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On the Hierarchy of Screw Systems

P. S. DONELAN
Department of Mathematics, Victoria University of Wellington, PO Box 600, Wellington, New Zealand, e-mail: *Peter.Donelan@vuw.ac.nz*

and

C. G. GIBSON
Department of Pure Mathematics, University of Liverpool, PO Box 147, Liverpool L69 3BX, England, e-mail: *su07@uk.ac.liverpool.ibm*

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Abstract. Let $E(3)$ be the Lie group of proper rigid motions of Euclidean 3-space. The adjoint action of $E(3)$ on its Lie algebra $e(3)$ induces an action on the Grassmannian of subspaces of given dimension d. Projectively, these subspaces are the screw systems of classical kinematics. The authors show that existing classifications of screw systems give rise to Whitney regular stratifications of the Grassmannians, and establish diagrams of specialisations for the strata. A list is given of the screw systems which can appear generically for motions of 3-space with at most three degrees of freedom.

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1. Introduction

Much of theoretical kinematics is concerned with smooth *spatial motions*, defined (for the purposes of this paper) as immersions μ: $M \to E(3)$, with M a smooth manifold of dimension m, and $E(3)$ the Lie group of proper rigid mappings of the ambient Euclidean space $\mathbb{R}^3$. Associated to such a motion is a family of trajectories Φ_μ: $\mathbb{R}^3 \times M \to \mathbb{R}^3$, one for each point of the ambient space.

A central problem of the subject is to understand the geometry of spatial motions, of their associated families of trajectories, and the connexions between these two geometric objects. At the time of writing, the first objective is represented by the first-order invariants of motion, namely the classical screw systems. Indeed, a 1-jet of the motion corresponds to a vector subspace of the Lie algebra $e(3)$ of $E(3)$, and this (projectively speaking) is a screw system. The second objective (at least in its local form) is represented by the growing theory of Kinematic Singularities in the hands of the present authors [2, 3, 8–10]. Some limited results towards the third objective appear in [3].

In pursuing these objectives, we came to the realisation that it was important to understand better the hierarchy of screw systems. A classification of screw systems based on physical reasoning first appeared in [12]. A decade later a more formal account appeared in [11] though the key geometric idea is implicit in Klein's work [13] a century before. The present authors clarified matters further in [4], spelling out the associated invariant theory and presenting a more geometric approach to the listing. The technical content of this paper is based largely on that approach: indeed it would be difficult to achieve the same results without it. The classifications involve *moduli*, in the sense of algebraic geometry, usually given by the pitches of screws, and provide natural partitions of the Grassmannians of subspaces of given dimension in $e(3)$. We show that these partitions are stratifications, find codimensions and diagrams of specialisations for the strata, and verify that the (Whitney) regularity conditions hold. A standard application of the Thom transversality lemma then tells us that a residual set of motions is transverse to the stratification, so that motions in this set only meet strata of codimension $\leqslant m$. In particular, for $m \leqslant 3$ we list explicitly the strata which can appear transversely for a 'generic' motion: so far as we are aware the special significance of these particular screw systems has never been recognised, bar the observation in [2] that for a 'generic' spatial motion with one degree of freedom screws of infinite pitch do not appear.

Quite apart from the technical objectives listed above, the content of this paper clarifies the sheer listing of screw systems, in different ways. First, we are able to give necessary and sufficient conditions for any two screw systems to be equivalent, a detail missing from previous accounts: the key here is the introduction in Section 2 of a new invariant (the *square invariant*) for a 2-system of type IB. Second, we achieve a rather better understanding of the choices of strata by enlarging the group $E(3)$ of proper rigid motions to the group $\mathrm{Sim}(3)$ of positive similarities. From a physical point of view, that makes considerable sense, since the geometric phenomena we seek to study are certainly invariant under the larger group. Under the similarity group there are just four distinct types of screw, namely those of positive, negative, zero or infinite pitch. This natural subdivision gives rise to a choice of strata refining those considered in [12, 11, 4], and reflecting better the reality of engineering application. Indeed our choice reflects the mathematical desirability of having connected strata, and the engineering desirability of distinguishing right-handed and left-handed screws. There is also a technical implication: certain of the strata become orbits under the larger group, and general results then enable us to establish Whitney regularity over such strata without further ado.

2. Screw Systems Reviewed

In understanding the movement of a rigid body in Euclidean n-space $\mathbb{R}^n$, the basic mathematical objects are *smooth motions* of $\mathbb{R}^n$, by which we mean immersions $\mu\colon M \to E(n)$, where M is a smooth manifold (the *parameter space*), and $E(n)$ is the Lie group of proper rigid motions of $\mathbb{R}^n$. We are only concerned with the local structure of these objects, so work with *local motions*, i.e. (mono)germs

μ: $(M, x) \to (E(n), \rho)$ of smooth motions with source a point $x \in M$ and target a rigid map $\rho \in E(n)$.

Two local motions μ_1: $(M_1, x_1) \to (E(n), \rho_1)$, and μ_2: $(M_2, x_2) \to (E(n), \rho_2)$ are said to be I-*equivalent* (the I stands for 'isometry') when there exist a local diffeomorphism g and rigid maps σ and τ for which the following diagram commutes, where $h_{\sigma,\tau}$ is the germ of the mapping $E(n) \to E(n)$ defined by $\phi \mapsto \sigma\phi\tau$:

$$\begin{tikzcd}
(M_1, x_1) \arrow[r, "\mu_1"] & (E(n), \rho_1) \\
(M_2, x_2) \arrow[u, "g"] \arrow[r, "\mu_2"'] & (E(n), \rho_2) \arrow[u, "h_{\sigma,\tau}"']
\end{tikzcd}$$

It is a trivial remark that any local motion is I-equivalent to one of the form

$$\mu\colon (\mathbb{R}^m, 0) \longrightarrow (E(n), 1),$$

where $m = \dim M$ and 1 is the identity element in $E(n)$. Two such local motions μ_1 and μ_2 are I-equivalent when there exist a local diffeomorphism h and a rigid map σ for which the following diagram commutes

$$\begin{tikzcd}
(\mathbb{R}^m, 0) \arrow[r, "\mu_1"] & (E(n), 1) \\
(\mathbb{R}^m, 0) \arrow[u, "g"] \arrow[r, "\mu_2"'] & (E(n), 1) \arrow[u, "h_\sigma"']
\end{tikzcd}$$

where h_σ denotes conjugation in $E(n)$ by the element σ. Taking differentials in the above diagram, we obtain a commuting diagram of linear mappings

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "T\mu_1"] & e(n) \\
\mathbb{R}^m \arrow[u, "Tg"] \arrow[r, "T\mu_2"'] & e(n) \arrow[u, "Th_\sigma"']
\end{tikzcd}$$

where the tangent space to $\mathbb{R}^m$ at the origin is identified with $\mathbb{R}^m$, and where $e(n)$ denotes the Lie algebra of $E(n)$. The map on the right of this diagram defines an action of $E(n)$ on its Lie algebra called the adjoint action: this action is linear so induces an action on the Grassmannian of subspaces of $e(n)$ of given dimension d. One therefore observes that for I-equivalent motions μ_1 and μ_2, the images of the differentials $T\mu_1$ and $T\mu_2$ are equivalent under this induced action, and one seeks natural finite stratifications of the Grassmannian invariant under the action.

$E(n)$ is isomorphic, via a choice of orthogonal coordinates in $\mathbb{R}^n$, to the semidirect product $\mathrm{SO}(n) \times_s T(n)$, where $\mathrm{SO}(n)$ is the special orthogonal group and $T(n)$ is the group of translations of $\mathbb{R}^n$. The Lie algebra $e(n)$ is then isomorphic to the semidirect product $\mathrm{so}(n) \times_s t(n)$ of the Lie algebras $\mathrm{so}(n)$ and $t(n)$ of the factors $\mathrm{SO}(n)$ and $T(n)$. We identify elements of $\mathrm{so}(n)$ with $n \times n$ skew-symmetric matrices, and elements of $t(n)$ with vectors in $\mathbb{R}^n$. In view of these identifications we can write elements of $E(n)$ in the form (A, a) with A an orthogonal $n \times n$ matrix, and a a vector in $\mathbb{R}^n$. Likewise, we can write elements of the algebra $e(n)$ in the form (B, b) where B is a skew-symmetric matrix, and b is a vector in $\mathbb{R}^n$. With these identifications the adjoint action of $E(n)$ on $e(n)$ is given by the following relation, where the superscript T denotes matrix transposition.

$$(A, a) \bullet (B, b) = \left(ABA^T, Ab - ABA^T a\right).$$

The case of greatest physical interest is $n = 3$, for which there is an established terminology. Elements of $e(3)$ are *motors*, $e(3)$ itself is the *motor space*, and vector subspaces of $e(3)$ are *motor systems*: elements of the projectivised Lie algebra are *screws*, the projective space itself is the *screw space*, and projective subspaces are *screw systems*. The convention in the engineering literature is that the screw system arising from a motor system of dimension n is called an n-system (though its projective dimension is $n - 1$). A listing of screw systems was obtained in [12], largely on the basis of physical intuition: formal classifications appear in [11] and [4]. We need to recall the key elements of the classification in [4] in order to carry out the computations in latter sections. A convenient notation for motors is obtained via the observation that a 3×3 skew-symmetric matrix U can be identified with the unique vector $u \in \mathbb{R}^3$ for which $Ux = u \wedge x$ for all vectors $x \in \mathbb{R}^3$. Thus, motors can be identified with pairs (u, v) of vectors in $\mathbb{R}^3$ (the *motor coordinates*), and the motor space can be identified in an obvious way with $\mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^6$. In terms of motor coordinates, the adjoint action of $E(3)$ on $e(3)$ can be written

$$(A, a) \bullet (u, v) = (Au, Av - Au \wedge a).$$

In [4] it was shown that the ring of invariant polynomials for the adjoint action of $E(3)$ on $e(3)$ is generated by the Klein form $\langle u, u\rangle$ and the Killing form $\langle u, v\rangle$ where $\langle\,,\,\rangle$ is the standard scalar product on $\mathbb{R}^3$. The ratio h defined by $h = \langle u, v\rangle / \langle u, u\rangle$ is the *pitch* of the motor (u, v). Since the invariant forms both have degree 2, the concept of 'pitch' is well-defined for screws and is an invariant of the induced action, where now (u, v) refers to homogeneous coordinates in the screw space. Screws with $u = 0$ are said to be of *infinite* pitch; screws with $u \neq 0$ are said to be of *finite* pitch.

Table I. Normal forms for screws

Type	Normal form
finite	$(1,0,0;h,0,0)$
infinite	$(0,0,0;1,0,0)$

Thus we have two natural types of screw under the adjoint action, whose normal forms are given by Table I.

The *axis* of a screw (u, v) is defined to be the line in $\mathbb{PR}^3$ with Plücker coordinates $(u, v - hu)$: when the screw is of infinite pitch, i.e. $u = 0$, the axis is interpreted as the screw $(0, v)$ itself. In the screw space, the generators for the ring of invariant polynomials define the Klein quadric $\langle u, v\rangle = 0$ and the Killing quadric $\langle u, u\rangle = 0$, which (in the real case) is simply a generating α-plane for the Klein quadric. These two quadrics form a base for a real quadratic complex with general member

$$Q_h(u,v) = \langle u,v\rangle - h\langle u,u\rangle$$

whose zero set comprises all screws of pitch h. For that reason Q_h is referred to as a *pitch quadric*.

Two screws are *reciprocal* when they are orthogonal with respect to the Klein form. And to any screw system S we can associate its *orthogonal* (or *reciprocal*) screw system $S^{\perp}$. Thus 1-, 2- and 3-systems are orthogonal, respectively, to 5-, 4- and 3-systems. It is easily verified that two screw systems S_1, S_2 are equivalent if and only if their orthogonal systems $S_1^{\perp}$, $S_2^{\perp}$ are equivalent. Thus the classification of 4- and 5-systems reduces to that of 2- and 1-systems, resulting in a considerable reduction of labour.

The general principle for distinguishing screw systems S is the way in which they lie in the screw space relative to the quadratic complex of pitch quadrics Q_h. For finite h we distinguish the two main types I (S not contained in any Q_h) and II (S is contained in some Q_h). For $h = \infty$ we distinguish four main types A (S does not meet Q_∞), B (S meets Q_∞ in a point), C (S meets Q_∞ in a line), D (S meets Q_∞ in a plane). The notation extends in an obvious way. Thus IA refers to the case when S is neither contained in any Q_h nor meets Q_∞. These crude distinctions alone are virtually sufficient to separate the six types of 2-systems [4] given in Table II. The exception is represented by 2-systems of type IB, characterised by the property that the axes of all finite screws in the system are parallel. For the general type ($p \neq 0$) the axes form a plane in $\mathbb{R}^3$, but when $p = 0$ we obtain the *coaxial* case IB^0 when all the axes coincide. For each entry in Table II a basis is given for the corresponding two-dimensional motor system.

Only types IB^0 and IIC are orbits under the induced action. A word is in order about the remaining normal forms, which have one or two moduli. For the general type IA the *principal pitches* h_α and h_β are distinct, and characterized as the extremal pitches of finite screws in the system. And for types IIA, IIB the modulus h is

Table II. Normal forms for 2-systems

Type	Basis	Type	Basis
IA	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$	IIA	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$
IB	$(1,0,0;0,0,0)$ $(0,0,0;1,p,0)$	IIB	$(1,0,0;h,0,0)$ $(0,0,0;0,1,0)$
IB^0	$(1,0,0;0,0,0)$ $(0,0,0;1,0,0)$	IIC	$(0,0,0;1,0,0)$ $(0,0,0;0,1,0)$

characterized as the common pitch of all (finite screws) in the system. It is however less obvious how to obtain complete invariants for 2-systems of type IB. First, some preliminary remarks. It is a standard fact [11] that a 2-system of type IB has a unique screw of any given pitch: in particular it contains unique screws of pitch zero and pitch infinity. (And note, for the proof of Lemma 2.1 below, that a change of basis in the system preserving the screws of zero and infinite pitch must be simply a scaling of these screws.) It is therefore always possible to choose screws of pitch zero and pitch infinity for a basis. By applying a pure translation we can suppose the screw of pitch zero has the shape $(u, 0)$ with $u \neq 0$; and, via a possible scaling, we can suppose that the screw of infinite pitch has the form $(0, u + v)$ where u, v are perpendicular vectors. We will call such a basis *standard.*

LEMMA 2.1. *Let* S_1, S_2 *be 2-systems of type* IB *with standard bases* $(u, 0)$, $(0, u+v_1)$ *and* $(u, 0)$, $(0, u + v_2)$. *Then* S_1, S_2 *are equivalent if and only if the vectors* v_1, v_2 *have the same length.*

Proof. If the systems are equivalent, there must be an element (A, a) of the group $E(3)$, and a change of basis, taking the basis screws for S_1 to the basis screws for S_2. By the above comments, (A, a) must preserve the basis screws (as opposed to motors) of the systems. That means we have

$$(A, a) \bullet (u, 0) = \lambda(u, 0) \quad \text{and} \quad (A, a) \bullet (0, u + v_1) = \mu(0, u + v_2)$$

for some scalars $\lambda, \mu \neq 0$. The first relation implies that $Au = \lambda u$: and since A is a rotation that means that $Au = \pm u$. The second relation is equivalent to $A(u + v_1) = \mu(u + v_2)$, i.e. to

$$(\pm 1 - \mu)u + (Av_1 - \mu v_2) = 0. \qquad (*)$$

When $Au = u$ the rotation A is about the line spanned by u, so preserves the plane V of vectors perpendicular to u: thus v_1, v_2, Av_1 all lie in V, and $(*)$ tells us that $\mu = 1$, $Av_1 = v_2$ and hence that v_1, v_2 have the same length. When $Au = -u$ the rotation A must be about a line in the plane V through an angle π: thus again v_1, v_2, Av_1 all lie in V, and $(*)$ tells us that $\mu = -1$, $Av_1 = -v_2$ and, hence, that v_1,

v_2 have the same length. For the converse, it is no restriction to suppose that the standard bases for S_1, S_2 are those given in the normal forms of Table II, i.e. that $u = (1,0,0)$, $v_1 = (0,p_1,0)$, $v_2 = (0,p_2,0)$. Then the supposition that v_1, v_2 have the same length is that $p_1 = \pm p_2$. Let A be the rotation of $\mathbb{R}^3$ through an angle π about the line spanned by the vector u, and let $a = 0$. Observe that the effect of the element (A, a) of the group $E(3)$ on the normal form with parameter p is to change the sign of p, so that S_1, S_2 are equivalent. □

In particular, Lemma 2.1 tells us that two-systems of type IB with normal forms involving parameters p, q are equivalent if and only if $p^2 = q^2$. We refer to the scalar p^2 as the *square invariant* attached to any 2-system of type IB.

For screw systems of higher dimension the detailed classification presented in [4] refines the broad distinctions described above. For any screw system S of type I the intersections of S with the pitch quadrics Q_h give rise to a pencil of real quadrics (of lower dimension) in the projective subspace S. And the projective type of this pencil is an invariant of S, giving rise to a natural partition of the screw systems of that dimension. For 3-systems we have a pencil of real conics, where the projective classification is rather well-known: for convenience we keep to the shorthand notation for pencils given in [4]. Only five main types of pencil arise, namely those given in the second column of Table III, and *defining* the main types of 3-system of type I in the first column.

The importance of the pencil type is first, that it provides a geometric rationale for the subdivision of the broad IA and IB classes already proposed in [11], and secondly, it provides a 'recognition principle' for 3-systems, i.e. a readily calculable criterion for distinguishing classes. This will be particularly useful later in analysing unfoldings. In [4] this geometric subdivision of type I is used to derive the first six normal forms in Table IV: the remaining four normal forms for systems of type II are easily established.

Here too something needs to be said about the various moduli which appear in the normal forms. For the general type IA_1 we have symmetries in the planes $h_\alpha = h_\beta$, $h_\beta = h_\gamma$, representing systems of type IA_2, so can suppose that $h_\alpha > h_\beta > h_\gamma$. By Table III the type IA_1 corresponds to the real pencil of conics of type 1b having singular members a real line-pair and two distinct complex conjugate line-pairs.

Table III. 3-Systems of type I

Type	Pencil	Degenerate members
IA_1	1b	one real line-pair & two complex conjugate line-pairs
IA_2	22b	one complex conjugate line-pair & a repeated line of multiplicity 2
IB_0	2b	one real line-pair & a complex conjugate line-pair of multiplicity 2
IB_3	12a	singular pencil with two distinct repeated lines
IC	11	singular pencil with a single repeated line

Table IV. Normal forms for 3-systems

Type	Basis	Type	Basis
IA_1	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,1;0,0,h_\gamma)$	IC^0	$(1,0,0;0,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;1,0,0)$
IA_2	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,1;0,0,h_\beta)$	IIA	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,1;0,0,h)$
IB_0	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,0;1,0,p)$	IIB	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,0;0,0,1)$
IB_3	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,0;0,0,1)$	IIC	$(1,0,0;h,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;0,0,1)$
IC	$(1,0,0;0,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;1,0,p)$	IID	$(0,0,0;1,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;0,0,1)$

The intermediate pitch h_β is the common pitch of all screws on the real line-pair, whilst h_α, h_γ are the pitches of the vertices of the complex conjugate line-pairs, and are characterized analytically as the extremal pitches of screws in the system. By Table III the type IA_2 corresponds to the real pencil of conics of type 22b having singular members a complex conjugate line–pair and a repeated line of multiplicity 2. h_α is the pitch of the vertex of the complex conjugate line-pair, and h_β is the common pitch of the screws on the repeated line. For type IB_0 the situation is summed up by

LEMMA 2.2. *Consider two normal forms for a 3-system of type* IB_0 *with pitches* g, h *and parameters* p, q. *A necessary and sufficient condition for these 3-systems to be equivalent is that* $g = h$ *and* $p^2 = q^2$.

Proof. Sufficiency is clear. For necessity, suppose the two 3-systems are equivalent under the action. By Table III the IB_0 type corresponds to the real pencil of conics of type 2b, containing a unique real line–pair (with vertex a screw $\$_\beta$ of finite pitch) and a unique complex conjugate line–pair of multiplicity 2 (with vertex a screw $\$_\gamma$ of finite pitch). Under the action the common pitch of all the (finite) screws on the real line–pair is preserved. However, for a normal form with pitch h the common pitch is precisely h: it follows that $g = h$. The vertex $\$_\gamma$, necessarily lies on one line of the real line–pair: and on the other line there is a unique screw $\$_\alpha$ reciprocal to is $\$_\beta$ [4, p. 249]. The line joining $\$_\alpha$, $\$_\gamma$ is then a 2-system of type IB, and the parameter p in its normal form is precisely the parameter p in the normal form for the 3-system of type IB_0. The result is immediate from Lemma 2.1, since an equivalence of the two 3-systems gives rise to an equivalence of the two 2-systems of type IB. □

Type IB$_3$ corresponds to the real singular pencil of type 12a having two distinct repeated lines on which the common pitches are h_α and h_β.

For a 3-system of type IC we require rather more discussion. By Table III a 3-system S of type IC corresponds to a pencil of real conics of type 11, namely the singular pencil comprising a fixed line together with a pencil of lines through a fixed screw $\$_2 = (0, v_2)$ on the fixed line. The only degenerate member of the pencil is therefore the (fixed) repeated line. The lines of the pencil represent 2-systems of type IIB, with the sole exception of the fixed line, which is of type IIC. Any other 2-system in S must be of type IB. Note that on the repeated line there is a *unique* screw $\$_3 = (0, v_3)$ for which the vectors v_2, v_3 are perpendicular. (The analytical significance of the screw $\$_3$ will become evident in the proof of Lemma 2.3 below.) There is no canonical choice for a third basis screw $\$_1$. However it is certainly natural to choose a screw $\$_1 = (u_1, v_1)$ of pitch zero: and by applying a suitable pure translation to the system one can assume that $v_1 = 0$, without affecting $\$_2$, $\$_3$.

LEMMA 2.3. *Consider two normal forms S_p, S_q for a 3-system of type* IC *with parameters p, q. A necessary and sufficient condition for these 3-systems to be equivalent is that $p^2 = q^2$.*

Proof. The key is to interpret the parameter p of the normal form in terms of the square invariants for the 2-systems of type IB contained in S_p. To compute these we write $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ for the standard basis vectors in $\mathbb{R}^3$. Then the screw basis for the normal form given in Table IV comprises $(e_1, 0)$, $(0, e_2)$, $(0, e_1 + pe_3)$. Thus, a screw basis for *any* IB 2-system in S_p is given by $(\lambda e_1, e_2)$, $(0, e_1 + \mu e_2 + pe_3)$ for some scalars λ, μ with $\mu \neq 0$. Noting that e_1, λe_2 are perpendicular vectors we can (by applying a suitable pure translation) assume that the screw basis is $(e_1, 0)$, $(0, e_1 + \mu e_2 + pe_3)$. This basis is in standard form, so by Lemma 2.1 the square invariant for the 2-system is $\mu^2 + p^2$. (Geometrically, through each screw of infinite pitch, except the vertex of the pencil, there is a pencil of 2-systems of type IB having the *same* square invariant.) We conclude that *the scalar p^2 is the minimal square invariant for any 2-system of type* IB *contained in a 3-system of type* IC.

Suppose now that S_p, S_q are equivalent 3-systems. Then any 2-system of type IB in S_p is equivalent to one in S_q, and vice-versa. It is then immediate from Lemma 2.1, and the above characterisation, that $p^2 = q^2$. For the converse, note that the sign of p (in the normal form for S_p) is changed by applying to the system a pure rotation about the e_1-axis through an angle π. It follows immediately that if $p^2 = q^2$, and hence $p = \pm q$, then S_p, S_q are equivalent. □

Lemma 2.3 clarifies the geometric reason for separating off the orbit IC^0 obtained when the parameter p in the normal form is zero: indeed for that value of the parameter the 3-system contains a pencil of coaxial 2-systems of type IB.

3. Screw Systems Under Similarities

From a purely physical point of view it seems clear that geometric properties of Euclidean motions should be preserved under rescaling. For this reason it is natural to enlarge the group $E(3)$ to Sim(3), the Lie group of (affine) similarities of $\mathbb{R}^3$. The enlargement will have two important consequences. First, it makes it abundantly clear why screws of zero pitch play a rather special role, leading to a finer classification than that given above. Second, it reduces the number of moduli in the normal forms by one. In particular, several families involving a single modulus become orbits under the action of Sim(3), reducing substantially the work involved in establishing Whitney regularity conditions.

Recall that a *(positive) similarity of ratio* $\sigma > 0$ on $\mathbb{R}^3$ is a bijection $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that for all $x, y \in \mathbb{R}^3$

$$\|f(y) - f(x)\| = \sigma\|y - x\|.$$

In other words all distances are preserved, up to a scaling factor σ. In fact, a similarity is an affine map and can be written [1] as the product of a dilation (with centre the origin) and an isometry. The similarities form a group and the isometries themselves appear as the subgroup of similarities of ratio 1. Thus Sim(3) is isomorphic (given a choice of orthogonal coordinates) to a product $\mathrm{SO}(3) \times T(3) \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the group of positive real numbers under the operation of multiplication. Similarities will be written (A, a, σ): with that notation the associated action of the group on $\mathbb{R}^3$ is given by

$$(A, a, \sigma)x = A(\sigma x) + a.$$

It is now an easy matter to verify that the operation in the group is given by

$$(A_2, a_2, \sigma_2) \cdot (A_1, a_1, \sigma_1) = \big(A_2 A_1, A_2(\sigma_2 a_1) + a_2, \sigma_2\sigma_1\big).$$

Differentiating the action by conjugation, we see that the adjoint action on the Lie algebra $\mathrm{sim}(3) \cong \mathrm{so}(3) \times t(3) \times \mathbb{R}$ is given by

$$(A, a, \sigma) \bullet (B, b, \beta) = \big(ABA^T, -ABA^T a - \beta a + A\sigma b, \beta\big).$$

It is important to observe that the subalgebra $\beta = 0$ is invariant under this action, so there is an induced action of Sim(3) on $e(3)$ given by

$$(A, a, \sigma) \bullet (B, b) = \big(ABA^T, -ABA^T a + A\sigma b\big).$$

Equivalently, using the motor coordinates (u, v) in $e(3)$ this can be written in the shape

$$(A, a, \sigma) \bullet (u, v) = (Au, \sigma Av - Au \wedge a).$$

Table V. Normal forms for screws under similarities.

$(1,0,0;1,0,0)$	right-handed screws	positive pitch
$(1,0,0;-1,0,0)$	left-handed screws	negative pitch
$(1,0,0;0,0,0)$	infinitesimal rotations	zero pitch
$(0,0,0;1,0,0)$	infinitesimal translations	infinite pitch

In particular note that the effect of a dilation (with centre the origin) of ratio σ is given by $(u,v) \mapsto (u,\sigma v)$: it simply scales the second motor coordinate by σ. Thus the effect of such a dilation on a finite screw is to scale the pitch by σ: screws of infinite pitch are left invariant. It follows that although pitch is no longer an invariant of finite screws under the similarity group, the *sign* of the pitch is invariant. Normal forms for screws under the similarity group are given in Table V.

Note that the action of Sim(3) on $e(3)$ is linear so induces an action on Grassmannians, as did $E(3)$. Moreover this action respects the broad subdivisions made to date:

LEMMA 3.1. *The partitions of* 2- *and* 3-*systems defined by Tables* II *and* IV *are invariant under the action of the similarity group.*

Proof. Since the partitions are invariant under the group $E(3)$ it suffices to check that they are invariant under dilations σ, with centre the origin. By the above remarks concerning the effect of dilations on pitch, the distinctions between types I and II, and amongst types A, B, C, and D, are invariant under σ. It remains to verify that for a screw system S of type I the projective type of the pencil of quadrics q_h, obtained by restricting the Q_h to S, is invariant as well. But (again by the above remarks) the pencil resulting from σ is simply $q_{\sigma h}$, i.e. an equivalent pencil. □

To reiterate, the consequences of working with the similarity group are first, that the number of moduli appearing in most of the normal forms for screw systems is reduced by one (the exceptions being those types in which the only modulus is a square invariant) and second, that the special role played by screws of pitch zero suggests a refinement of the partitions described in Section 2.

For 2-systems the finer partition is presented in Table VI below. The general type IA splits naturally into subtypes, corresponding to subsets of the (h_α, h_β) plane,

Table VI. The finer partition of 2-systems

Type	Finer subtypes
IA	IA^{++}, IA^{+0}, IA^{+-}, IA^{0-}, IA^{--}
IB	IB, IB^0
IIA	IIA^+, IIA^0, IIA^-
IIB	IIB^+, IIB^0, IIB^-
IIC	no finer subtypes

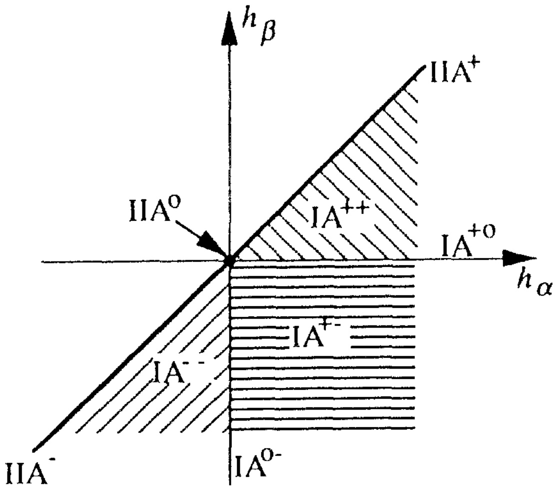


Fig. 1. Partition of type IA 2-systems.

depicted in Figure 1. Note that the orbits are invariant under reflexion in the line $h_\alpha = h_\beta$, corresponding to type IIA, so we can suppose $h_\alpha > h_\beta$. Moreover, under the similarity group (h_α, h_β) is invariant only up to positive scalar multiples, so the signs (positive, zero or negative) of h_α, h_β provide a convenient notation for the finer partition of IA into IA^{++}, IA^{+0}, IA^{+-}, IA^{0-}, and IA^{--}, of which only IA^{+0} and IA^{0-} are orbits. For type IB, similarities leave the modulus p invariant, so there are no subtypes bar the orbit IB^0. Types IIA and IIB split naturally into IIA^+, IIA^0, IIA^- and IIB^+, IIB^0, IIB^- according as the common pitch h of all the finite screws in the system is positive, zero or negative and all of these are orbits of the similarity action.

For 3-systems the finer partition is presented in Table VII below. The general type IA_1 splits into subtypes, corresponding to subsets of $(h_\alpha, h_\beta, h_\gamma)$ space. Under the similarity group $(h_\alpha, h_\beta, h_\gamma)$ is invariant only up to positive scalar multiples, so again the signs of h_α, h_β, h_γ provide a convenient notation for the finer partition of IA_1 into subtypes IA_1^{+++}, IA_1^{++0}, and so on. The IA_2 type can be regarded in exactly the same way, i.e. the subtypes correspond to subsets of $(h_\alpha, h_\beta, h_\gamma)$ space with the proviso that there is a coincidence $h_\alpha = h_\beta$ or $h_\beta = h_\gamma$: we shall therefore use the same notation as for IA_1, save that we will bracket together the signs corresponding to the coincident pitches. Thus $IA_2^{(++)+}$ represents the subtype with h_α, h_β, h_γ all positive and $h_\alpha = h_\beta$. By Lemma 2.2 the equivalence type for a 3-system of type IB_0 is determined by the pitch h and the absolute value of the parameter p appearing in the normal form. For $p \neq 0$ we use the sign of h to distinguish IB_0^+, IB_0^0 and IB_0^-: and for $p = 0$ we distinguish $IB_0^{0,+}$, $IB_0^{0,0}$, $IB_0^{0,-}$ in the same way. For the types IIA, IIB and IIC we distinguish subtypes according to the sign of the common pitch h of all the finite screws in the system. And finally, IC^0 and IID are orbits so have no finer subtypes.

Table VII. The finer partition of 3-systems

Type	Finer subtypes
IA_1	IA_1^{+++}, IA_1^{++0}, IA_1^{++-}, IA_1^{+0-}, IA_1^{+--}, IA_1^{0--}, IA_1^{---}
IA_2	$\mathrm{IA}_2^{(++)+}$, $\mathrm{IA}_2^{+(++)}$, $\mathrm{IA}_2^{(++)0}$, $\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(++)-}$, $\mathrm{IA}_2^{+(--)}$, $\mathrm{IA}_2^{(00)-}$, $\mathrm{IA}_2^{0(--)}$, $\mathrm{IA}_2^{(--)-}$, $\mathrm{IA}_2^{-(--)}$
IB_0	IB_0^{+}, IB_0^{0}, IB_0^{-}, $\mathrm{IB}_0^{0,+}$, $\mathrm{IB}_0^{0,0}$, $\mathrm{IB}_0^{0,-}$
IB_3	IB_3^{++}, IB_3^{+0}, IB_3^{+-}, IB_3^{0-}, IB_3^{--}
IC	no finer subtypes
IC^0	no finer subtypes
IIA	IIA^+, IIA^0, IIA^-
IIB	IIB^+, IIB^0, IIB^-
IIC	IIC^+, IIC^0, IIC^-
IID	no finer subtypes

In view of the comments at the end of the previous section, 4-systems are classified (up to equivalence) by their orthogonal 2-systems. We shall therefore extend the notation used for 2-systems to the orthogonal 4-systems. Likewise, 5-systems are classified by their orthogonal 1-systems, so we obtain four types, according as the pitch of the orthogonal screw is positive, negative, zero, or infinity.

4. Unfoldings of Screw Systems

The types of screw systems described in this paper define subsets of the appropriate Grassmannians invariant under the action induced by the adjoint action on $e(3)$. In this section we will show that all these invariant subsets are smooth, so we are justified in referring to them as *strata*. The strata described in Section 2 are the *basic strata*, whilst those described in Section 3 are the *finer strata*.

The main tool in establishing smoothness, specialisations and regularity conditions is the construction of unfoldings. A *p-parameter unfolding* of a screw system S is a germ of an embedding of $\mathbb{R}^p$ into the Grassmannian of screw systems with source O and target S, transverse to the orbit through S. A representative of the image is sometimes known as a *transversal* at S. It is a standard fact [5] that for the questions of smoothness, specialisation and regularity it suffices to work in an unfolding. More precisely, let X be an invariant subset of the Grassmannian, let S be a screw system in X, and let T be a transversal at S: then X is smooth if and only if $X \cap T$ is smooth. Further, let Y be another invariant subset of the Grassmannian: then Y contains the orbit through X in its closure if and only if $Y \cap T$ contains S in its closure, and in that case Y is smooth (in a neighbourhood of X) if and only if $Y \cap T$ is smooth. Finally, assuming Y is smooth and contains X in its closure, Y is Whitney regular over X if and only if $Y \cap T$ is Whitney regular over $X \cap T$. Specializations are determined in Sections 5 and 6 while regularity is considered in Section 7.

By the homogeneity property of the orbits it is sufficient to construct unfoldings at normal forms for screw systems, a straightforward computational task. Moreover, since the orbits of the similarity action are unions of orbits under the action of $E(3)$ induced by the adjoint action, it will be sufficient for the study of both stratifications to find unfoldings only for the $E(3)$ action. For this it is necessary to construct a supplement to the tangent space to an orbit in the Grassmannian. However, since k-systems are most easily represented by a basis of motors in $e(3) \cong \mathbb{R}^6$, that is to say a k-tuple of motors, it is more straightforward to construct unfoldings in $(\mathbb{R}^6)^k$, taking into account the possibility of a change of basis. The appropriate group action is thus that of the direct product $\mathrm{GL}(k) \times E(3)$ on $(\mathbb{R}^6)^k$.

At a given k-system $S = (\mathbf{u}_1; \ldots; \mathbf{u}_k)$ the tangent space is the image of the derivative of the map $\mathrm{GL}(k) \times E(3) \to (\mathbb{R}^6)^k$, hence a subspace of dimension $\leqslant k^2 + 6$. There are k^2 vectors in the tangent space of the form $(\mathbf{0}; \ldots; \mathbf{u}_i; \ldots; \mathbf{0})$ (where the $\mathbf{u}_i$ occurs in the jth position, and $i, j = 1, \ldots, k$) arising from the action of $\mathrm{GL}(k)$. It is a standard part of the theory of Lie groups that for a group G with Lie algebra g, the derivative with respect to $g \in G$ at the identity $e \in G$ of the adjoint mapping Ad_g on a fixed element $\mathbf{u} \in g$ is simply given by the Lie bracket in the Lie algebra, that is $T_e \mathrm{Ad}_g \mathbf{u} = [\mathbf{u}, \cdot]$. Writing $\mathbf{u} = (u, u')$ and $\mathbf{v} = (v, v')$ in $e(3)$, the Lie bracket is given by

$$[\mathbf{u}, \mathbf{v}] = (u \wedge v, u \wedge v' + u' \wedge v).$$

A further six vectors in the tangent space to the orbit are found by allowing $\mathbf{v}$ to range over the standard basis for $\mathbb{R}^6$. It is then, in principle, a routine calculation to find the rank of a $6k \times (k^2 + 6)$ matrix and to choose a basis for a supplement to its image.

Tables VIII–XI give transversals for the normal forms of the 1-, 2- and 3-systems appearing in Tables I, II and IV. Unfolding parameters are denoted systematically by the symbols a, b, $c, \ldots$. The third column of each table gives the codimension, namely the number of unfolding parameters minus the number of moduli in the normal form. The final column of each table gives explicit equations defining the stratum in the transversal, obtained by checking whether the general screw $\sum_{i=1}^{k} \mu_i \mathbf{u}_i(a, b, \ldots)$ in the unfolding satisfies the appropriate conditions. The simplest possible situation is illustrated by 1-systems.

Table VIII. Unfoldings for 1-systems

Stratum	Unfolding	Codim	Equation
h finite	$(1, 0, 0; h + a, 0, 0)$	0	–
h infinite	$(a, b, c; \quad 1, \quad 0, 0)$	3	$a = b = c = 0$

Table IX. Unfoldings for 2-systems

Stratum	Unfolding	Codim	Equation
IA	$(1,0,0;h_\alpha + a,\ 0,\ 0)$ $(0,1,0;\ 0,\ h_\beta + b, 0)$	0	–
IB	$(1,0,0;0,\ 0,\ 0)$ $(0,b,c;1,p+a,0)$	2	$b = c = 0$
IB^0	$(1,0,0;0,0,0)$ $(0,a,b;1,c,d)$	4	$a = b = c = d = 0$
IIA	$(1,0,0;h+a,\ 0,\ 0)$ $(0,1,0;\ c,\ h+b,0)$	2	$a = b,\ c = 0$
IIB	$(1,0,0;h+a,0,0)$ $(0,b,c;\ d,\ 1,0)$	3	$b = c = d = 0$
IIC	$(a,b,c;1,0,0)$ $(d,e,f;0,1,0)$	6	$a = b = c = 0,$ $d = e = f = 0$

Table X. Unfoldings for 3-systems of type I

Stratum	Unfolding	Codim	Equation
IA_1	$(1,0,0;h_\alpha + a,\ 0,\ 0)$ $(0,1,0;\ 0,\ h_\beta + b,\ 0)$ $(0,0,1;\ 0,\ 0,\ h_\gamma + c)$	0	–
IA_2	$(1,0,0;h_\alpha + a,\ 0,\ 0)$ $(0,1,0;\ 0,\ h_\beta + b,\ d)$ $(0,0,1;\ 0,\ d,\ h_\beta + c)$	2	$b = c,\ d = 0$
IB_0	$(1,0,0;h,\ 0,\ 0)$ $(0,1,0;0,h+a,\ 0)$ $(0,0,c;1,\ 0,\ p+b)$	1	$c = 0$
IB_3	$(1,0,0;h_\alpha + a,\ 0,\ 0)$ $(0,1,0;\ 0,\ h_\beta + b, 0)$ $(0,0,c;\ d,\ e,\ 1)$	3	$c = d = e = 0$
IC	$(1,0,0;0,0,\ 0)$ $(0,b,c;0,1,\ 0)$ $(0,d,e;1,0,p+a)$	4	$b = c = d = e = 0$
IC^0	as IC, $p = 0$	5	$a = b = c = 0,$ $d = e = 0$

Table XI. Unfoldings for 3-systems of type II

Stratum	Unfolding	Codim	Equation
IIA	$(1,0,0;h+a,\ d,\ e)$ $(0,1,0;\ d,\ h+b,\ f)$ $(0,0,1;\ e,\ f,\ h+c)$	5	$a=b=c$, $d=e=f=0$
IIB	$(1,0,0;h+a,\ 0,\ 0)$ $(0,1,0;\ f,\ h+b,0)$ $(0,0,c;\ d,\ e,\ 1)$	5	$a=b$, $c=d=e=f=0$
IIC	$(1,0,0;h+a,0,0)$ $(0,b,c;\ d,\ 1,0)$ $(0,e,f;\ g,\ 0,1)$	6	$b=c=d=0$, $e=f=g=0$
IID	$(a,b,c;1,0,0)$ $(d,e,f;0,1,0)$ $(g,h,i;0,0,1)$	9	$a=b=c=0$ $d=e=f=0$, $g=h=i=0$

EXAMPLE 1. The unfolding of the normal form for a screw of infinite pitch is $(a,b,c;1,0,0)$. Such a screw has infinite pitch if and only if $a=b=c=0$, representing the origin in the unfolding parameter space. And it has finite pitch f if and only if $(a^2+b^2+c^2)h=a$. For $h\neq 0$ this is the sphere through the origin centred at the point $(1/2h,0,0)$ on the a-axis; and for $h=0$ it is the plane $a=0$. Thus we obtain a good picture of the pitch quadrics Q_h close to the 2-plane Q_∞. Figure 2 illustrates this by means of a cross-section of the image of the unfolding by $c=0$. It is also clear that the plane $a=0$ corresponds to the finer stratum of screws of pitch zero while the half-space $a>0$ (respectively, $a<0$) is the intersection of screws of positive (respectively, negative) pitch with the unfolding.

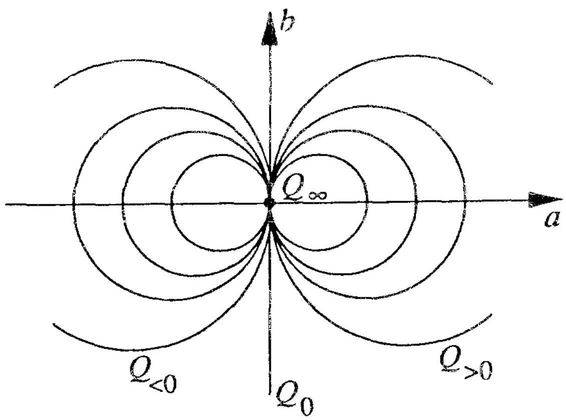


Fig. 2. Cross-sections of the unfolding of a screw of infinite pitch.

EXAMPLE 2. We will check that the 2-systems of type IIA form a smooth submanifold of the Grassmannian having codimension 2. According to Table IX a transversal at the normal form $(1,0,0;h,0,0)$, $(0,1,0;0,h,0)$ is given by $(1,0,0;h+a,0,0)$, $(0,1,0;c,h+b,0)$, where the unfolding parameters a,b,c are small. The general screw in the resulting 2-system is an arbitrary linear combination of these two basis screws, i.e.

$$(\mu_1,\mu_2,0;\ \mu_1(h+a)+\mu_2 c, \mu_2(h+b), 0).$$

A 2-system in the unfolding will be of type IIA if and only if the pitch of the general screw in that 2-system is constant, i.e. independent of μ_1, μ_2. By computation, the pitch is

$$k=\frac{\mu_1^2(h+a)+\mu_1\mu_2 c+\mu_2^2(h+b)}{\mu_1^2+\mu_2^2}$$

which is constant if and only if $a=b$ and $c=0$.

This example illustrate a general phenomenon, common to screw systems of all dimensions. In every case the stratum appears as a *linear* subspace of the transversal and, hence, (by the introductory remarks) is smooth.

It should be noted that strata from the finer partitions of Tables V, VI and VII may have higher codimension, according to the number of moduli set to zero. For example the 3-system subtype $\mathrm{IA}_2^{+(00)}$ has codimension 3 and appears in its unfolding as $b=c=d=0$.

In the final section of this paper, we will use the fact that the codimension of a screw system equals that of its orthogonal system. Indeed the mapping that takes a d-system S to its orthogonal $(6-d)$-system $S^\perp$, yields a diffeomorphism between the Grassmannians of d- and $(6-d)$-systems, which (in view of the comments at the end of Section 2) preserves orbits, under the actions induced by the adjoint actions. In particular, the orbits of S and $S^\perp$ have the same dimension.

5. Specialisations for 2-Systems

For the purposes of this paper a stratum Y *specialises* to a stratum X, and we write $Y \to X$, when the frontier ∂Y of Y contains X. Since the strata are all semialgebraic sets (a point we will expand upon in Section 7) a necessary condition for Y to specialise to X is that $\operatorname{codim} Y < \operatorname{codim} X$, so that numerous possibilities can be excluded solely on codimensional grounds. Note also that the relation of specialisation is transitive. The problem of specialisation is to determine all pairs of strata X, Y for which $Y \to X$. A practical motivation is to determine which screw-systems are realised under small perturbations of a given one. Suppose $Y \to X$. Since the strata are semialgebraic the Curve Selection Lemma [14] ensures (in principle) that for any point $x \in X$ there exists an analytic curve p: $[0,\varepsilon) \to X \cup Y$ such that $p(t) \in Y$ for $t>0$, and $p(0)=x$. As we observed in the previous section,

Table XII. Specialisations for basic 2-systems

IA $\longrightarrow$ IB	$(0; t, 0)$
IA $\longrightarrow$ IIA	$(t, 0; 0)$
IB $\longrightarrow$ IB^0	$(0, 0, t, 0)$
IB $\longrightarrow$ IIB	$(0; 0, 0, t)$
IB^0 $\longrightarrow$ IIC	$(0, t, 0, 0, 0, 0)$
IIA $\longrightarrow$ IIB	$(t, b(t^2), c(t^2), 0)$
IIB $\longrightarrow$ IIC	$(t, 0, 0, 0, 0, 0)$

it suffices to establish specialisations in transversals to normal forms. Thus we shall establish specialisations $Y \to X$ by exhibiting explicit analytic curves of the form just mentioned in the transversal, independent of the choice of $x \in X$. It is worth making the general observation that certain specialisations are not possible on continuity grounds. For instance systems of type II cannot specialise to systems of type I, systems of type B cannot specialise to systems of type A, and systems of type C cannot specialise to systems of type B.

Table XII lists specialisations $Y \to X$ for the basic stratification of 2-systems, together with explicit analytic paths into the transversal of the lower dimensional stratum X at its normal form (see Table IX) realising the specialisation. In each case the curve has been exhibited in the space of unfolding parameters $(a, b, c, \ldots)$ with the origin $(0, 0, 0, \ldots)$ as the limit. The components have been grouped in a way which reflects the appearance of the moduli in the unfoldings. For instance the unfolding parameters for the IB stratum are written $(a; b, c)$ indicating that a unfolds the pitch modulus. The definitions of the analytic functions $b(t^2)$ and $c(t^2)$ are given in Example 1 below: for all the remaining cases the reader is left to check that the exhibited curves realise the corresponding specialisations.

EXAMPLE 3. Consider specialisations to IIB. From Table VIII, the general screw in the unfolding has the form

$$\$ = \big(\lambda, \mu b, \mu c; \lambda(h + a) + \mu d, \mu, 0\big).$$

We ask for the conditions on the unfolding parameters a, b, c, d for this 2-system to be of type IIA. For the 2-system to be of type A, at least one of b, c must be nonzero. The pitch of the general screw in the system is

$$k = \frac{\lambda^2(h + a) + \lambda\mu d + \mu^2 b}{\lambda^2 + \mu^2(b^2 + c^2)}.$$

And for the system to be of type II we require the pitch to be constant, i.e. independent of λ, μ. Clearing denominators, and setting the coefficients of λ^2, $\lambda\mu$, μ^2

equal to zero, we see that we require

$$d = 0, \qquad h + a - k = 0, \qquad b - kb^2 - kc^2 = 0.$$

Choosing $d = 0$, this requires $(h + a)(b^2 + c^2) - b = 0$, defining a cubic surface in (a, b, c) space. For fixed a this defines a circle through the origin in the (b, c) plane, possibly degenerating to a point. The circle has the rational parametrisation

$$b(s) = \frac{s^2}{(h + a)(1 + s^2)},$$
$$c(s) = \frac{s}{(h + a)(1 + s^2)}.$$

Note that for sufficiently small $a \neq 0$ we have $h + a \neq 0$, no matter what the value of h. Set $a = t$ and $s = t^2$ to realise the required specialisation.

On the basis of the above we obtain the specialisation diagram for basis 2-systems (Figure 3).

Specialisations for the finer strata may be found in the same way. It is necessary to determine the signs of principal pitches in the unfoldings. The existence of the stratification preserving involution $(u, v) \mapsto (u, -v)$ in the screw space, which has the effect of reversing the sign of the pitch, reduces the number of calculations and is revealed in the symmetry of the specialisation diagram (Figure 4).

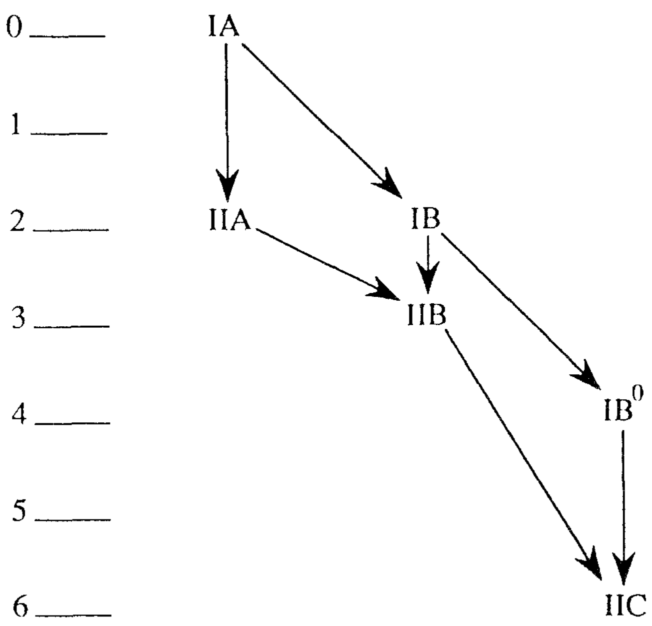


Fig. 3. Diagram of basic 2-systems.

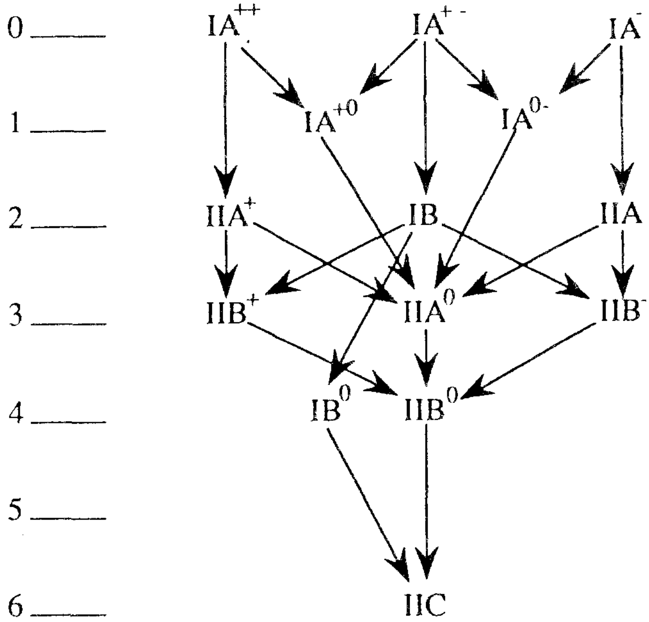


Fig. 4. Diagram of finer 2-systems.

6. Specialisations for 3-Systems

Specialisations for 3-systems can be established in much the same way as for 2-systems. The results appear in Table XIII in the same format as for 2-systems, that is with an analytic path into the transversal representing the specialisation, valid for every choice of the moduli in the lower stratum, and the unfolding parameters grouped in a natural way.

In establishing the table of specialisations, we may begin by ruling out certain cases. As for 2-systems, type II systems cannot specialise to type I nor can type B specialise to A, C to B or A, nor D to C, B or A. Moreover since the type of the associated pencil of conics (as in Table III) is an invariant, the lack of a specialisation among pencils rules out the corresponding specialisation for 3-systems (see Figure 5). Thus IA_2 cannot specialise to IA_1 nor IB_3 to IB_0.

EXAMPLE 4. Consider specialisations to IC. The general screw for a 3-system in the unfolding of IC is

$$\$ = \big(\lambda, \mu b + \nu d, \mu c + \nu e;\ \nu, \mu, \nu(p+a)\big).$$

We recognise types via the associated pencil. To realise the specialisation $IA_2 \to IC$, we must look for pencils of type 22b (see Table III). For this, it is sufficient to ensure the 3-system has a nonsingular pencil and meets some Q_h in a repeated line. We can simplify the search for such 3-systems by choosing $a = 0$. The symmetric matrix associated with the corresponding pencil of conics is:

Table XIII. Specialisations of basic 3-systems

$IA_2 \longleftarrow IA_1$	$(0,0,t;0)$		
$IB_0 \longleftarrow IA_1$	$(0,0;t)$		
$IB_3 \longleftarrow IA_1$	$(0,0;t,0,0)$		
$IB_3 \longleftarrow IA_2$	$(0,0;c(t)t,0,c(t))$ $\quad c(t)=\frac{(h_\beta-h_\alpha)t}{1+h_\alpha(h_\beta-h_\alpha)t^2}$		
$IB_3 \longleftarrow IB_0$	$(0,0;0,0,t)$		
$IC \longleftarrow IA_1$	$(0;t,0,0,t)$		
$IC \longleftarrow IA_2$	$(0;t,t,\gamma(p)t,\delta(p)t)$ $\quad \gamma(p)=\frac{1}{10}\left(3p+\sqrt{9p^2+10}\right)$, $\quad \delta(p)=2p-3\gamma(p)$		
$IC \longleftarrow IB_0$	$(0;t,0,0,0)$		
$IC \longleftarrow IB_3$	$(0;0,0,0,t)$		
$IC^0 \longleftarrow IA_1, IA_2, IB_0, IB_3$ as IC			
$IC^0 \longleftarrow IC$	$(t;0,0,0,0)$		
$IIA \longleftarrow IA_1$	$(0,t,2t;0,0,0)$	$IIA \longleftarrow IA_2$	$(0,t,t;0,0,0)$
$IIB \longleftarrow IA_1$	$(t,0;t,0,0,0)$	$IIB \longleftarrow IA_2$	$(0,0;t,0,0,0)$
$IIB \longleftarrow IB_0$	$(0,0;0,t,0,0)$	$IIB \longleftarrow IB_3$	$(t,0;0,0,0,0)$
$IIC \longleftarrow IA_1$	$(0;t,0,0,2t,0,0)$	$IIC \longleftarrow IA_2$	$(0;t,0,0,t,0,0)$
$IIC \longleftarrow IB_0$	$(0;t,0,0,0,0,t)$	$IIC \longleftarrow IB_3$	$(0;t,0,0,0,0,0)$
$IC \longleftarrow IC$	$(0;t,0,0,0,0,0)$		
$IIC \longleftarrow IIA$	$(t;b(t)t^2,-b(t),b(t),b(t)t^2,0,0)$ $\quad b(t)=\frac{t}{(h+t)(1+t^4)}$		
$IID \longleftarrow IA_1$	$(t,0,0,0,2t,0,0,0,3t)$	$IID \longleftarrow IA_2$	$(t,0,0,0,t,0,0,0,2t)$
$IID \longleftarrow IB_0$	$(t,0,0,0,t,0,t,0,0)$	$IID \longleftarrow IB_3$	$(t,0,0,t,t,0,0,0,0)$
$IID \longleftarrow IC$	$(t,0,0,t,0,0,0,0,0)$	$IID \longleftarrow IC^0$	$(0,0,0,t,0,0,0,0,0)$
$IID \longleftarrow IIA$	$(t,0,0,0,t,0,0,0,t)$	$IID \longleftarrow IIB$	$(t,0,0,0,t,0,0,0,0)$
$IID \longleftarrow IIC$	$(t,0,0,0,0,0,0,0,0)$		

$$
\begin{tikzcd}
1b \arrow[rr] \arrow[dr] & & 22b \arrow[dr] & & \\
& 2b \arrow[rr] & & 12a \arrow[r] & 11
\end{tikzcd}
$$

Fig. 5. Diagram for pencils of conics.

$$
\begin{pmatrix} 2h & 0 & -1 \\ 0 & 2h(b^2+c^2)-2b & h(bd+ce)-cp-d \\ -1 & h(bd+ce)-cp-d & 2h(d^2+e^2)-2ep \end{pmatrix}.
$$

The pencil is singular only if the determinant vanishes identically. By examining the coefficient of h^3, and the constant term, necessary conditions are found to be either

$b = c = 0$ or $b = d = e = 0$. In both cases the corresponding system must be of type B or C. Avoiding these possibilities, the matrix has rank 1 (and so intersects a Q_h in a repeated line) precisely when

$$0 = h(b^2 + c^2) - b,$$

$$0 = h(bd + ce) - cp - d,$$

$$0 = 4h^2(d^2 + e^2) - 4eph - 1.$$

Eliminating h from these equations:

$$0 = bce + (2b^2 + c^2)d - cp(b^2 + c^2),$$

$$0 = 4b^2d^2 + \left[2be - p(b^2 + c^2)\right]^2 - (b^2 + c^2)(1 + p^2),$$

which for fixed b, c define a line and a circle in the (d, e) plane. Setting $b = c = t \neq 0$ they may be solved simultaneously for d and e in terms of t.

Notice that IC^0 cannot specialise to IC. Although these types are not distinguished by pencils of conics, the continuity of the square invariant (Lemma 2.3) for IC systems ensures this. Moreover, IC^0 does not specialise to type IIC. It is evident that for a 3-system in the unfolding of IIC to be of type C requires $b = c = e = f = 0$. Under that assumption, the general screw in the system is given by

$$\left(\lambda, 0, 0;\ \lambda(h + a) + \mu d + \nu g, \mu, \nu\right)$$

with axis $(\lambda, 0, 0; 0, \mu, \nu)$. The axes thus span a plane whereas a IC^0 system has only a line of axes. We may now construct from Table XIII the specialisation diagram for 3-systems (Figure 6).

The finer stratification for 3-systems has a great many classes so a full specialisation diagram is unwieldy to construct. We are however able to give a complete answer, represented in two stages. Initially all substrata of a given codimension are amalgamated into an *intermediate* stratification listed in Table XIV and a specialisation diagram constructed for this (see Figure 7). Then each arrow in that diagram is expanded into a subdiagram showing all possible specialisations between finer strata (Table XV). This can be done in such a way that whenever a specialisation exists between intermediate strata via the transitivity property, all corresponding specialisations in the finer stratification can be found by applying transitivity to the subdiagrams.

For type A systems progress can be made in establishing specialisations by considering the principle pitches. These are the zeros of a cubic polynomial (the determinant of a symmetric 3×3 matrix), so are continuous functions of the coefficients. This criterion is sufficient to determine all specialisations amongst 3-systems of type A. It follows that for each pair of strata the moduli are continuous functions on the set of normal forms. This idea can be extended to certain other pairs of strata by working in an unfolding of the lower stratum at a normal form.

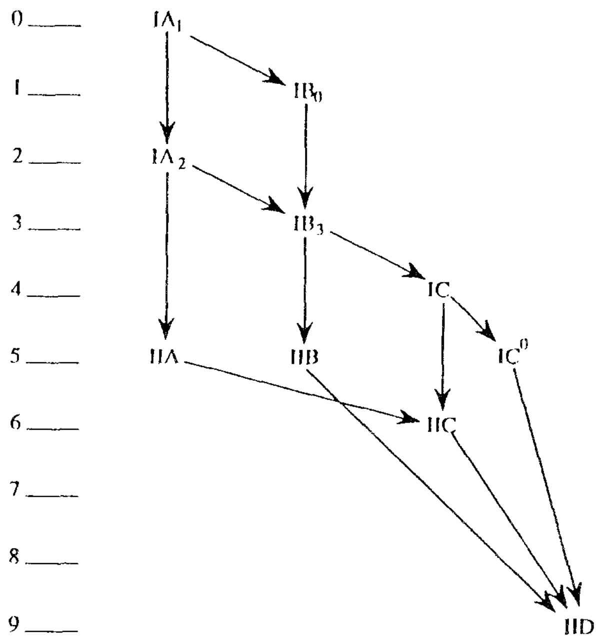


Fig. 6. Diagram of basic 3-systems.

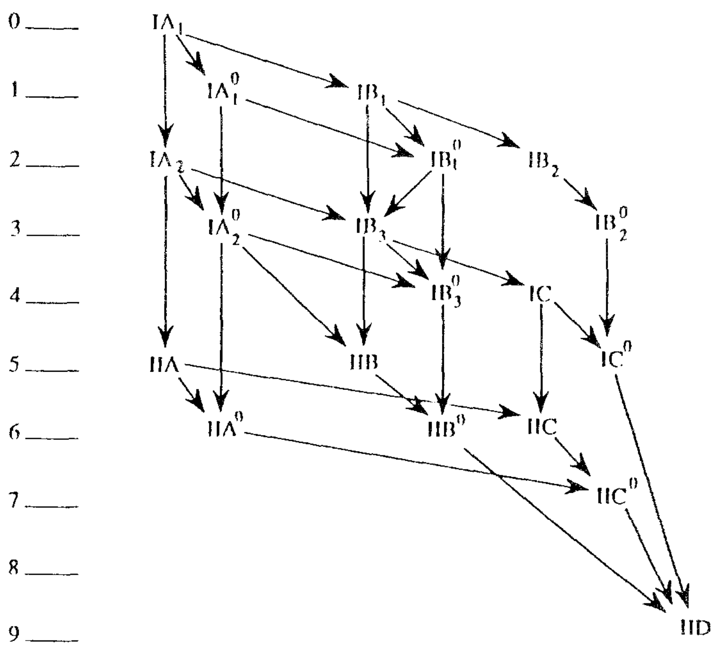


Fig. 7. Diagram of intermediate 3-systems.

Table XIV. Intermediate strata for 3-systems

Primary type	Intermediate subtypes	Finer subtypes
IA_1	$\mathrm{IA}_1^{(\pm)}$	IA_1^{+++}, IA_1^{++-}, IA_1^{+--}, IA_1^{---}
	IA_1^0	IA_1^{++0}, IA_1^{+0-}, IA_1^{0--}
IA_2	$\mathrm{IA}_2^{(\pm)}$	$\mathrm{IA}_2^{(++)+}$, $\mathrm{IA}_2^{+(++)}$, $\mathrm{IA}_2^{(++)-}$, $\mathrm{IA}_2^{+(--)}$, $\mathrm{IA}_2^{(--)-}$, $\mathrm{IA}_2^{-(--)}$
	IA_2^0	$\mathrm{IA}_2^{(++)0}$, $\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(00)-}$, $\mathrm{IA}_2^{0(--)}$
IB_0	$\mathrm{IB}_1^{(\pm)}$	IB_0^+, IB_0^-
	IB_1^0	IB_0^0
	$\mathrm{IB}_2^{(\pm)}$	$\mathrm{IB}_0^{0,+}$, $\mathrm{IB}_0^{0,-}$
	IB_2^0	$\mathrm{IB}_0^{0,0}$
IB_3	$\mathrm{IB}_3^{(\pm)}$	IB_3^{++}, IB_3^{+-}, IB_3^{--}
	IB_3^0	IB_3^{+0}, IB_3^{0-}
IC	no finer subtypes	
IC^0	no finer subtypes	
IIA	$\mathrm{IIA}^{(\pm)}$	IIA^+, IIA^-
	IIA^0	IIA^0
IIB	$\mathrm{IIB}^{(\pm)}$	IIB^+, IIB^-
	IIB^0	IIB^0
IIC	$\mathrm{IIC}^{(\pm)}$	IIC^+, IIC^-
	IIC^0	IIC^0
IID	no finer subtypes	

EXAMPLE 5. For any 3-system of type A in the unfolding of a IB_0 system (in normal form) the principal pitches are $h + a$ and

$$\frac{1}{2c}\left[(p+b) \pm \sqrt{(p+b)^2+1}\right].$$

In particular, the modulus of the IB_0 system is the limit of a principal pitch of any sequence of IA systems converging to it, while the other two principal pitches must be nonzero, and of opposite sign. It follows that neither IA_1^{+++} nor IA_1^{---} can specialise to IB_0.

Where an arrow from the diagram of intermediate strata (Figure 7) is not listed in Table XV, all specialisations between finer strata are possible. Note that as for 2-systems all the subdiagrams are symmetric with respect to sign reversal of the moduli.

Table XV. Specialisations of finer 3-systems

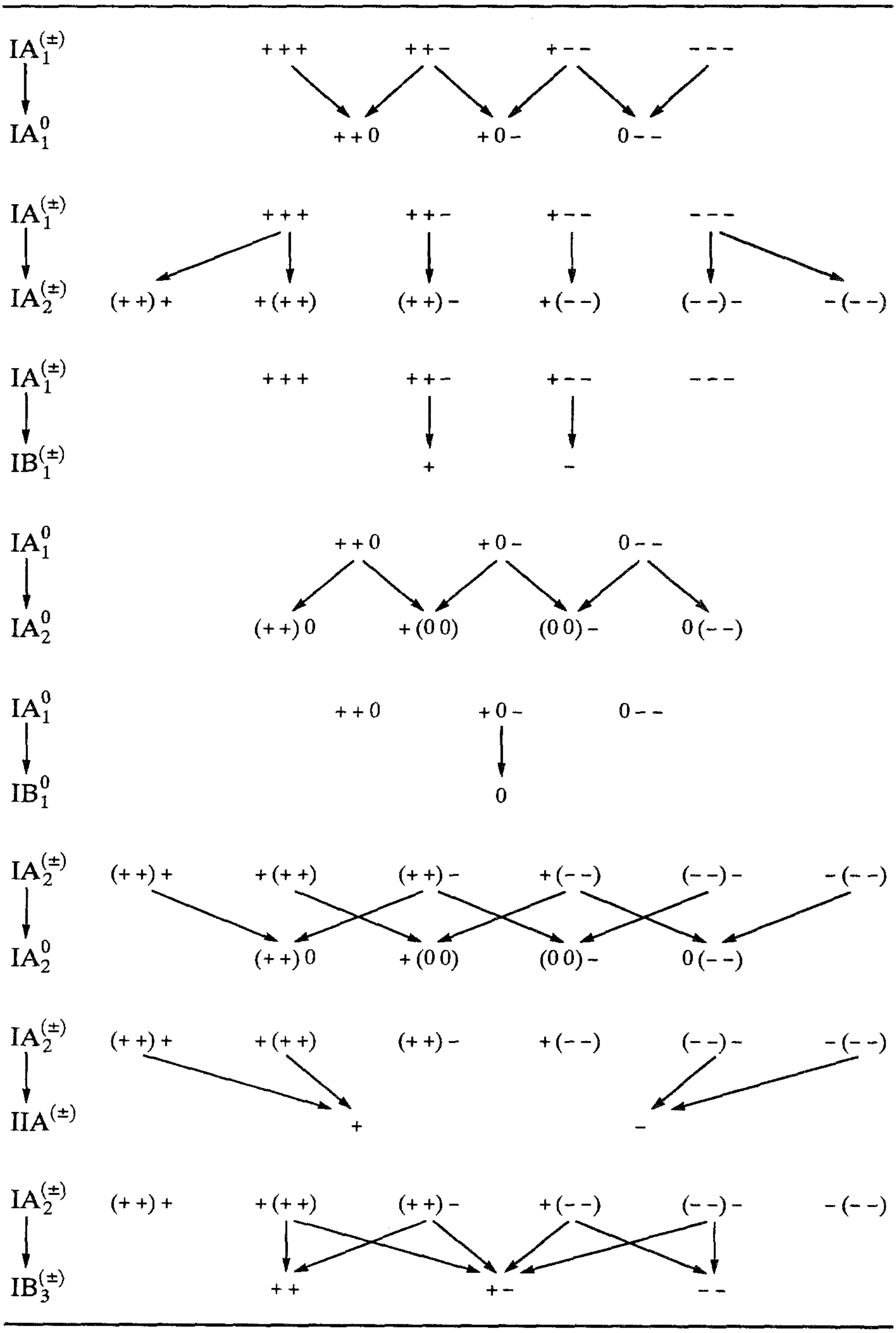
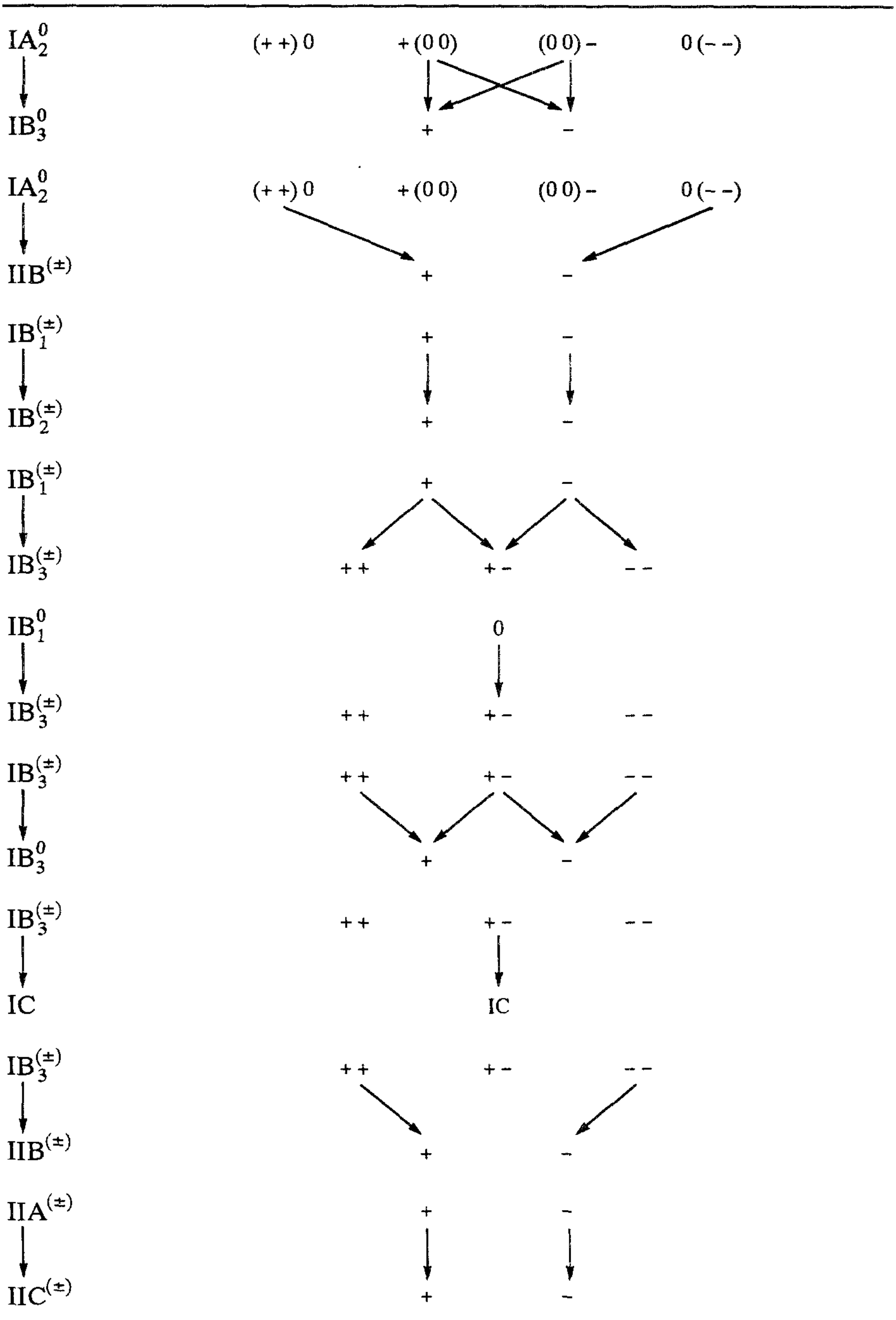


Table XV. Continued



7. Regularity of the Finer Stratifications

In this section we verify that the finer partitions discussed in this paper stratify 2- and 3-systems, and are (Whitney) regular. We can then use the Thom Transversality lemma to make simple deductions for 'generic' motions of 3-space. A precise statement of the regularity conditions can be found in [7]. However we do not need direct recourse to the definitions, relying instead on standard results and techniques for establishing regularity.

A preliminary observation is that the orbits are semialgebraic, since (clearly) the adjoint action is algebraic [6]. (Incidentally, this applies both to the *Euclidean* action induced by the adjoint action of the Euclidean group, and the *similarity* action induced by the adjoint action of the similarity group.) Moreover, the same basic argument shows that the strata are semialgebraic: one just uses the extra fact that for a given normal form, the set of moduli is itself (trivially) semialgebraic. Then, if X, Y are strata with $Y \to X$, and X is an orbit, it follows from standard results [7] that Y is regular over X. These observations alone establish that the finer partition of 2-systems is a regular stratification. The only specialisation in which the lower stratum is not a (similarity) orbit is $\mathrm{IA}^{+-} \to \mathrm{IB}$: but in that case regularity is trivial since the upper stratum is open. However for 3-systems we have to work a little harder. Table XVI lists the specialisations for the finer strata, where either the lower stratum fails to be a (similarity) orbit, or the upper stratum fails to be open.

As we have already noted above, it is sufficient to establish regularity in an unfolding. Two techniques suffice for us to establish regularity in unfoldings. The first is when the lower stratum is a submanifold of its union with each of a finite number of open subsets of the upper stratum: regularity then follows immediately from the observation that the tangent space to a smooth manifold varies smoothly across the manifold. (In particular, this is the case when the upper stratum appears linearly in the unfolding.) The second technique applies when the equations defining a stratum within an unfolding are independent of the moduli in the lower stratum. In that case the stratification is locally a product of the moduli space with a semialgebraic stratification in which the lower stratum is a point, the latter being regular. It then follows from a standard lemma [7, p. 12] that the original stratification is regular.

Example 6 is a typical application of the first technique, which establishes regularity for all pairs of strata in Table XVI, except for IB_0^+, IB_0^- over IC (dealt with in Example 7 below via the second technique).

Table XVI. Specialisations of some finer strata

IB_0^0	IA_1^{+0-}, IB_0^+, IB_0^-
IB_3^{++}	$\mathrm{IA}_2^{+(++)}$, $\mathrm{IA}_2^{(++)-}$, IB_0^+
IB_3^{+-}	$\mathrm{IA}_2^{+(++)}$, $\mathrm{IA}_2^{(++)-}$, $\mathrm{IA}_2^{+(--)}$, $\mathrm{IA}_2^{(--)-}$, IB_0^+, IB_0^0, IB_0^-
IB_3^{--}	$\mathrm{IA}_2^{+(--)}$, $\mathrm{IA}_2^{(--)-}$, IB_0^-
IC	IB_3^{+-} and all those specialising to it

EXAMPLE 6. Consider the specialisations from the substrata of IA_2 to IB_3^{+-}. We will use the first technique to establish regularity. First, we require equations for the upper stratum in the unfolding. As in Example 4 of Section 6, a IA_2 system can be identified by its associated pencil of conics having type 22b. In particular, the quadratic form associated with the general screw in the system must have rank 1 for some value of the pitch k. This gives rise to six equations, one for each 2×2 minor of the 3×3 matrix:

$$(h_\alpha + a - k)(h_\beta + b - k) = 0,$$
$$(h_\alpha + a - k)e = 0,$$
$$(h_\beta + b - k)d = 0,$$
$$de = 0,$$
$$c(h_\alpha + a - k)(1 - ck) - d^2 = 0,$$
$$c(h_\beta + b - k)(1 - ck) - e^2 = 0.$$

We claim that the variety defined by these equations is nonsingular on the lower stratum. For $d = e = 0$ there are no solutions for small values of b, c: thus $d = 0$, $e \neq 0$ and $k = h_\alpha$, or $d \neq 0$, $e = 0$ and $k = h_\beta$. In the former case, the system of equations is equivalent to:

$$c(h_\alpha - h_\beta + a - b)(1 - ch_\alpha - ac) - e^2 = 0$$

which is of the form $F(a, b, c, e) = 0$ where F is a polynomial of degree four. Along the IB_3^{+-} stratum, given by $c = d = e = 0$, the derivative

$$\frac{\partial F}{\partial c}(a, b, 0, 0) = h_\alpha - h_\beta + a - b$$

which, given $h_\alpha \neq h_\beta$, is nonzero for small a, b. A similar argument applies to the latter case. Thus, each of the (open) substrata of IA_2 is nonsingular along the lower stratum, establishing the claim.

EXAMPLE 7. A necessary condition for a type B system in the unfolding of IC is $be - cd = 0$ and at least one of b, c, d, e should be nonzero to avoid type C. One can deduce from consideration of the associated pencils of conics that systems of type IB_3 are given by $b = c = d = 0$ and $e \neq 0$. Thus the (basic) stratum IB_0 appears in the subspace $a = 0$ of the unfolding as a semialgebraic set. Within this the (finer) substrata IB_0^+, IB_0^- are open and are thus product stratifications (with the e-axis), hence regular. The substratum IB_0^0 requires the intersection with the pitch quadric Q_0 to be a singular conic, from which it may be derived that $e = 0$. This, together with the initial equation, gives $c = 0$, $d \neq 0$ or $c \neq 0$, $d = 0$ which may be seen to be regular over IC by either technique.

8. Generic Motions

That brings us to the question of which screw systems can appear 'generically' for spatial motions. To this end we recall some standard facts. Let M be a smooth manifold of dimension m. Any motion μ: $M \to E(n)$, yields a 1-jet extension $j^1\mu$: $M \to J^1(M, E(n))$. For each possible natural number k let $\mathcal{S}_k$ be a (regular) stratification of the submanifold Σ^k of jets of corank k in $J^1(M, E(n))$, and let $\mathcal{S}$ be the union of the $\mathcal{S}_k$. By the Thom Transversality Lemma we have the following

PROPOSITION. *The m-dimensional motions* μ: $M \to E(n)$ *with* $j^1\mu$ *transverse to* $\mathcal{S}$ *form a residual set in* $C^\infty(M, E(n))$, *endowed with the Whitney* C^∞ *topology.*

We wish to apply this proposition to the case $p = 3$. For $1 \leqslant k \leqslant 6$ we have described (regular) stratifications of the Grassmannian of k-dimensional subspaces of the Lie algebra $e(3)$, namely the finer stratifications. These stratifications induce (regular) stratifications $\mathcal{S}_k$ of Σ^k. A motion μ is 1-*generic* when it is transverse to the resulting stratification $\mathcal{S}$. By the proposition, the 1-generic motions form a residual subset of $C^\infty(M, E(3))$, and only strata of codimension $\leqslant m$ can be met transversely. The simplest situation is provided by $m \leqslant 3$, when only 1-jets of maximal rank m can be met transversely. Note that in these cases the 1-generic motions form an open and dense subset of $C^\infty(M, E(3))$, in view of the regularity of the finer stratifications. Explicitly, we have the following cases.

Case $m = 1$. A 1-generic one-dimensional motion of space is an embedding, exhibiting only 1-systems of codimension $\leqslant 1$ transversely: thus only the strata of positive, zero and negative pitches can appear transversely. In particular, 1-systems of pitch zero can only occur at isolated points of M. (This result appeared in [2].)

Case $m = 2$. A 1-generic two-dimensional motion of space is an embedding, exhibiting only 2-systems of codimension $\leqslant 2$ transversely, namely

codim 0: IA^{++}, IA^{+-}, IA^{--}

codim 1: IA^{+0}, IA^{0-}

codim 2: IIA^{+}, IB, IIA^{-}

Two-systems of codimension 1 occur only along curves and those of codimension 2 at isolated points on those curves.

Case $m = 3$. A 1-generic three-dimensional motion of space is an immersion, exhibiting only 3-systems of codimension $\leqslant 3$ transversely, namely

codim 0: IA_1^{+++}, IA_1^{++-}, IA_1^{+--}, IA_1^{---}

codim 1: IA_1^{++0}, IA_1^{+0-}, IA_1^{0--}, IB_1^{+}, IB_1^{-}

codim 2: $IA_2^{(++)+}$, $IA_2^{+(++)}$, $IA_2^{(++)-}$, $IA_2^{+(--)}$, $IA_2^{(--)-}$, $IA_2^{-(--)}$, IB_1^0, IB_2^+, IB_2^-

codim 3: $IA_2^{(++)0}$, $IA_2^{+(00)}$, $IA_2^{(00)-}$, $IA_2^{0(--)}$, IB_3^{++}, IB_3^{+-}, IB_3^{--}, IB_2^0

Three-systems of codimension 1 occur on surfaces, those of codimension 2 along curves within those surface and those of codimension 3 at isolated points on those curves.

We finish this study by raising a problem. What are the conditions for the 1-generic screw-systems (described above) to be assumed transversely? The condition depends solely on the 2-jet of the motion, so raises the interesting question of studying *second order* invariants of motions. So far as we are aware this area of kinematics is untouched.

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