\mathbb{R}$. Taking the inner product of both sides of equation 4.1) with $\mathbf{e}^*$ we get

$$\mu=(\dot{A}(t_0)\mathbf{x}\cdot\mathbf{e}^*)+(\dot{\mathbf{a}}(t_0)\cdot\mathbf{e}^*)$$

Now the inner product of two real vectors is the same as multiplying the transpose of one vector by the other vector, so

$$\begin{aligned}\dot{A}(t_0)\mathbf{x}\cdot\mathbf{e}^* &= (\dot{A}(t_0)\mathbf{x})^T\mathbf{e}^*\\ &= (\mathbf{x}^T\dot{A}(t_0)^T)\mathbf{e}^*\\ &= \mathbf{x}^T(\dot{A}(t_0)^T\mathbf{e}^*)\\ &= \mathbf{x}\cdot\dot{A}(t_0)^T\mathbf{e}^*\\ &= 0.\end{aligned}$$

Thus the locus of points $\mathbf{x}$ having a velocity vector with stationary norm must satisfy

$$\dot{\Phi}_{\mathbf{x}}(t_0)=\dot{A}(t_0)\mathbf{x}+\dot{\mathbf{a}}(t_0)=(\dot{\mathbf{a}}(t_0)\cdot\mathbf{e}^*)\mathbf{e}^* \tag{4.2}$$

where $\mathbf{e}^*$ is a unit vector for which $\dot{A}(t_0)^T\mathbf{e}^*=0$.

It can be shown that the line defined by equation (4.2) is in fact precisely the locus of points with velocity vector having minimal norm. The velocity of a point on this line is $\mathbf{v} = (\dot{\mathbf{a}}(t_0)\cdot\mathbf{e}^*)\mathbf{e}^*$. The velocity of an arbitrary point is

$$\begin{aligned}\dot{A}(t_0)\mathbf{x} + \dot{\mathbf{a}}(t_0) &= \dot{A}(t_0)\mathbf{x} + \dot{\mathbf{a}}(t_0) + (\dot{\mathbf{a}}(t_0)\cdot\mathbf{e}^*)\mathbf{e}^* - (\dot{\mathbf{a}}(t_0)\cdot\mathbf{e}^*)\mathbf{e}^* \\ &= \mathbf{v} + \dot{A}(t_0)\mathbf{x} + \dot{\mathbf{a}}(t_0) - (\dot{\mathbf{a}}(t_0)\cdot\mathbf{e}^*)\mathbf{e}^* \\ &= \mathbf{v} + \mathbf{w}, \text{ say.}\end{aligned}$$

We have

$$\begin{aligned}\mathbf{w}\cdot\mathbf{e}^* = (\dot{A}\mathbf{x} + \dot{\mathbf{a}} - (\dot{\mathbf{a}}\cdot\mathbf{e}^*)\mathbf{e}^*)\cdot\mathbf{e}^* &= (\dot{A}\mathbf{x}\cdot\mathbf{e}^*) + (\dot{\mathbf{a}}\cdot\mathbf{e}^*) - (\dot{\mathbf{a}}\cdot\mathbf{e}^*) \\ &= (\dot{A}\mathbf{x}\cdot\mathbf{e}^*). \end{aligned} \tag{4.3}$$

We have seen before that $(\dot{A}\mathbf{x}\cdot\mathbf{e}^*) = 0$ and substituting this into equation (4.3) tells us that the velocity of an arbitrary point is the sum of two orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$. So the norm of the velocity vector of an arbitrary point satisfies

$$\|\dot{\Phi}_{\mathbf{x}}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 \geq \|\mathbf{v}\|^2$$

with equality if and only if $\mathbf{w} = 0$, i.e. $\mathbf{x}$ is on the line in E^3 defined by equation (4.2).

If $\dot{A}(t_0) = 0$ the velocity is $\dot{\mathbf{a}}(t_0)$ for all $\mathbf{x} \in \mathbb{R}^3$, so all points have a velocity vector with minimal norm.

Instantaneously, the motion can be thought of as a screw motion whose axis is this line, called the *instantaneous screw axis*, or *I.S.A.*

As t varies, the I.S.A. also varies, generating a ruled surface called the *moving axode* of the motion.

Suppose $\mu(t) = (A(t), \mathbf{a}(t))$ is a smooth regular rigid–body motion with Taylor expansion (at $t_0 = 0$) as in Section 1.5:

$$\begin{aligned}A(t) &= I + B_1 t + \tfrac{1}{2}(B_1^2 + B_2)t^2 + \tfrac{1}{6}(\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)t^3 + \ldots \\ \mathbf{a}(t) &= \mathbf{b}_1 t + \tfrac{1}{2}\mathbf{b}_2 t^2 + \tfrac{1}{6}\mathbf{b}_3 t^3 + \ldots\end{aligned}$$

Let

$$B_1 = \begin{pmatrix} 0 & -w_1 & v_1 \\ w_1 & 0 & -u_1 \\ -v_1 & u_1 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix}.$$

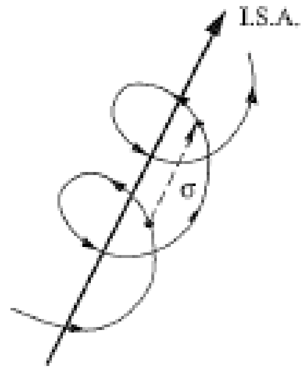

Figure 4.1: The pitch σ is the distance travelled parallel to the I.S.A. in one revolution about it

Then, provided $B_1 \neq 0$, the *pitch* of the motion at $t = 0$ is defined by

$$\sigma(0) = \frac{(u_1, v_1, w_1) \cdot (b_{11}, b_{12}, b_{13})}{(u_1, v_1, w_1) \cdot (u_1, v_1, w_1)}.$$

If $B_1 = 0$ (and $\mathbf{b}_1 \neq 0$) we define the pitch to be *infinite*. Intuitively the pitch can be thought of as the distance travelled parallel to the I.S.A. in one revolution about it, see fig. 4.1. It is invariant under a change of coordinates as is the existence of the I.S.A.

4.3 Normal Forms

Using the change of coordinates $(G, \mathbf{g})$ described in Section 1.5 and given by equation (1.1), we can reduce the matrices B_i and the vectors $\mathbf{b}_i$ that determine the motion up to third order, to simpler 'normal' forms. The particular normal form that a motion has depends on whether or not certain quantities vanish. Accordingly, motions are grouped under nine classifications of increasing degree of 'degeneracy'.

Class I:	$B_1 \neq 0$, $B_2 \neq 0$, $B_2 \neq \alpha B_1$ for any $\alpha \in \mathbb{R}$
Class IIa:	$B_1 \neq 0$, $B_2 = \alpha B_1$ for some $\alpha \in \mathbb{R} - \{0\}$ and $B_3 \neq \beta B_1$ for any $\beta \in \mathbb{R}$
Class IIb:	$B_1 \neq 0$, $B_2 = \alpha B_1$ for some $\alpha \in \mathbb{R} - \{0\}$ and $B_3 = \beta B_1$ for some $\beta \in \mathbb{R}$
Class IIIa:	$B_1 \neq 0$, $B_2 = 0$ and $B_3 \neq \beta B_1$ for any $\beta \in \mathbb{R}$
Class IIIb:	$B_1 \neq 0$, $B_2 = 0$ and $B_3 = \beta B_1$ for some $\beta \in \mathbb{R}$
Class IVa:	$B_1 = 0$, $B_2 \neq 0$ and $B_3 \neq \gamma B_2$ for any $\gamma \in \mathbb{R}$
Class IVb:	$B_1 = 0$, $B_2 \neq 0$ and $B_3 = \gamma B_2$ for some $\gamma \in \mathbb{R}$
Class Va:	$B_1 = 0$, $B_2 = 0$ and $B_3 \neq 0$
Class Vb:	$B_1 = 0$, $B_2 = 0$ and $B_3 = 0$.

Within each class the zero–curvature curve and zero–torsion surface of a motion have common geometries. Moreover the normal forms allow us to describe them in terms of a reduced set of parameters.

Class I: $\mathbf{B_1 \neq 0}$, $\mathbf{B_2 \neq 0}$, $\mathbf{B_2 \neq \alpha\, B_1}$ for any $\boldsymbol{\alpha} \in \mathbb{R}$.

Theorem 4.1. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class I can be reduced to*

$$B_1 = \begin{pmatrix} 0 & -\rho_1 & 0 \\ \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^{*} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & 0 \\ -\rho_2 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & -\rho_4 \\ -\rho_5 & \rho_4 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. For motions in class I B_1 can be reduced to the normal form

$$\begin{pmatrix} 0 & -\rho_1 & 0 \\ \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.4}$$

via a change of coordinates $(G_1, \mathbf{g}_1)$ where

$$G_1 = \begin{pmatrix} \frac{v_1}{\sqrt{u_1^2+v_1^2}} & \frac{-u_1}{\sqrt{u_1^2+v_1^2}} & 0 \\ \frac{u_1 w_1}{\sqrt{(u_1^2+v_1^2)(u_1^2+v_1^2+w_1^2)}} & \frac{v_1 w_1}{\sqrt{(u_1^2+v_1^2)(u_1^2+v_1^2+w_1^2)}} & \frac{-\sqrt{u_1^2+v_1^2}}{\sqrt{u_1^2+v_1^2+w_1^2}} \\ \frac{-u_1}{\sqrt{u_1^2+v_1^2+w_1^2}} & \frac{-v_1}{\sqrt{u_1^2+v_1^2+w_1^2}} & \frac{-w_1}{\sqrt{u_1^2+v_1^2+w_1^2}} \end{pmatrix} \quad (4.5)$$

$G_1^{-1} = G_1^T$, $\rho_1 = \sqrt{u_1^2+v_1^2+w_1^2} \neq 0$ and $\mathbf{g}_1$ is an arbitrary vector in $\mathbb{R}^3$.
Now, under this change of coordinates, the 3×3 skew–symmetric matrix

$$B_2 = \begin{pmatrix} 0 & -w_2 & v_2 \\ w_2 & 0 & -u_2 \\ -v_2 & u_2 & 0 \end{pmatrix}$$

is transformed to

$$G_1 B_2 G_1^T =$$
$$\begin{pmatrix} 0 & \frac{-(u_1u_2+v_1v_2+w_1w_2)}{\sqrt{u_1^2+v_1^2+w_1^2}} & \frac{-(u_1u_2+v_1v_2)w_1+(u_1^2+v_1^2)w_2}{\sqrt{(u_1^2+v_1^2)(u_1^2+v_1^2+w_1^2)}} \\ \frac{u_1u_2+v_1v_2+w_1w_2}{\sqrt{u_1^2+v_1^2+w_1^2}} & 0 & \frac{u_2v_1-u_1v_2}{\sqrt{u_1^2+v_1^2}} \\ \frac{(u_1u_2+v_1v_2)w_1-(u_1^2+v_1^2)w_2}{\sqrt{(u_1^2+v_1^2)(u_1^2+v_1^2+w_1^2)}} & \frac{-u_2v_1+u_1v_2}{\sqrt{u_1^2+v_1^2}} & 0 \end{pmatrix}$$

We now want to reduce this matrix to a simpler form without affecting the reduced B_1.
If we apply the change of coordinates $(G_2, \mathbf{g}_2)$ with $\mathbf{g}_2$ an arbitrary vector in $\mathbb{R}^3$ and

$$G_2 = \begin{pmatrix} \frac{(u_1u_2+v_1v_2)w_1-(u_1^2+v_1^2)w_2}{k} & \frac{\sqrt{u_1^2+v_1^2+w_1^2}\,(-u_2v_1+u_1v_2)}{k} & 0 \\ \frac{-\sqrt{u_1^2+v_1^2+w_1^2}\,(-u_2v_1+u_1v_2)}{k} & \frac{(u_1u_2+v_1v_2)w_1-(u_1^2+v_1^2)w_2}{k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.6)$$

where

$$k = \sqrt{((u_1u_2 + v_1v_2)w_1 - (u_1^2 + v_1^2)w_2)^2 + (u_1^2 + v_1^2 + w_1^2)(-u_2v_1 + u_1v_2)^2},$$

we can reduce B_2 as follows:

$$G_2G_1B_2G_1^TG_2^T = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & 0 \\ -\rho_2 & 0 & 0 \end{pmatrix} \tag{4.7}$$

where

$$\rho_2 = -\frac{k}{\sqrt{(u_1^2 + v_1^2)(u_1^2 + v_1^2 + w_1^2)}}$$

$$\rho_3 = \frac{u_1u_2 + v_1v_2 + w_1w_2}{\sqrt{u_1^2 + v_1^2 + w_1^2}}.$$

This is only possible if $k \neq 0$, equivalently $B_2 \neq \alpha B_1$ ($\alpha \in \mathbb{R} - \{0\}$). Note that $\rho_2 \neq 0$ but $\rho_3 = 0$ is possible. So the resultant change of coordinates that has been performed is given by

$$(G, \mathbf{g}) = (G_2, \mathbf{g}_2)(G_1, \mathbf{g}_1) = (G_2G_1, \mathbf{g}_2 + G_2\mathbf{g}_1)$$

where $\mathbf{g}_1$ and $\mathbf{g}_2$ are, at present, arbitrary vectors in $\mathbb{R}^3$.

We now want to make suitable choices for $\mathbf{g}_1$ and $\mathbf{g}_2$ in order to simplify $\mathbf{b}_1$ and $\mathbf{b}_2$ under the coordinate transformation $(G, \mathbf{g})$.

As in section 1, under the coordinate change $(G, \mathbf{g})$, the vector $\mathbf{b}_1$ gets mapped to $-GB_1G^{-1}\mathbf{g} + G\mathbf{b}_1$. If $\mathbf{g} = (g_1, g_2, g_3)^T$ and $\mathbf{b}_1 = (b_{11}, b_{12}, b_{13})^T$ then

$$\begin{aligned} \mathbf{b}_1 &\mapsto -GB_1G^{-1}\mathbf{g} + G\mathbf{b}_1 \\ &= \begin{pmatrix} 0 & \rho_1 & 0 \\ -\rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} + \begin{pmatrix} b_{11}^* \\ b_{12}^* \\ b_{13}^* \end{pmatrix} \\ &= \begin{pmatrix} \rho_1 g_2 + b_{11}^* \\ -\rho_1 g_1 + b_{12}^* \\ b_{13}^* \end{pmatrix} \end{aligned}$$

(where $\mathbf{b}_1^* = G\mathbf{b}_1$). As $\rho_1 \neq 0$ we can choose

$$g_1 = \frac{b_{12}^*}{\rho_1}, \quad g_2 = -\frac{b_{11}^*}{\rho_1} \tag{4.8}$$

so that $\mathbf{b}_1 \mapsto (0,0,b_{13}^*)^T$. Note that g_3 is still arbitrary.

So up to first order the motion is determined by two invariants: ρ_1 and b_{13}^*. The pitch of the motion is now given simply by the ratio of the two invariants: $\sigma = b_{13}^*/\rho_1$. In particular, the pitch of the motion is 0 if $b_{13}^* = 0$.

With respect to the new coordinates the unit vector $\mathbf{e}^*$ for which $\dot{A}(0)^T\mathbf{e}^* = B_1\mathbf{e}^* = 0$ is $(0,0,1)^T$, so, from equation (4.2), the I.S.A. consists of all $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\begin{pmatrix} -\rho_1 x_2 \\ \rho_1 x_1 \\ b_{13}^* \end{pmatrix} = b_{13}^* \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence the I.S.A. of the motion is the x_3–axis.

Consider what happens under the preceding coordinate transformations to $\mathbf{b}_2$. If $\mathbf{g} = (b_{12}^*/\rho_1, -b_{11}^*/\rho_1, g_3)^T$ and $\mathbf{b}_2 = (b_{21}, b_{22}, b_{23})^T$ then

$$\begin{aligned} \mathbf{b}_2 &\mapsto -G(B_1^2 + B_2)G^{-1}\mathbf{g} + G\mathbf{b}_2 \\ &= \begin{pmatrix} \rho_1^2 & \rho_3 & -\rho_2 \\ -\rho_3 & \rho_1^2 & 0 \\ \rho_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{12}^*/\rho_1 \\ -b_{11}^*/\rho_1 \\ g_3 \end{pmatrix} + \begin{pmatrix} b_{21}^* \\ b_{22}^* \\ b_{23}^* \end{pmatrix} \\ &= \begin{pmatrix} \rho_1 b_{12}^* - \frac{\rho_3}{\rho_1} b_{11}^* - \rho_2 g_3 + b_{21}^* \\ -\frac{\rho_3}{\rho_1} b_{12}^* - \rho_1 b_{11}^* + b_{22}^* \\ \frac{\rho_2}{\rho_1} b_{12}^* + b_{23}^* \end{pmatrix} \end{aligned}$$

As $\rho_1, \rho_2 \neq 0$ we can choose

$$g_3 = \frac{\rho_1^2 b_{12}^* - \rho_3 b_{11}^* + \rho_1 b_{21}^*}{\rho_1 \rho_2} \tag{4.9}$$

so that

$$\mathbf{b}_2 \mapsto \begin{pmatrix} 0 \\ -\frac{\rho_3}{\rho_1} b_{12}^* - \rho_1 b_{11}^* + b_{22}^* \\ \frac{\rho_2}{\rho_1} b_{12}^* + b_{23}^* \end{pmatrix} = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}.$$

Thus, up to second order the motion is determined by four further parameters ρ_2, ρ_3, b_{22}^{**} and b_{23}^{**}.

The skew–symmetric matrix B_3 will also be affected by the coordinate transformation, but it cannot be reduced without affecting the reduced forms of B_1 and B_2. If B_3 has the form

$$B_3 = \begin{pmatrix} 0 & -w_3 & v_3 \\ w_3 & 0 & -u_3 \\ -v_3 & u_3 & 0 \end{pmatrix}$$

under the coordinate transformation it will have the form

$$\begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & -\rho_4 \\ -\rho_5 & \rho_4 & 0 \end{pmatrix} \tag{4.10}$$

where

$$\begin{aligned} \rho_4 &= \frac{\sqrt{u_1^2+v_1^2}\,(u_1(v_2w_3-v_3w_2)+v_1(u_3w_2-u_2w_3)+w_1(u_2v_3-u_3v_2))}{k}, \\ \rho_5 &= \frac{\sqrt{u_1^2+v_1^2}\,\Big(-u_1^2(v_2v_3+w_2w_3)-v_1^2(u_2u_3+w_2w_3)-w_1^2(u_2u_3+v_2v_3) \\ \qquad +u_1v_1(u_2v_3+u_3v_2)+u_1w_1(u_2w_3+u_3w_2)+v_1w_1(v_2w_3+v_3w_2)\Big)}{k\sqrt{u_1^2+v_1^2+w_1^2}}, \\ \rho_6 &= \frac{u_1u_3+v_1v_3+w_1w_3}{\sqrt{u_1^2+v_1^2+w_1^2}}. \end{aligned} \tag{4.11}$$

Similarly $\mathbf{b}_3$ cannot be reduced and will be transformed to

$$\begin{aligned} &-G(\tfrac{3}{2}(B_1B_2+B_2B_1)+B_3)G^{-1}\mathbf{g}+G\mathbf{b}_3 = \\ &\begin{pmatrix} \frac{(\rho_3\rho_5-\rho_6)}{\rho_1}b_{11}+(3\rho_3-\rho_1\rho_5)b_{12}-\frac{\rho_5}{\rho_2}b_{21}+b_{31}^* \\ \frac{\rho_3(\frac{3}{2}\rho_1\rho_2-3\rho_1-\rho_4)}{\rho_1}b_{11}+\frac{(\rho_1^2\rho_4-\frac{3}{2}\rho_1^3\rho_2-\rho_6)}{\rho_1}b_{12}+\frac{(\rho_4-\frac{3}{2}\rho_1\rho_2)}{\rho_2}b_{21}+b_{32}^* \\ \frac{(\frac{3}{2}\rho_1\rho_2+\rho_4)}{\rho_1}b_{11}+\frac{\rho_5}{\rho_1}b_{12}+b_{33}^* \end{pmatrix} \\ &= \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}. \end{aligned}$$

This completes the derivation of the normal form for motions in class I. □

The geometry of the point–paths of the motions $\Phi_{\mathbf{x}}(t) = A(t)\mathbf{x} + \mathbf{a}(t)$ which we are interested in depends on the first three derivatives at $t = 0$. In class I under the change of coordinates $(G, \mathbf{g})$ these are now:

$$\begin{aligned}
\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\
&= B_1\mathbf{x} + \mathbf{b}_1 \\
&\mapsto \begin{pmatrix} -\rho_1 x_2 \\ \rho_1 x_1 \\ b_{13}^* \end{pmatrix} \\
\ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\
&= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\
&\mapsto \begin{pmatrix} -\rho_1^2 x_1 - \rho_3 x_2 + \rho_2 x_3 \\ \rho_3 x_1 - \rho_1^2 x_2 + b_{22}^{**} \\ -\rho_2 x_1 + b_{23}^{**} \end{pmatrix} \\
\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\
&= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\
&\mapsto \begin{pmatrix} -3\rho_1\rho_3 x_1 - \rho_6 x_2 + \rho_5 x_3 + b_{31}^{**} \\ \rho_6 x_1 - 3\rho_1\rho_3 x_2 + (\tfrac{3}{2}\rho_1\rho_2 - \rho_4)x_3 + b_{32}^{**} \\ -\rho_5 x_1 + (\tfrac{3}{2}\rho_1\rho_2 + \rho_4)x_2 + b_{33}^{**} \end{pmatrix}.
\end{aligned} \tag{4.12}$$

Class IIa: $\mathbf{B_1 \neq 0}$, $\mathbf{B_2 = \alpha B_1}$ for some $\boldsymbol{\alpha} \in \mathbb{R} - \{\mathbf{0}\}$ and $\mathbf{B_3 \neq \boldsymbol{\beta}\, B_1}$ for any $\boldsymbol{\beta} \in \mathbb{R}$.

Theorem 4.2. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class IIa can be reduced to*

$$B_1 = \begin{pmatrix} 0 & -\rho_1 & 0 \\ \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & -\rho_3 & 0 \\ \rho_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & -\rho_4 \\ -\rho_5 & \rho_4 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. For motions in this class, under the coordinate transformation $(G_1, \mathbf{g}_1)$ where G_1 is as given by (4.5), B_1 has the same normal form as for class I and B_2 has the normal form

$$\begin{pmatrix} 0 & -\rho_3 & 0 \\ \rho_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\rho_3 = \sqrt{u_2^2 + v_2^2 + w_2^2} = \alpha\rho_1$.

We can choose G_2 and $\mathbf{g}$ so that under the coordinate transformation $(G_2G_1, \mathbf{g})$ the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ have the same normal forms as for class I.

Let G_2 be the orthogonal matrix

$$\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned}
a &= \frac{l}{\sqrt{l^2 + m^2}} \\
b &= \frac{m}{\sqrt{l^2 + m^2}} \\
l &= w_1(u_1b_{21} + v_1b_{22}) - (u_1^2 + v_1^2)b_{23} - \alpha(w_1(u_1b_{11} + v_1b_{12}) - (u_1^2 + v_1^2)b_{13}) \\
&\quad - (u_1^2 + v_1^2 + w_1^2)(v_1b_{11} - u_1b_{12}) \\
m &= \sqrt{u_1^2 + v_1^2 + w_1^2}\,\big(w_1(u_1b_{11} + v_1b_{12}) - (u_1^2 + v_1^2)b_{13} \\
&\qquad + \alpha(u_1b_{12} - v_1b_{11}) - u_1b_{22} + v_1b_{21})
\end{aligned}$$

and let $\mathbf{g} = (g_1, g_2, g_3)^T$ where

$$\begin{aligned}
g_1 &= \frac{a(u_1w_1b_{11} + v_1w_1b_{12} - (u_1^2 + v_1^2)b_{13}) + b\sqrt{u_1^2 + v_1^2 + w_1^2}(v_1b_{11} - u_1b_{12})}{\sqrt{u_1^2 + v_1^2}(u_1^2 + v_1^2 + w_1^2)} \\
g_2 &= \frac{a\sqrt{u_1^2 + v_1^2 + w_1^2}(u_1b_{12} - v_1b_{11}) + b(u_1w_1b_{11} + v_1w_1b_{12} - (u_1^2 + v_1^2)b_{13})}{\sqrt{u_1^2 + v_1^2}(u_1^2 + v_1^2 + w_1^2)}.
\end{aligned}$$

Under this coordinate transformation

$$\mathbf{b}_1 \mapsto \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$
$$\mathbf{b}_2 \begin{pmatrix} \rho_1^2 g_1 + \rho_3 g_2 + b_{21}^* \\ -\rho_3 g_1 + \rho_1^2 g_2 + b_{22}^* \\ b_{23}^* \end{pmatrix} = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}$$

(where $\mathbf{b}_i^* = G\mathbf{b}_i$).

Once again the motion is determined up to first order by the two invariants b_{13}^* and ρ_1 and the pitch of the motion is the ratio of these two invariants. In the new coordinates the I.S.A. of the motion is the x_3–axis.

As in class I, B_3 cannot be reduced and has the normal form given by (4.10) where

$$\begin{aligned}
\rho_4 &= \frac{a(u_1 v_3 - v_1 u_3)}{\sqrt{u_1^2 + v_1^2}} + \frac{b(u_1(w_1 u_3 - u_1 w_3) + v_1(w_1 v_3 - v_1 w_3))}{\sqrt{(u_1^2 + v_1^2)(u_1^2 + v_1^2 + w_1^2)}} \\
\rho_5 &= \frac{a(u_1(u_1 w_3 - w_1 u_3) + v_1(v_1 w_3 - w_1 v_3))}{\sqrt{(u_1^2 + v_1^2)(u_1^2 + v_1^2 + w_1^2)}} + \frac{b(u_1 v_3 - v_1 u_3)}{\sqrt{u_1^2 + v_1^2}} \\
\rho_6 &= \frac{u_1 u_3 + v_1 v_3 + w_1 w_3}{\sqrt{u_1^2 + v_1^2 + w_1^2}}.
\end{aligned} \tag{4.13}$$

The third coordinate of $\mathbf{g}$ is still arbitrary, so we can use this to reduce $\mathbf{b}_3$. If $\rho_4 \neq 0$, choose $g_3 = (\rho_6 g_1 - 3\rho_1\rho_3 g_2 - b_{32}^*)/\rho_4$ so that $\mathbf{b}_3 \mapsto (b_{31}^{**}, 0, b_{33}^{**})$. If $\rho_4 = 0$ and $\rho_5 \neq 0$, choose $g_3 = (3\rho_1\rho_3 g_1 + \rho_6 g_2 + b_{31}^*)/\rho_5$ so that $\mathbf{b}_3 \mapsto (0, b_{32}^{**}, b_{33}^{**})^T$. If both ρ_4 and ρ_5 are zero then we are in class IIb and $\mathbf{b}_3$ cannot be reduced (see below). □

Thus for motions in class IIa, the first three derivatives of the point–paths of the motion at $t = 0$ are given by

$$\begin{aligned}
\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\
&= B_1\mathbf{x} + \mathbf{b}_1 \\
&\mapsto \begin{pmatrix} -\rho_1 x_2 \\ \rho_1 x_1 \\ b_{13}^* \end{pmatrix} \\
\ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\
&= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\
&\mapsto \begin{pmatrix} -\rho_1^2 x_1 - \rho_3 x_2 \\ \rho_3 x_1 - \rho_1^2 x_2 + b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\
&= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\
&\mapsto \begin{pmatrix} -3\rho_1\rho_3 x_1 - \rho_6 x_2 + \rho_5 x_3 + b_{31}^{**} \\ \rho_6 x_1 - 3\rho_1\rho_3 x_2 - \rho_4 x_3 \\ -\rho_5 x_1 + \rho_4 x_2 + b_{33}^{**} \end{pmatrix} \quad \text{if } \rho_4 \neq 0 \\
&\mapsto \begin{pmatrix} -3\rho_1\rho_3 x_1 - \rho_6 x_2 + \rho_5 x_3 \\ \rho_6 x_1 - 3\rho_1\rho_3 x_2 + b_{32}^{**} \\ -\rho_5 x_1 + b_{33}^{**} \end{pmatrix} \quad \text{if } \rho_4 = 0
\end{aligned} \tag{4.14}$$

Class IIb: $\mathbf{B_1 \neq 0}$, $\mathbf{B_2 = \boldsymbol{\alpha} B_1}$ for some $\boldsymbol{\alpha} \in \mathbb{R} - \{0\}$ and $\mathbf{B_3 = \boldsymbol{\beta} B_1}$ for some $\boldsymbol{\beta} \in \mathbb{R}$.

Theorem 4.3. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class IIb can be reduced to*

$$B_1 = \begin{pmatrix} 0 & -\rho_1 & 0 \\ \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & -\rho_3 & 0 \\ \rho_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & 0 \\ \rho_6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. In this class, if we use exactly the same coordinate transformation as in class IIa, the normal forms for B_1, $\mathbf{b}_1$, B_2 and $\mathbf{b}_2$ will be the same as for class IIa. The normal form for B_3 will be as given by (4.10) with $\rho_4 = \rho_5 = 0$ and $\rho_6 = \sqrt{u_3^2 + v_3^2 + w_3^2} = \beta\rho_1$. In this class $\mathbf{b}_3$ cannot be reduced. □

The pitch and the I.S.A. for motions in this class are exactly the same as for motions in class IIa and the first three derivatives of the point–paths of the motion at $t = 0$ are given by

$$\begin{aligned}
\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\
&= B_1\mathbf{x} + \mathbf{b}_1 \\
&\mapsto \begin{pmatrix} -\rho_1 x_2 \\ \rho_1 x_1 \\ b_{13}^* \end{pmatrix} \\
\ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\
&= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\
&\mapsto \begin{pmatrix} -\rho_1^2 x_1 - \rho_3 x_2 \\ \rho_3 x_1 - \rho_1^2 x_2 + b_{22}^{**} \\ b_{23}^{**} \end{pmatrix} \\
\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\
&= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\
&\mapsto \begin{pmatrix} -3\rho_1\rho_3 x_1 - \rho_6 x_2 + b_{31}^{**} \\ \rho_6 x_1 - 3\rho_1\rho_3 x_2 + b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}
\end{aligned} \tag{4.15}$$

Class IIIa: $\mathbf{B_1 \neq 0}$, $\mathbf{B_2 = 0}$ and $\mathbf{B_3 \neq \boldsymbol{\beta} B_1}$ for any $\boldsymbol{\beta} \in \mathbb{R}$.

Theorem 4.4. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class IIIa can be* *reduced to*

$$B_1 = \begin{pmatrix} 0 & -\rho_1 & 0 \\ \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & -\rho_4 \\ -\rho_5 & \rho_4 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. In this class B_2 is the zero matrix and we perform the coordinate transformation $(G, \mathbf{g})$ where $G = G_2 G_1$ with G_1 as given by (4.5),

$$G_2 = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} a &= \frac{l}{\sqrt{l^2 + m^2}} \\ b &= \frac{m}{\sqrt{l^2 + m^2}} \\ l &= w_1(u_1 b_{21} + v_1 b_{22}) - (u_1^2 + v_1^2) b_{23} - (u_1^2 + v_1^2 + w_1^2)(v_1 b_{11} - u_1 b_{12}) \\ m &= \sqrt{u_1^2 + v_1^2 + w_1^2}(w_1(u_1 b_{11} + v_1 b_{12}) - (u_1^2 + v_1^2) b_{13} + v_1 b_{21} - u_1 b_{22}) \end{aligned}$$

and $\mathbf{g} = (g_1, g_2, g_3)^T$ with g_1 and g_2 as given for class IIa. This gives the normal forms for B_1, $\mathbf{b}_1$ and $\mathbf{b}_2$ as already given and the normal form for B_3 is as given by (4.10) with ρ_4, ρ_5 and ρ_6 as given by (4.13).

The pitch and the I.S.A. for motions in this class are as before.

Once again, we can choose g_3 so that $\mathbf{b}_3$ can be reduced. If $\rho_4 \neq 0$ choose

$$g_3 = \frac{1}{\rho_4}(\rho_6 g_1 - b_{32}^*)$$

so that

$$\mathbf{b}_3 \mapsto \begin{pmatrix} -\frac{\rho_5 \rho_6}{\rho_4} g_1 + \rho_6 g_2 + \frac{\rho_5}{\rho_4} b_{32}^* + b_{31}^* \\ 0 \\ \rho_5 g_1 - \rho_4 g_2 + b_{33}^* \end{pmatrix} = \begin{pmatrix} b_{31}^{**} \\ 0 \\ b_{33}^{**} \end{pmatrix}$$

(where $\mathbf{b}_3^* = G\mathbf{b}_3$). If $\rho_4 = 0$ and $\rho_5 \neq 0$, choose

$$g_3 = \frac{1}{\rho_5}(\rho_6 g_2 + b_{31}^*)$$

so that

$$\mathbf{b}_3 \mapsto \begin{pmatrix} 0 \\ -\rho_6 g_1 + b_{32}^* \\ \rho_5 g_1 + b_{33}^* \end{pmatrix} = \begin{pmatrix} 0 \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

This completes the derivation of the normal form for motions in class IIIa. □

The first three derivatives of the point–paths of the motion at $t = 0$ are:

$$\begin{aligned}
\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\
&= B_1\mathbf{x} + \mathbf{b}_1 \\
&\mapsto \begin{pmatrix} -\rho_1 x_2 \\ \rho_1 x_1 \\ b_{13}^* \end{pmatrix} \\
\ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\
&= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\
&\mapsto \begin{pmatrix} -\rho_1^2 x_1 \\ -\rho_1^2 x_2 + b_{22}^{**} \\ b_{23}^{**} \end{pmatrix} \\
\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\
&= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\
&\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 + b_{31}^{**} \\ \rho_6 x_1 - \rho_4 x_3 \\ -\rho_5 x_1 + \rho_4 x_2 + b_{33}^{**} \end{pmatrix} \quad \text{if } \rho_4 \neq 0 \\
&\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 \\ \rho_6 x_1 + b_{32}^{**} \\ -\rho_5 x_1 + b_{33}^{**} \end{pmatrix} \quad \text{if } \rho_4 = 0.
\end{aligned} \tag{4.16}$$

Class IIIb: $\mathbf{B_1} \neq \mathbf{0}$, $\mathbf{B_2} = \mathbf{0}$ and $\mathbf{B_3} = \boldsymbol{\beta}\mathbf{B_1}$ for some $\boldsymbol{\beta} \in \mathbb{R}$.

Theorem 4.5. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class IIIb can be* *reduced to*

$$B_1 = \begin{pmatrix} 0 & -\rho_1 & 0 \\ \rho_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^{**} \\ b_{23}^{**} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & 0 \\ \rho_6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. This is exactly the same as class IIIa except that under the coordinate transformation $(G, \mathbf{g})$ as given for class IIIa, the normal form for B_3 has $\rho_4 = \rho_5 = 0$ and $\rho_6 = \sqrt{u_3^2 + v_3^2 + w_3^2} = \beta\rho_1$ and $\mathbf{b}_3$ cannot be reduced, leaving g_3 still arbitrary. □

The pitch and the I.S.A. are the same as before and the first three derivatives of the point–paths of the motion at $t = 0$ are:

$$\begin{aligned}
\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\
&= B_1\mathbf{x} + \mathbf{b}_1 \\
&\mapsto \begin{pmatrix} -\rho_1 x_2 \\ \rho_1 x_1 \\ b_{13}^* \end{pmatrix} \\
\ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\
&= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\
&\mapsto \begin{pmatrix} -\rho_1^2 x_1 \\ -\rho_1^2 x_2 + b_{22}^{**} \\ b_{23}^{**} \end{pmatrix} \\
\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\
&= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\
&\mapsto \begin{pmatrix} -\rho_6 x_2 + b_{31}^{**} \\ \rho_6 x_1 + b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}
\end{aligned} \tag{4.17}$$

Class IVa: $\mathbf{B_1 = 0}$, $\mathbf{B_2 \neq 0}$ and $\mathbf{B_3 \neq \gamma B_2}$ for any $\boldsymbol{\gamma} \in \mathbb{R}$.

Theorem 4.6. *There exists a change of coordinates* $(G, \mathbf{g}) \in SE(3)$ *so that the matrices* B_i *and the vectors* $\mathbf{b}_i$ *that determine, up to third order, a motion in class IVa can be reduced to*

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b^*_{13} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & 0 \\ -\rho_2 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 0 \\ b^{**}_{23} \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & -\rho_4 \\ -\rho_5 & \rho_4 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b^{**}_{31} \\ b^{**}_{32} \\ b^{**}_{33} \end{pmatrix}.$$

Proof. Motions in this class have $\dot{A}(0) = 0$, but as we restricted ourselves to regular motions $\dot{\mathbf{a}}(0) \neq 0$. We wish to find an orthogonal matrix G such that $G\mathbf{b}_1 = \mathbf{b}^*_1 = (0, 0, b^*_{13})^T$ and GB_2G^T has the normal form given by (4.7). If

$$G = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

where

$$
\begin{aligned}
a &= \frac{-b_{12}\|\mathbf{b}_1\|^2 f - b_{11}b_{13}\sqrt{b_{11}^2 + b_{12}^2 - \|\mathbf{b}_1\|^2 f^2}}{(b_{11}^2 + b_{12}^2)\|\mathbf{b}_1\|} \\
b &= \frac{b_{11}\|\mathbf{b}_1\|^2 f - b_{12}b_{13}\sqrt{b_{11}^2 + b_{12}^2 - \|\mathbf{b}_1\|^2 f^2}}{(b_{11}^2 + b_{12}^2)\|\mathbf{b}_1\|} \\
c &= \frac{\sqrt{b_{11}^2 + b_{12}^2 - \|\mathbf{b}_1\|^2 f^2}}{\|\mathbf{b}_1\|} \\
d &= \frac{-b_{11}b_{13}f - b_{12}\sqrt{b_{11}^2 + b_{12}^2 - \|\mathbf{b}_1\|^2 f^2}}{(b_{11}^2 + b_{12}^2)} \\
e &= \frac{-b_{12}b_{13}f + b_{11}\sqrt{b_{11}^2 + b_{12}^2 - \|\mathbf{b}_1\|^2 f^2}}{(b_{11}^2 + b_{12}^2)} \\
f &= \frac{\sqrt{b_{11}^2 + b_{12}^2}((b_{11}^2 + b_{12}^2)w_2 - b_{13}(b_{11}u_2 + b_{12}v_2))}{\|\mathbf{b}_1\|\sqrt{((b_{11}^2 + b_{12}^2)w_2 - b_{13}(b_{11}u_2 + b_{12}v_2))^2 + \|\mathbf{b}_1\|^2(b_{11}v_2 - b_{12}u_2)^2}} \\
g &= \frac{b_{11}}{\|\mathbf{b}_1\|} \\
h &= \frac{b_{12}}{\|\mathbf{b}_1\|} \\
i &= \frac{b_{13}}{\|\mathbf{b}_1\|}
\end{aligned}
$$

then these requirements are fulfilled, with

$$
\begin{aligned}
\rho_2 &= u_2(ch - bi) - v_2(cg - ai) + w_2(bg - ah) \\
\rho_3 &= u_2(bf - ce) - v_2(af - cd) + w_2(ae - bd).
\end{aligned}
$$

If, in addition we choose $\mathbf{g} = (g_1, g_2, g_3)^T$ with

$$
\begin{aligned}
g_1 &= \frac{1}{\rho_3} b_{22}^* \\
g_2 &= \frac{1}{\rho_3}(\rho_2 g_3 - b_{21}^*)
\end{aligned}
$$

with g_3 still arbitrary, under the coordinate transformation $(G, \mathbf{g})$ the vector $\mathbf{b}_2$ gets mapped to $(0, 0, \frac{\rho_2}{\rho_3} b_{22}^* + b_{23}^*)^T = (0, 0, b_{23}^{**})^T$.

Motions in this class have infinite pitch and the I.S.A. does not exist.

Under this transformation B_3 is not reduced and has the normal form given by (4.10) where

$$\begin{aligned}\rho_4 &= u_3(ei - fh) - v_3(di - fg) + w_3(dh - eg)\\ \rho_5 &= u_3(ch - bi) - v_3(cg - ai) + w_3(bg - ah)\\ \rho_6 &= u_3(bf - ce) - v_3(af - cd) + w_3(ae - bd)\end{aligned}$$

but we can choose g_3 in order to reduce $\mathbf{b}_3$. If $\rho_2\rho_6 - \rho_3\rho_5 \neq 0$, choose $g_3 = \frac{\rho_3 b^*_{31} - \rho_6 b^*_{21}}{\rho_2\rho_6 - \rho_3\rho_5}$ to get $\mathbf{b}_3 \mapsto (0, b^{**}_{32}, b^{**}_{33})^T$. If $\rho_2\rho_6 - \rho_3\rho_5 = 0$, choose $g_3 = \frac{\rho_6}{\rho_3\rho_4} - \frac{1}{\rho_4}b^*_{32}$ so that $\mathbf{b}_3 \mapsto (b^{**}_{31}, 0, b^{**}_{33})^T$. □

The first three derivatives of the point–paths of the motion at $t = 0$ are:

$$\begin{aligned}\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0)\\ &= B_1\mathbf{x} + \mathbf{b}_1\\ &\mapsto \begin{pmatrix} 0 \\ 0 \\ b^*_{13}\end{pmatrix}\\ \ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0)\\ &= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2\\ &\mapsto \begin{pmatrix} -\rho_3 x_2 + \rho_2 x_3 \\ \rho_3 x_1 \\ -\rho_2 x_1 + b^{**}_{23}\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0)\\ &= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3\\ &\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 \\ \rho_6 x_1 - \rho_4 x_3 + b^{**}_{32} \\ -\rho_5 x_1 + \rho_4 x_2 + b^{**}_{33}\end{pmatrix} \quad \text{if } \rho_2\rho_6 - \rho_3\rho_5 \neq 0\\ &\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 + b^{**}_{31} \\ \rho_6 x_1 - \rho_4 x_3 \\ -\rho_5 x_1 + \rho_4 x_2 + b^{**}_{33}\end{pmatrix} \quad \text{if } \rho_2\rho_6 - \rho_3\rho_5 = 0.\end{aligned} \tag{4.18}$$

Class IVb: $\mathbf{B_1 = 0}$, $\mathbf{B_2 \neq 0}$ and $\mathbf{B_3 = \boldsymbol{\gamma} B_2}$ for some $\boldsymbol{\gamma} \in \mathbb{R}$.

Theorem 4.7. *There exists a change of coordinates* $(G, \mathbf{g}) \in SE(3)$ *so that the matrices* B_i *and the vectors* $\mathbf{b}_i$ *that determine, up to third order, a motion in class IVb can be reduced to*

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$
$$B_2 = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & 0 \\ -\rho_2 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 0 \\ b_{23}^{**} \end{pmatrix}$$
$$B_3 = \begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & 0 \\ -\rho_5 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. With the same coordinate transformation as for class IVa we get the same normal forms for B_1, $\mathbf{b}_1$, B_2, $\mathbf{b}_2$ and B_3 with $\rho_4 = 0$ and $\rho_5 = \gamma\rho_2$, $\rho_6 = \gamma\rho_3$. The vector $\mathbf{b}_3$ cannot be reduced and g_3 is still arbitrary. □

This class of motions also has infinite pitch and does not possess an I.S.A.

The first three derivatives of the point–paths of the motion at $t = 0$ are given by:

$$\begin{aligned} \dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\ &= B_1\mathbf{x} + \mathbf{b}_1 \\ &\mapsto \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\ &= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\ &\mapsto \begin{pmatrix} -\rho_3 x_2 + \rho_2 x_3 \\ \rho_3 x_1 \\ -\rho_2 x_1 + b_{23}^{**} \end{pmatrix} \\ \dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\ &= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\ &\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 + b_{31}^{**} \\ \rho_6 x_1 + b_{32}^{**} \\ -\rho_5 x_1 + b_{33}^{**} \end{pmatrix}. \end{aligned} \tag{4.19}$$

Class Va: $\mathbf{B_1 = 0}$, $\mathbf{B_2 = 0}$ and $\mathbf{B_3 \neq 0}$.

Theorem 4.8. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class Va can be reduced to*

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^* \\ b_{23}^* \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\rho_6 & \rho_5 \\ \rho_6 & 0 & -\rho_4 \\ -\rho_5 & \rho_4 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^{**} \\ b_{32}^{**} \\ b_{33}^{**} \end{pmatrix}.$$

Proof. For motions in this class, if we perform the coordinate transformation $(G, \mathbf{g})$ where G is the orthogonal matrix

$$\begin{pmatrix} \left(\frac{\mathbf{b}_1 \times \mathbf{b}_2}{\|\mathbf{b}_1 \times \mathbf{b}_2\|}\right)^T \\ \left(\frac{\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2)}{\|\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2)\|}\right)^T \\ \left(\frac{\mathbf{b}_1}{\|\mathbf{b}_1\|}\right)^T \end{pmatrix}$$

then $\mathbf{b}_1 \mapsto G\mathbf{b}_1 = (0, 0, b_{13}^*)^T$ and $\mathbf{b}_2 \mapsto G\mathbf{b}_2 = (0, b_{22}^*, b_{23}^*)^T$. B_3 has the normal form given by (4.10) where

$$\rho_4 = -\frac{(u_3, v_3, w_3) \cdot (\mathbf{b}_1 \times (\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2)))^T}{\|\mathbf{b}_1\| \|\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2)\|}$$
$$\rho_5 = \frac{(u_3, v_3, w_3) \cdot (\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2))^T}{\|\mathbf{b}_1\| \|\mathbf{b}_1 \times \mathbf{b}_2\|}$$
$$\rho_6 = -\frac{(u_3, v_3, w_3) \cdot ((\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2)) \times (\mathbf{b}_1 \times \mathbf{b}_2))^T}{\|\mathbf{b}_1 \times \mathbf{b}_2\| \|\mathbf{b}_1 \times (\mathbf{b}_1 \times \mathbf{b}_2)\|}$$

Under this coordinate change

$$\mathbf{b}_3 \mapsto -GB_3G^{-1}\mathbf{g} + G\mathbf{b}_3 = \begin{pmatrix} \rho_6 g_2 - \rho_5 g_3 + b_{31}^* \\ -\rho_6 g_1 + \rho_4 g_3 + b_{32}^* \\ \rho_5 g_1 - \rho_4 g_2 + b_{33}^* \end{pmatrix}$$

If $\rho_4 \neq 0$ we can choose $g_1 = 0$, $g_2 = \frac{1}{\rho_4} b_{33}^*$ and $g_3 = -\frac{1}{\rho_4} b_{32}^*$ so that $\mathbf{b}_3 \mapsto (b_{31}^{**}, 0, 0)^T$. If $\rho_4 = 0$ and $\rho_5 \neq 0$ we can choose $g_1 = -\frac{1}{\rho_5} b_{33}^*$, $g_2 = 0$ and $g_3 = \frac{1}{\rho_5} b_{31}^*$ so that $\mathbf{b}_3 \mapsto (0, b_{32}^{**}, 0)^T$ and if $\rho_4 = \rho_5 = 0$ (and $\rho_6 \neq 0$) we can choose $g_1 = \frac{1}{\rho_6} b_{32}^*$, $g_2 = -\frac{1}{\rho_6} b_{31}^*$ so that $\mathbf{b}_3 \mapsto (0, 0, b_{33}^*)^T$. □

The pitch and the I.S.A. are the same as for motions in class IVa.

The first three derivatives of the point–paths at $t = 0$ of motions in this class are given by:

$$\begin{aligned}
\dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\
&= B_1\mathbf{x} + \mathbf{b}_1 \\
&\mapsto \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix} \\
\ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\
&= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\
&\mapsto \begin{pmatrix} 0 \\ b_{22}^* \\ b_{23}^* \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\
&= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\
&\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 + b_{31}^{**} \\ \rho_6 x_1 - \rho_4 x_3 \\ -\rho_5 x_1 + \rho_4 x_2 \end{pmatrix} \quad \text{if } \rho_4 \neq 0 \\
&\mapsto \begin{pmatrix} -\rho_6 x_2 + \rho_5 x_3 \\ \rho_6 x_1 + b_{32}^{**} \\ -\rho_5 x_1 \end{pmatrix} \quad \text{if } \rho_4 = 0 \text{ and } \rho_5 \neq 0 \\
&\mapsto \begin{pmatrix} -\rho_6 x_2 \\ \rho_6 x_1 \\ b_{33}^* \end{pmatrix} \quad \text{if } \rho_4 = \rho_5 = 0.
\end{aligned} \tag{4.20}$$

Class Vb: $\mathbf{B_1 = 0}$, $\mathbf{B_2 = 0}$ and $\mathbf{B_3 = 0}$.

Theorem 4.9. *There exists a change of coordinates $(G, \mathbf{g}) \in SE(3)$ so that the matrices B_i and the vectors $\mathbf{b}_i$ that determine, up to third order, a motion in class Vb can be reduced to*

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 0 \\ b_{22}^* \\ b_{23}^* \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} b_{31}^* \\ b_{32}^* \\ b_{33}^* \end{pmatrix}.$$

Proof. This is exactly the same as class Va except that as B_3 is the zero matrix we have $\rho_4 = \rho_5 = \rho_6 = 0$ and consequently $\mathbf{b}_3$ cannot be reduced. □

The pitch and the I.S.A. of motions in this class are the same as for motions in class IVa. The first three derivatives at $t = 0$ of the point–paths of the motions in this class are independent of $\mathbf{x}$ and are given by:

$$\begin{aligned} \dot{\Phi}_{\mathbf{x}}(0) &= \dot{A}(0)\mathbf{x} + \dot{\mathbf{a}}(0) \\ &= B_1\mathbf{x} + \mathbf{b}_1 \\ &\mapsto \begin{pmatrix} 0 \\ 0 \\ b_{13}^* \end{pmatrix} \\ \ddot{\Phi}_{\mathbf{x}}(0) &= \ddot{A}(0)\mathbf{x} + \ddot{\mathbf{a}}(0) \\ &= (B_1^2 + B_2)\mathbf{x} + \mathbf{b}_2 \\ &\mapsto \begin{pmatrix} 0 \\ b_{22}^* \\ b_{23}^* \end{pmatrix} \\ \dddot{\Phi}_{\mathbf{x}}(0) &= \dddot{A}(0)\mathbf{x} + \dddot{\mathbf{a}}(0) \\ &= (\tfrac{3}{2}(B_1B_2 + B_2B_1) + B_3)\mathbf{x} + \mathbf{b}_3 \\ &\mapsto \begin{pmatrix} b_{31}^* \\ b_{32}^* \\ b_{33}^* \end{pmatrix}. \end{aligned} \tag{4.21}$$

4.4 Differential Geometry of the Trajectories

We are interested in the points $\mathbf{x}$ whose point–paths $\Phi_{\mathbf{x}}(t)$ exhibit special features such as vanishing curvature and torsion. From Section 2.8, $\kappa = 0$ when $\dot{\Phi}_{\mathbf{x}}(0) \parallel \ddot{\Phi}_{\mathbf{x}}(0)$, or when

$$\ddot{\Phi}_{\mathbf{x}}(0) = u\dot{\Phi}_{\mathbf{x}}(0) \tag{4.22}$$

for some $u \in \mathbb{R}$, unless $\dot{\Phi}_{\mathbf{x}}(0) = 0$ in which case the curves are singular.

For motions in class I, solving equation (4.22) with (4.12), the zero–curvature curve is given parametrically by:

$$\begin{aligned}
x_1 &= -\frac{1}{\rho_2}b^*_{13}u + \frac{1}{\rho_2}b^{**}_{23} \\
x_2 &= \frac{1}{\rho_1\rho_2}b^*_{13}u^2 - \frac{1}{\rho_1^2\rho_2}(\rho_1 b^{**}_{23} + \rho_3 b^*_{13})u + \frac{1}{\rho_1^2\rho_2}(\rho_2 b^{**}_{22} + \rho_3 b^{**}_{23}) \\
x_3 &= -\frac{1}{\rho_2^2}b^*_{13}u^3 + \frac{1}{\rho_1\rho_2^2}(\rho_1 b^{**}_{23} + 2\rho_3 b^*_{13})u^2 \\
&\quad - \frac{1}{\rho_1^2\rho_2^2}((\rho_1^4 + \rho_3^2)b^*_{13} + \rho_1\rho_2 b^{**}_{22} + 2\rho_1\rho_3 b^{**}_{23})u \\
&\quad + \frac{\rho_3}{\rho_1^2\rho_2}b^{**}_{22} + \frac{1}{\rho_1^2\rho_2^2}(\rho_1^4 + \rho_3^2)b^{**}_{23}
\end{aligned} \tag{4.23}$$

This is a twisted cubic if $b^*_{13} \neq 0$ (i.e. the pitch is non–zero) and if $b^*_{13} = 0$ (i.e the motion has zero pitch) it becomes

$$\begin{aligned}
x_1 &= \frac{1}{\rho_2}b^{**}_{23} \\
x_2 &= -\frac{1}{\rho_1\rho_2}b^{**}_{23}u + \frac{1}{\rho_1^2\rho_2}(\rho_2 b^{**}_{22} + \rho_3 b^{**}_{23}) \\
x_3 &= \frac{1}{\rho_2^2}b^{**}_{23}u^2 - \frac{1}{\rho_1\rho_2^2}(\rho_2 b^{**}_{22} + 2\rho_3 b^{**}_{23})u + \frac{\rho_3}{\rho_1^2\rho_2}b^{**}_{22} + \frac{1}{\rho_1^2\rho_2^2}(\rho_1^4 + \rho_3^2)b^{**}_{23}
\end{aligned} \tag{4.24}$$

which is a parabola in the plane $x_1 = \frac{1}{\rho_2}b^{**}_{23}$ which is parallel to the I.S.A., the x_3–axis.

If, in addition, $b^{**}_{23} = 0$ the zero–curvature curve becomes

$$\begin{aligned}
x_1 &= 0 \\
x_2 &= \frac{1}{\rho_1^2}b^{**}_{22} \\
x_3 &= -\frac{1}{\rho_1\rho_2}b^{**}_{22}u + \frac{\rho_3}{\rho_1^2\rho_2}b^{**}_{22}
\end{aligned} \tag{4.25}$$

which is a line parallel to the I.S.A.

If we also have $b^{**}_{22} = 0$ the zero–curvature curve degenerates to the single point $(0, 0, 0)$.

For motions in classes IIa and IIb, we solve (4.22) with (4.14) to get the zero–curvature curve. If $b^*_{13} \neq 0$ the zero–curvature curve is a line parallel to the x_3–axis with

$$\begin{aligned} x_1 &= \frac{b^*_{13} b^{**}_{22} (\rho_1 b^{**}_{23} - \rho_3 b^*_{13})}{\rho_1^4 {b^*_{13}}^2 + (\rho_1 b^{**}_{23} - \rho_3 b^*_{13})^2} \\ x_2 &= \frac{\rho_1^2 {b^*_{13}}^2 b^{**}_{22}}{\rho_1^4 {b^*_{13}}^2 + (\rho_1 b^*_{23} - \rho_3 b^*_{13})^2} \end{aligned} \tag{4.26}$$

unless $b^{**}_{22} = 0$ in which case it is the x_3–axis itself.

If $b^*_{13} = 0$ and $b^{**}_{23} \neq 0$, the zero–curvature curve is the x_3–axis. If $b^*_{13} = 0$ and $b^{**}_{23} = 0$, the zero–curvature curve has equation

$$\rho_1^2 x_1^2 + \rho_1^2 x_2^2 - b^{**}_{22} x_2 = 0 \tag{4.27}$$

which is a cylinder whose axis is parallel to the x_3–axis, unless $b^{**}_{22} = 0$ in which case the zero–curvature curve is the x_3–axis.

For motions in classes IIIa and IIIb, the zero–curvature curve is obtained by solving (4.22) with (4.16). If $b^*_{13} \neq 0$, this is a line parallel to the x_3–axis with

$$\begin{aligned} x_1 &= \frac{b^{**}_{23} b^{**}_{22} b^*_{13}}{\rho_1 (\rho_1^2 {b^*_{13}}^2 + {b^{**}_{23}}^2)} \\ x_2 &= \frac{b^{**}_{22} {b^*_{13}}^2}{\rho_1^2 {b^*_{13}}^2 + {b^{**}_{23}}^2} \end{aligned} \tag{4.28}$$

If $b^{**}_{22} = 0$ this becomes the x_3–axis.

If $b^*_{13} = 0$ and $b^{**}_{23} \neq 0$ the zero–curvature curve again degenerates to the x_3–axis and if $b^*_{13} = b^{**}_{23} = 0$ it has equation given by (4.27) which is a cylinder with axis parallel to the x_3–axis unless $b^{**}_{22} = 0$ in which case it is the x_3–axis itself.

The zero–curvature curve of motions in classes IVa and IVb is found by solving (4.22) with (4.18). As $b^*_{13} \neq 0$, this gives a line in the $x_2 x_3$ plane with $\rho_3 x_2 = \rho_2 x_3$ which degenerates to the x_3–axis if $\rho_2 = 0$.

For motions in classes Va and Vb, from (4.20) we see that $\dot{\Phi}_{\mathbf{x}}(0)$ and $\ddot{\Phi}_{\mathbf{x}}(0)$ are independent of $\mathbf{x}$, so either all points of $\mathbb{R}^3$ have zero curvature at $t = 0$ (if $b^*_{22} = 0$), or there are no points with zero curvature (if $b^*_{22} \neq 0$).

We are also interested in the points where $\tau = 0$. From equation (2.7), we can see that a curve has zero torsion when the triple scalar product of its first three derivatives is zero, provided the cross product of the first two derivatives is non–zero. Indeed, if the cross product of its first two derivatives is zero then the curve has zero curvature and the torsion is undefined. Thus the zero–torsion surface of the motion $\Phi_{\mathbf{x}}(t)$ at $t = 0$ is given by all $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$(\dot{\Phi}_{\mathbf{x}}(0) \times \ddot{\Phi}_{\mathbf{x}}(0)) \cdot \dddot{\Phi}_{\mathbf{x}}(0) = 0 \tag{4.29}$$

provided $\mathbf{x}$ is not on the zero–curvature curve of the motion at $t = 0$.

For motions in class I, solving (4.29) with (4.12) gives the cubic surface

$$\begin{aligned}
&\rho_1^2(x_1^2 + x_2^2)\big((3\rho_2\rho_3 - \rho_1\rho_5)x_1 + (\tfrac{3}{2}\rho_1^2\rho_2 + \rho_1\rho_4)x_2\big) - 3\rho_1^2\rho_2^2 x_1x_2x_3 \\
&\quad + \rho_1(\rho_1^2 b_{33}^{**} - \rho_2 b_{31}^{**} - 3\rho_1\rho_3 b_{23}^{**} + 3\rho_3^2 b_{13}^{*} - \rho_1\rho_6 b_{13}^{*})x_1^2 + \rho_1(\rho_5 b_{22}^{**} - \rho_2 b_{32}^{**})x_1x_2 \\
&\qquad + (\rho_1\rho_5 b_{23}^{**} - \rho_3\rho_5 b_{13}^{*} + \rho_1^2\rho_4 b_{13}^{*} - \tfrac{3}{2}\rho_1^3\rho_2 b_{13}^{*} + \rho_2\rho_6 b_{13}^{*} - \rho_1\rho_2 b_{33}^{**})x_1x_3 \\
&\qquad + \rho_1(\rho_1^2 b_{33}^{**} - \rho_1\rho_6 b_{13}^{*} + 3\rho_3^2 b_{13}^{*} - 3\rho_1\rho_3 b_{23}^{**} - \rho_4 b_{22}^{**} - \tfrac{3}{2}\rho_1\rho_2)x_2^2 \\
&\qquad + (\rho_1^2\rho_5 b_{13}^{*} + \rho_3\rho_4 b_{13}^{*} - \tfrac{9}{2}\rho_1\rho_2\rho_3 b_{13}^{*} + \tfrac{3}{2}\rho_1^2\rho_2 b_{23}^{**} - \rho_1\rho_4 b_{23}^{**})x_2x_3 \\
&\quad + \rho_2(\tfrac{3}{2}\rho_1\rho_2 b_{13}^{*} - \rho_4 b_{13}^{*})x_3^2 + (\rho_1 b_{23}^{**} b_{31}^{**} - \rho_3 b_{13}^{*} b_{31}^{**} + 3\rho_1\rho_3 b_{13}^{*} b_{22}^{**} - \rho_1^2 b_{13}^{*} b_{32}^{**})x_1 \\
&\qquad + (\rho_1^2 b_{13}^{*} b_{31}^{**} + \rho_6 b_{13}^{*} b_{22}^{**} - \rho_3 b_{13}^{*} b_{32}^{**} + \rho_1 b_{23}^{**} b_{32}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 \\
&\qquad\qquad + (\rho_2 b_{13}^{*} b_{32}^{**} - \rho_5 b_{13}^{*} b_{22}^{**})x_3 - b_{13}^{*} b_{22}^{**} b_{31}^{**} = 0
\end{aligned} \tag{4.30}$$

So the zero–torsion surface is this surface minus its intersection with the zero–curvature curve.

Motions in class IIa with $\rho_4 \neq 0$ also have a cubic zero–torsion surface, given by

$$\begin{aligned}
&\rho_1^3(x_1^2 + x_2^2)(-\rho_5 x_1 + \rho_4 x_2) + \rho_1(\rho_1^2 b_{33}^{**} - 3\rho_1\rho_3 b_{23}^{**} + 3\rho_3^2 b_{13}^{*} - \rho_1\rho_6 b_{13}^{*})x_1^2 \\
&\qquad + \rho_1\rho_5 b_{22}^{**} x_1x_2 + (\rho_1\rho_5 b_{23}^{**} - \rho_3\rho_5 b_{13}^{*} + \rho_1^2\rho_4 b_{13}^{*})x_1x_3 \\
&\qquad + \rho_1(\rho_1^2 b_{33}^{**} - \rho_1\rho_6 b_{13}^{*} + 3\rho_3^2 b_{13}^{*} - 3\rho_1\rho_3 b_{23}^{**} - \rho_4 b_{22}^{**})x_2^2 \\
&\quad + (\rho_1^2\rho_5 b_{13}^{*} + \rho_3\rho_4 b_{13}^{*} - \rho_1\rho_4 b_{23}^{**})x_2x_3 + (3\rho_1\rho_3 b_{13}^{*} b_{22}^{**} + \rho_1 b_{23}^{**} b_{31}^{**} - \rho_3 b_{13}^{*} b_{31}^{**})x_1 \\
&\qquad + (\rho_6 b_{13}^{*} b_{22}^{**} + \rho_1^2 b_{13}^{*} b_{31}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 - \rho_5 b_{13}^{*} b_{22}^{**} x_3 - b_{13}^{*} b_{22}^{**} b_{31}^{**} = 0
\end{aligned} \tag{4.31}$$

If $\rho_4 = 0$ the zero–torsion surface becomes

$$\begin{aligned}
&-\rho_1^3\rho_5(x_1^2 + x_2^2)x_1 + \rho_1(\rho_1^2 b_{33}^{**} - 3\rho_1\rho_3 b_{23}^{**} + 3\rho_3^2 b_{13}^{*} - \rho_1\rho_6 b_{13}^{*})(x_1^2 + x_2^2) \\
&\qquad + \rho_1\rho_5 b_{22}^{**} x_1x_2 + (\rho_1\rho_5 b_{23}^{**} - \rho_3\rho_5 b_{13}^{*})x_1x_3 + \rho_1^2\rho_5 b_{13}^{*} x_2x_3 \\
&\qquad\qquad + (3\rho_1\rho_3 b_{13}^{*} b_{22}^{**} - \rho_1^2 b_{13}^{*} b_{32}^{**})x_1 \\
&\qquad + (\rho_6 b_{13}^{*} b_{22}^{**} - \rho_3 b_{13}^{*} b_{32}^{**} + \rho_1 b_{23}^{**} b_{32}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 - \rho_5 b_{13}^{*} b_{22}^{**} x_3 = 0
\end{aligned} \tag{4.32}$$

For motions in class IIb, solving (4.29) with (4.15), the zero–torsion surface is a quadratic surface given by

$$\begin{aligned}\rho_1(\rho_1^2 b_{33}^{**} + 3\rho_3^2 b_{13}^{*} - 3\rho_1\rho_3 b_{23}^{**} - \rho_1\rho_6 b_{13}^{*})(x_1^2 + x_2^2)\\ + (3\rho_1\rho_3 b_{13}^{*} b_{22}^{**} + \rho_1 b_{23}^{**} b_{31}^{**} - \rho_3 b_{13}^{*} b_{31}^{**} - \rho_1^2 b_{13}^{*} b_{32}^{**})x_1\\ + (\rho_6 b_{13}^{*} b_{22}^{**} + \rho_1^2 b_{13}^{*} b_{31}^{**} - \rho_3 b_{13}^{*} b_{32}^{**} + \rho_1 b_{23}^{**} b_{32}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 - b_{13}^{*} b_{22}^{**} b_{31}^{**} = 0 \quad (4.33)\end{aligned}$$

If a motion is in class IIIa with $\rho_4 \neq 0$, the zero–torsion surface is given by

$$\begin{aligned}\rho_1^3(x_1^2 + x_2^2)(\rho_4 x_2 - \rho_5 x_1) + \rho_1^2(x_1^2 + x_2^2)(\rho_1 b_{33}^{**} - \rho_6 b_{13}^{*})\\ - \rho_1\rho_4 b_{22}^{**} x_2^2 + \rho_1\rho_5 b_{22}^{**} x_1 x_2 + \rho_1(\rho_1\rho_4 b_{13}^{*} + \rho_5 b_{23}^{**})x_1 x_3 + \rho_1(\rho_1\rho_5 b_{13}^{*} - \rho_4 b_{23}^{**})x_2 x_3\\ + \rho_1 b_{23}^{**} b_{31}^{**} x_1 + (\rho_1^2 b_{13}^{*} b_{31}^{**} + \rho_6 b_{13}^{*} b_{22}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 - \rho_5 b_{13}^{*} b_{22}^{**} x_3 - b_{13}^{*} b_{22}^{**} b_{31}^{**} = 0 \quad (4.34)\end{aligned}$$

If $\rho_4 = 0$ it becomes

$$\begin{aligned}-\rho_1^3\rho_5(x_1^2 + x_2^2)x_1 + \rho_1^2(x_1^2 + x_2^2)(\rho_1 b_{33}^{**} - \rho_6 b_{13}^{*})\\ + \rho_1\rho_5 b_{22}^{**} x_1 x_2 + \rho_1\rho_5 b_{23}^{**} x_1 x_3 + \rho_1^2\rho_5 b_{13}^{*} x_2 x_3 - \rho_1^2 b_{13}^{*} b_{32}^{**} x_1\\ + (\rho_1 b_{23}^{**} b_{32}^{**} + \rho_6 b_{13}^{*} b_{22}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 - \rho_5 b_{13}^{*} b_{22}^{**} x_3 = 0 \quad (4.35)\end{aligned}$$

If the motion is in class IIIb, solving (4.29) with (4.17), we see that the zero–torsion surface degenerates to a quadratic surface given by

$$\begin{aligned}\rho_1^2(\rho_1 b_{33}^{**} - \rho_6 b_{13}^{*})(x_1^2 + x_2^2) + \rho_1(b_{23}^{**} b_{31}^{**} - \rho_1 b_{13}^{*} b_{32}^{**})x_1\\ + (\rho_1^2 b_{13}^{*} b_{31}^{**} + \rho_6 b_{13}^{*} b_{22}^{**} + \rho_1 b_{23}^{**} b_{32}^{**} - \rho_1 b_{22}^{**} b_{33}^{**})x_2 - b_{13}^{*} b_{22}^{**} b_{31}^{**} = 0 \quad (4.36)\end{aligned}$$

For motions in class IVa, if $\rho_2\rho_6 - \rho_3\rho_5 \neq 0$ the zero–torsion surface is a quadratic surface given by

$$(\rho_2\rho_6 - \rho_3\rho_5)x_1 x_3 + \rho_3\rho_4 x_2 x_3 - \rho_2\rho_4 x_3^2 - \rho_3 b_{32}^{**} x_2 + \rho_2 b_{32}^{**} x_3 = 0 \quad (4.37)$$

On the other hand, if $\rho_2\rho_6 - \rho_3\rho_5 = 0$ the zero–torsion surface is given by

$$\rho_3\rho_4 x_2 x_3 - \rho_2\rho_4 x_3^2 - \rho_3 b_{31}^{**} x_1 = 0 \quad (4.38)$$

If the motion is in class IVb, solving (4.29) with (4.19), we get

$$(\rho_2\rho_6 - \rho_3\rho_5)x_1 x_3 - \rho_3 b_{31}^{**} x_1 - \rho_3 b_{32}^{**} x_2 + \rho_2 b_{32}^{**} x_3 = 0$$

For motions in this class, $\rho_5 = \gamma\rho_2$ and $\rho_6 = \gamma\rho_3$ so $\rho_2\rho_6 - \rho_3\rho_5 = 0$ giving a zero–torsion surface that is linear:

$$\rho_3 b_{31}^{**} x_1 + \rho_3 b_{32}^{**} x_2 - \rho_2 b_{32}^{**} x_3 = 0 \tag{4.39}$$

For motions in class Va and Vb, $\dot{\Phi}_{\mathbf{x}}(0) \times \ddot{\Phi}_{\mathbf{x}}(0) = (-b_{13}^* b_{22}^*, 0, 0)^T$, so if $b_{22}^* = 0$ then the curvature is zero and the torsion is undefined. If $b_{22}^* \neq 0$, motions in class Va with $\rho_4 \neq 0$ have a zero–torsion surface given by

$$\rho_6 x_2 - \rho_5 x_3 - b_{31}^{**} = 0$$

If $\rho_4 = 0$ and $\rho_5 \neq 0$ the zero–torsion surface becomes the plane

$$\rho_6 x_2 - \rho_5 x_3 = 0.$$

If the motion is in class Va with $\rho_4 = \rho_5 = 0$ the zero–torsion surface is the x_1x_3–plane, $x_2 = 0$. If the motion is in class Vb, the torsion is zero if and only if $b_{31}^* = 0$, so either all points in $\mathbb{R}^3$ have zero torsion or none do.

These results are summarised in the following tables.

Class	Zero-Torsion Surface		
I	cubic surface		
IIa	cubic surface		
IIb	quadratic surface		
IIIa	cubic surface		
IIIb	quadratic surface		
IVa	quadratic surface		
IVb	linear surface		
Va	$b_{22}^* \neq 0$		plane
	$b_{22}^* = 0$		torsion undefined
Vb	$b_{22}^* \neq 0$	$b_{31}^* \neq 0$	no points of zero torsion
		$b_{31}^* = 0$	$\mathbb{R}^3$
	$b_{22}^* = 0$		torsion undefined

Class	Zero-Curvature Curve			
I	$b^*_{13} \neq 0$			twisted cubic
	$b^*_{13} = 0$	$b^{**}_{23} \neq 0$		parabola in plane parallel to the I.S.A.
		$b^{**}_{23} = 0$	$b^{**}_{22} \neq 0$	line parallel to the I.S.A.
			$b^{**}_{22} = 0$	single point on the I.S.A.
IIa	$b^*_{13} \neq 0$	$b^{**}_{22} \neq 0$		line parallel to the I.S.A.
IIb		$b^{**}_{22} = 0$		I.S.A.
IIIa	$b^*_{13} = 0$	$b^{**}_{23} \neq 0$		I.S.A.
IIIb		$b^{**}_{23} = 0$	$b^{**}_{22} \neq 0$	cylinder with axis parallel to the I.S.A.
			$b^{**}_{22} = 0$	I.S.A.
IVa			$\rho_2 \neq 0$	line in x_2x_3–plane
IVb			$\rho_2 = 0$	x_3–axis
Va			$b^*_{22} \neq 0$	no points of zero curvature
Vb			$b^*_{22} = 0$	$\mathbb{R}^3$

4.5 Examples

Example 1
Suppose we take the motion given by

$$\begin{aligned} A(t) &= \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 t & -\sin t & -\sin t \cos t \\ \sin t \cos t & \cos t & -\sin^2 t \\ \sin t & 0 & \cos t \end{pmatrix} \\ \mathbf{a}(t) &= \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \end{aligned}$$

which is a rotation about the x_2–axis of angle t followed by a rotation about the x_3–axis of angle t followed by a translation of t units in the x_2–direction.

Then we have

$$B_1 = \dot{A}(0) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B_2 = \ddot{A}(0) - B_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B_3 = \dddot{A}(0) - \tfrac{3}{2}(B_1B_2 + B_2B_1) = \begin{pmatrix} 0 & \frac{5}{2} & \frac{5}{2} \\ -\frac{5}{2} & 0 & 0 \\ -\frac{5}{2} & 0 & 0 \end{pmatrix}$$

$$\mathbf{b}_1 = \dot{\mathbf{a}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \ddot{\mathbf{a}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \dddot{\mathbf{a}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so the motion is in class I.

Using the change of coordinates given by (4.5), (4.6), (4.8) and (4.9) we have

$$\begin{array}{ll} \rho_1 = \sqrt{2} & b_{13}^* = \frac{1}{\sqrt{2}} \\ \rho_2 = -1 & b_{22}^{**} = 1 \\ \rho_3 = 0 & b_{23}^{**} = 0 \\ \rho_4 = 0 & b_{31}^{**} = -\frac{5}{2\sqrt{2}} \\ \rho_5 = 0 & b_{32}^{**} = 0 \\ \rho_6 = -\frac{5}{\sqrt{2}} & b_{33}^{**} = \frac{3}{2\sqrt{2}} \end{array}$$

Thus the pitch of this motion is $\frac{1}{\sqrt{2}}/\sqrt{2} = \frac{1}{2}$, the I.S.A. of the motion (in the new coordinates) is the x_3–axis and by (4.23) the zero–curvature curve is given parametrically by

$$\begin{aligned} x_1 &= \frac{u}{\sqrt{2}} \\ x_2 &= -\frac{1}{2}u^2 + \frac{1}{2} \\ x_3 &= -\frac{1}{\sqrt{2}}u^3 - \frac{1}{\sqrt{2}}u \end{aligned}$$

We can see that this is the twisted cubic represented parametrically in terms of x_1 by $(x_1, -x_1^2 + \frac{1}{2}, -2x_1^3 - x_1)$.

The zero torsion surface is given by, from (4.30), the equation

$$-6(x_1^2 + x_2^2)x_2 - 6x_1x_2x_3 + \tfrac{11}{2}x_1^2 + 7x_1x_3 + 11x_2^2 + \tfrac{3}{2}x_3^2 - \tfrac{13}{2}x_2 + \tfrac{5}{4} = 0$$

minus its intersection with the zero curvature curve.
If we consider this equation as a quadratic in x_3, we can solve it in terms of x_1 and x_2:

$$x_3 = \frac{1}{3}\left(6x_1x_2 - 7x_1 \pm \sqrt{(6x_1x_2 - 7x_1)^2 + 36(x_1^2 + x_2^2)x_2 - 33x_1^2 - 66x_2^2 + 39x_2 - \tfrac{15}{2}}\right)$$

So the zero–torsion surface consists of the union of the graphs of these two functions minus the zero–curvature curve. A plot of the zero–torsion surface and the zero–curvature curve is shown in fig. 4.2.

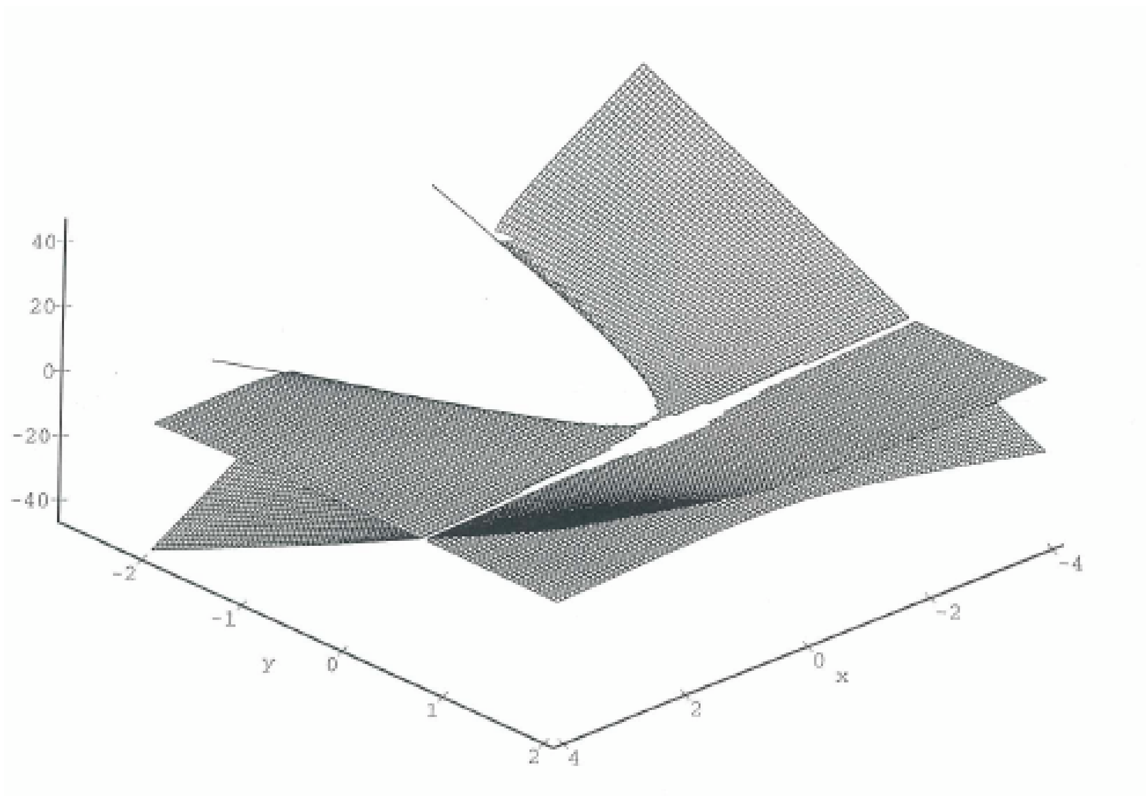

Figure 4.2: The zero–torsion surface and zero–curvature curve for example 1

Every vertical line will intersect the surface at two points, corresponding to the positive and negative values of the square root when the discriminant of the quadratic is greater

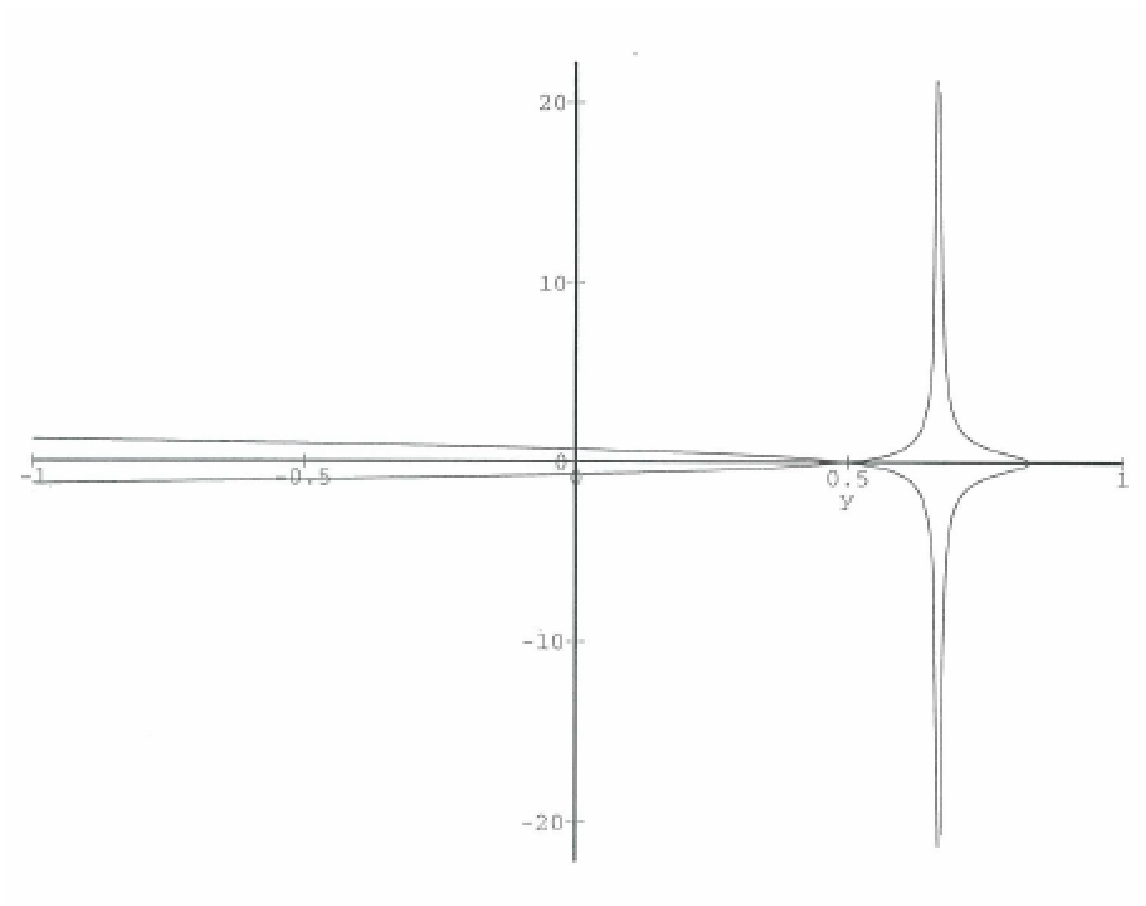

Figure 4.3: The zero–discriminant curve from example 1

than zero. When the discriminant is zero, there is only one solution to the quadratic, corresponding to either a self–intersection or a folding of the surface. When the discriminant is negative, there is no real solution to the quadratic, so there is no point on the surface corresponding to these values of (x_1, x_2).

The zero–discriminant curve is quadratic in x_1 and is given by

$$x_1^2 = \frac{-36x_2^3 + 66x_2^2 - 39x_2 + \frac{15}{2}}{4(3x_2 - 2)^2}$$

A plot of this curve is shown in fig. 4.3. The curve has an asymptote at $x_2 = \frac{2}{3}$ and this corresponds to the wedge–shaped gap running parallel to the x_1–axis in the surface shown in fig. 4.2.

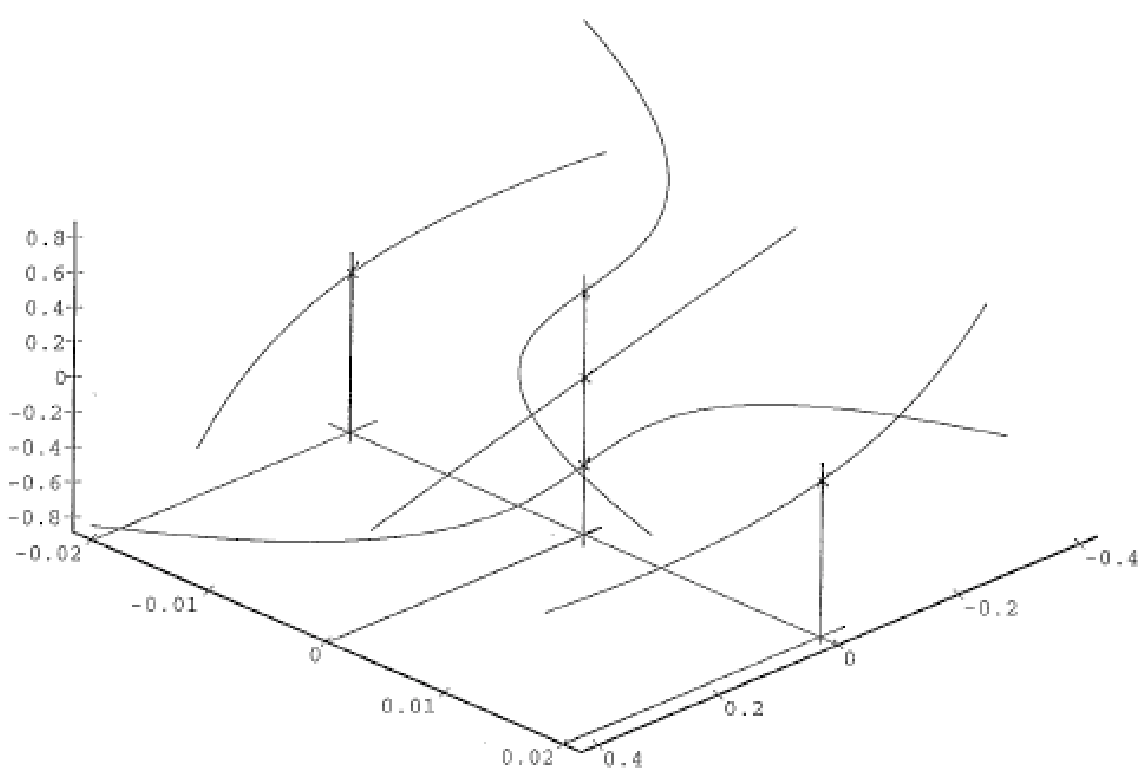

Figure 4.4: Trajectories of a point on the zero–curvature curve and four others close to it for example 1

The zero–torsion surface has only one singularity at the point $(0, \frac{1}{2}, 0)^T$.

The trajectory of the point on the zero–curvature curve $(0, \frac{1}{2}, 0)^T$ and those of four points close to it, but not on the zero–curvature curve, $(0, \frac{1}{2}, -\frac{1}{2})^T$, $(0, \frac{1}{2}, \frac{1}{2})^T$, $(0, \frac{12}{25}, 0)^T$ and $(0, \frac{13}{25}, 0)^T$, from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$ are shown in fig. 4.4.

Example 2

Let us now look at the motion given by $\Phi_{\mathbf{x}}(t) = A(t)\mathbf{x} + \mathbf{a}(t)$ where

$$A(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{a}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

which consists of a rotation about the x_3–axis of angle t followed by a translation by the twisted cubic curve $(t, t^2, t^3)^T$.

Here the axis of rotation is fixed in the direction of the x_3-axis for all t and

$$B_1 = \dot{A}(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$B_2 = \ddot{A}(0) - B_1^2 = 0$$
$$B_3 = \dddot{A}(0) - \tfrac{3}{2}(B_1 B_2 + B_2 B_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathbf{b}_1 = \dot{\mathbf{a}}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \ddot{\mathbf{a}}(0) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \dddot{\mathbf{a}}(0) = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

Thus we have $B_2 = 0$ and $B_3 = -B_1$ so the motion is in class IIIb. In this case B_1 and $\mathbf{b}_2$ are already in the required normal form, so the matrix G in the coordinate transformation $(G, \mathbf{g})$ is simply the identity matrix and the vector $\mathbf{g}$ is given by $(0, -1, g_3)^T$ resulting in the following:

$$\mathbf{b}_1 \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 \mapsto \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}$$

Hence $b_{13}^* = 0$ so the motion has zero pitch and $b_{23}^{**} = 0$ also, so the zero–curvature curve is given, from (4.27), as

$$x_1^2 + x_2^2 - x_2 = 0$$

which is a cylinder of revolution with axis parallel to the I.S.A. (the x_3–axis).

The surface of zero torsion is given, from (4.36), by the surface

$$6(x_1^2 + x_2^2 - x_2) = 0$$

minus its intersection with the zero–curvature curve which means there are no points $\mathbf{x} \in \mathbb{R}^3$ whose point–paths have zero torsion at $t = 0$.

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