Proceedings of
A Symposium Commemorating the
Legacy, Works, and Life of
Sir Robert Stawell Ball
Upon the 100th Anniversary of
A Treatise on the Theory of Screws
July 9-11, 2000
University of Cambridge, Trinity College

Screw Systems, Singular Trajectories and Darboux-Type Motions

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Abstract

The instantaneously singular trajectories of points of a rigid body, able to move with several degrees of freedom, are determined by the associated screw system. We present a classification of screw systems based on a Lie group representation of motions and deduce the corresponding forms of the instantaneous singular sets. The idea of a generic property of motions is discussed in this context. The analysis of a particular class of motions, those of Darboux–type determined by constraining points of a rigid body to lie on surfaces, shows that the generic approach may fail for significant examples, yet this has practical advantages in the design of mechanisms.

AMS 1991 Classification: 70B10, 57R45

1 Introduction

Singularities of point trajectories of rigid body motions abound in theory and practice. The instantaneous centre of rotation for a continuous (or, more strictly, differentiable) planar motion provides the simplest and most familiar example for kinematicians. The loci of instantaneous centres, the fixed and moving centrodes, play a central role in understanding planar motions. Their practical significance arises in many mechanisms from steam engines to robot arms.

The trajectory under the motion of the instantaneous centre is singular in the sense that *at that instant* its derivative vanishes. Typically, the trajectory exhibits a cusp, such as the trajectory of a contact point in cycloidal motion. We are here assuming that the motion is parametrised by a single real variable, t, most easily interpreted as time. However singular trajectories are equally important for motions with several degrees of freedom (dof). Now the set that parametrises the motion is a higher–dimensional set. One way to approach such motions is to consider all the 1 dof motions that are included in them, but in many ways it is preferable to treat the motion as a single entity. A trajectory of such a motion is singular if the rank of its differential (Jacobian) falls below its maximum possible value. For 2 dof planar motions the singular trajectories typically form a line at each instant [11], a result known to Blaschke and Müller [2, 3] in terms of the instantaneous centres of embedded 1 dof motions.

The situation for spatial motions is somewhat different and ultimately more interesting. Typically the instantaneous motion associated to a 1 dof motion is that of a screw and unless the pitch is zero, there are no singular trajectories. In the special pitch zero case, however, all points along the axis have singularities on their trajectories. For a spatial rigid body motion with several degrees of freedom, the instantaneous motion is described by a screw system. In [5], we showed that the singular trajectories form, at each configuration, a ruled set (possibly empty)—the *instantaneous singular set* or ISS—and that it is determined by the screws of pitch zero in the corresponding screw system. The nature of this set can be determined in terms of the classification of screw systems due originally to Hunt [14] and subsequently given a mathematical basis by Hunt and Gibson [10]. We briefly review the history of this classification in Section 4 and present the results on ISSs in Section 5.

In this paper we have taken the point of view that much is to be gained by using the language of differential topology and singularity theory from pure mathematics. While the terminology is now widely used by physicists, it is less evident in the engineering literature. It incorporates the use of Lie groups in describing motions and also treats the parametrising sets for motions, such as joint spaces and configuration spaces, as *manifolds*. A number of papers in the subject have adopted this approach, for example [23, 24]. Yet it remains a significant obstacle that researchers from different backgrounds develop independent notations for the same underlying ideas. However we hope that our approach is not an impediment to an engineering audience, for in most cases there is a ready translation into familiar ideas. For example, the homogeneous Plücker coordinates traditionally used to describe screws convert simply to coordinates for an element of the Lie algebra of the Euclidean isometry group. We introduce the relevant ideas and results in Section 2. The setting of manifolds enables us to invoke the techniques and results of *singularity theory* and thereby tackle problems related to motions with several degrees of freedom for which no adequate framework previously existed.

One of the guiding principles of singularity theory has been to concentrate on the 'typical'. Thus we seek properties of sets of functions—in our case, motions or trajectories—that are *generic*, that is, are possessed by almost all members of the set in question. Genericity for motions operates at two levels: at the level of the motion itself and at the level of the motion's trajectories. A generic approach was taken by Pai and Leu [23], base on the simple classification of singularity types by corank, described in Section 5, and also considered by Tchoń and Muszyński [25]. However both were concerned only with singularities of the motion itself. We are interested in singularities of the family of trajectories and therefore require a more subtle notion of genericity. The classification of screw systems provides the ideal basis for this refined generic property of motions [8]. The classification of singularity types by corank, on the other hand, then provides a basis for defining generic properties of motions related to their trajectories (though more refined classifications are also available).

These two levels of genericity are connected by a Genericity Theorem which states that the property of a motion that the family of *all* the associated trajectories is generic, is itself a generic property of motions [9]. In other words, among the set of all motions (in a given ambient space and of a given number of degrees of freedom) almost all possess trajectories whose singularity types are of a specified low order of degeneracy. Moreover, the local structure of how the singular trajectories of various classes are organised is known. We discuss genericity in more detail in Section 3. It is this connection between the central geometric object, the motion, and its associated family of trajectories that is evident in the ISSs, and that gives rise to questions which have not previously been addressed to our knowledge.

In principle, this Genericity Theorem is a powerful result, allowing us to deduce valuable qualitative information about general motions. In the theory of mechanisms, however, we are often interested in specific families of motions determined, for example, by a particular robot arm architecture, or a particular form of constraint. There is no guarantee that the Genericity Theorem carries over to this restricted setting. Indeed, experience suggests that it does not—perhaps because practical devices frequently incorporate symmetry or other special features.

To illustrate this we explore, in Section 6, Darboux–type rigid body motions. These are defined by the requirement that each of a finite set of points in the body is constrained to lie on some smooth surface. The case considered by Darboux was where the constraining surfaces are three planes. Darboux–type motions also arise in robotics, where some parallel devices such as shoulder joints and remote centre compliance devices are so configured, the constraining surfaces being spheres. Finally, in Section 7, we show that in the three–constraint case, these motions do display unexpectedly degenerate singular behaviour, but that this is precisely what contributes to their usefulness.

2 Manifolds and motions

The underlying set which parametrises a rigid body motion is usually either a joint space, comprising a list of values for each of a number of joints in series, or else is described by the solution set of a collection of equations, as is typical for motions embodying constraints or closed loops (sometimes characterised as parallel devices). For example, the configurations of a typical 6 dof robot arm are given by sextuples $(\theta_1, \ldots, \theta_6)$ where each θ_i, $i = 1, \ldots, 6$, represents the position of one joint of the arm. In the case of revolute joints, we may prefer to regard θ_i as denoting a point on a circle, denoted S^1, while for a prismatic joint $\theta_i \in \mathbb{R}$, the real line, or some closed interval in $\mathbb{R}$. On the other hand, the configurations of the Gough–Stewart platform, a parallel mechanism, are described by a set of polynomial equations involving the coordinates of the joints and the lengths of the retractable legs [19]. In most configurations of the platform, the leg lengths provide a suitable *local* parametrisation for all nearby configurations, though not for the entire space of configurations, as a given set of leg lengths may correspond to a number of different configurations.

A *differentiable manifold* is a subset M of a Euclidean space $\mathbb{R}^n$ such that there is an integer k and, in some neighbourhood V of each point $x \in M$, an open subset $U \subseteq \mathbb{R}^k$ and a one-to-one parametrisation $\phi : U \to V$. Moreover the transformations between overlapping parametrisations should be differentiable. The number k is the *dimension* of the manifold. A function $f : M \to N$ between (differentiable) manifolds can be represented locally by a function between parametrisations. We are interested in *smooth* functions for which every such local representation is (infinitely often) differentiable. If M and N have dimensions m and n respectively, we say $x \in M$ is a *regular point* of the smooth function f if the differential at x of any parametric representation of f on a neighbourhood of x has maximum rank, i.e. whichever is the smaller of m, n; otherwise x is a *singular point* or *singularity* of f.

Thus the property that determines if x is a singular point is that the rank of the differential at x should fall below its maximum possible value. For example, for a smooth curve $f : \mathbb{R} \to \mathbb{R}^n$, $x \in \mathbb{R}$ is singular if and only if the derivative of f vanishes (has rank 0) at x. It is often easier to categorise singular points by their *corank*, that is the difference between the rank of the differential and its maximum possible value. Thus a regular point has corank zero.

Many manifolds arise as solution sets of a collection of equations, the necessary conditions being given in Theorem 1. The set $\{x \in \mathbb{R}^m : f(x) = c\}$ is called the *fibre* of f over c and is denoted $f^{-1}(c)$.

Theorem 1 *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ with $m \geq n$, and for any $c \in \mathbb{R}^n$ let $M = f^{-1}(c)$. If*
(a) M is non-empty and
(b) for all $x \in M$, x is regular (i.e. c is a regular value *of f),*
then M is a manifold of dimension $n - m$.

A proof of this fundamental result may be found in most books on differential topology, for example [13, 20]. The Theorem of Sard, also found in these texts, asserts that for any function, almost all values $c \in \mathbb{R}^n$ are regular, so their inverse images are manifolds or else empty. 'Almost all' has the technical meaning: *except for a set which has measure zero*. This leads to a valuable result concerning families of functions. Suppose $F : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$. We may think of this as a family of functions in the following way: for each $b \in \mathbb{R}^p$, define $f_b : \mathbb{R}^m \to \mathbb{R}^n$ by $f_b(x) = F(x, b)$.

Theorem 2 *Suppose c is a regular value of F, as above. Then for almost all $b \in \mathbb{R}^p$, $M_b = f_b^{-1}(c)$ is a manifold or empty.*

Key examples of manifolds are the transformation groups used to describe rigid body motions. These are the rotation groups $SO(p)$ consisting of $p \times p$ orthogonal matrices of determinant 1. We may realise $SO(p)$ as a manifold by identifying the set of $M(p)$ of $p \times p$ matrices with $\mathbb{R}^{p^2}$. Then $SO(p) = \{A \in M(p) : A^t A = I \text{ and } \det A > 0\}$. The Euclidean isometry groups $SE(p)$ combine rotations and translations in a (semi-direct) product $SO(p) \times \mathbb{R}^p$. The cases of interest to us are $p = 2$ and 3. The dimension of $SO(p)$ is $\frac{1}{2}p(p-1)$ and for $p = 2$ the angle of rotation provides a local parametrisation, while for $p = 3$ Euler angles or quaternions (suitably scaled) may be used. The dimension of $SE(p) = \frac{1}{2}p(p-1) + p = \frac{1}{2}p(p+1)$. Manifolds such as these which also have the structure of a group are known as *Lie groups*.

Definition 1 *A* **rigid body motion** *in $\mathbb{R}^p$ is a smooth function $\lambda : M \to SE(p)$, where M is a manifold, representing the space of configurations of the rigid body, and its rank k at a configuration $x \in M$ is the number of (infinitesimal) degrees of freedom of the motion at x.*

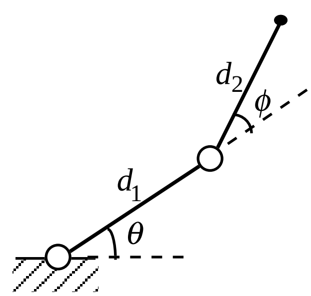

Figure 1: A planar two–bar arm.

To take a simple example in the planar case, consider a two–bar robot arm, with arm lengths d_1, d_2, and revolute joints as in Figure 1. The space of configurations can be taken to be the jointspace consisting of pairs of angles (θ, ϕ). Hence M is a *torus* (product of two circles). The motion λ of the end–effector, relative to an obvious choice of home position, is given by

$$\lambda(\theta, \phi) = \left(\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}, \begin{pmatrix} d_1 \cos(\theta) \\ d_1 \sin(\theta) \end{pmatrix} \right) \in SE(2).$$

In terms of parameters on M and $SE(2)$, we may represent the motion by

$$(\theta, \phi) \mapsto (\theta + \phi, d_1 \cos(\theta), d_1 \sin(\theta)).$$

The example illustrates how we may represent an element of $SE(p)$ by a pair $(A, \mathbf{a})$ where $A \in SO(p)$ and $\mathbf{a} \in \mathbb{R}^p$. Thus a rigid body motion $\lambda : M \to SE(p)$ may be represented by $\lambda(x) = (A(x), \mathbf{a}(x))$, $x \in M$. Given a point $\mathbf{w} \in \mathbb{R}^p$ of the rigid body (in moving coordinates), the *trajectory* of $\mathbf{w}$ is given by the action of the Euclidean isometry group on $\mathbb{R}^p$, that is by the function

$$\tau_w : M \to \mathbb{R}^p, \qquad \tau_w(x) = \lambda(x).\mathbf{w} = A(x)\mathbf{w} + \mathbf{a}(x). \tag{1}$$

It is useful to think of the trajectories forming a family parametrised by $\mathbf{w} \in \mathbb{R}^p$:

$$\tau : M \times \mathbb{R}^p \to \mathbb{R}^p; \qquad \tau(x, \mathbf{w}) = \tau_w(x). \tag{2}$$

In the example above, the trajectory of a point $\mathbf{w} = (w_1, w_2)$ in the end-effector is given by

$$\begin{aligned} \tau_w(\theta, \phi) &= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} d_1 \cos(\theta) \\ d_1 \sin(\theta) \end{pmatrix} \\ &= \begin{pmatrix} w_1 \cos(\theta+\phi) - w_2 \sin(\theta+\phi) + d_1 \cos(\theta) \\ w_1 \sin(\theta+\phi) + w_2 \cos(\theta+\phi) + d_1 \sin(\theta) \end{pmatrix}. \end{aligned}$$

For a given (θ, ϕ), the determinant of the Jacobian matrix of this function reduces to

$$d_1 w_1 \sin \phi + d_1 w_2 \cos \phi.$$

The singular trajectories correspond to the vanishing of this expression, illustrating the result that for a 2 dof planar motion the singular trajectories $\mathbf{w} = (w_1, w_2)$ at a given configuration (θ, ϕ) lie on a line. Note that the endpoint of the end–effector, whose home coordinates are $(d_2, 0)$, has a singularity in its trajectory if and only if $\phi = n\pi$ for some integer n.

3 Genericity

The key to defining generic properties of families of maps is *transversality*, an extension of the idea of regularity embodied in Theorem 1 and Sard's Theorem.

If $f : M \to N$ is a smooth map between differentiable manifolds and $Q \subseteq N$ is a submanifold, we say f is *transverse* to Q if, whenever $x \in M$ is such that $f(x) \in Q$, then the image of the differential of f spans a complement to the tangent space to Q at $f(x)$ in the tangent space to N at $f(x)$. Suppose $\dim M = m$, $\dim N = n$ and $\dim Q = q$. We define the *codimension* of Q in N to be $\dim N - \dim Q = n - q$. To check the condition for transversality is, in principle, a matter of showing that the rank of an $n \times (m+q)$ matrix is n. In the matrix, the first m columns represent the differential (in terms of local parameters on M and N) and the last q columns form a basis for the tangent space to Q. Note that if $m < n - q = \operatorname{codim} Q$, then f can only be transverse to Q if the image of f is disjoint from Q. So, for example, a smooth space curve cannot transversely meet a line in $\mathbb{R}^3$, while if it meets a plane transversely then at each point of intersection the tangent to the curve must have a non-zero component in the normal direction to the plane. Note that if Q is a single point c (a 0–dimensional manifold) then f is transverse to Q if and only if c is a regular value. Denote $\{x \in M : f(x) \in Q\}$ by $f^{-1}(Q)$. The following result extends Theorem 1:

Theorem 3 *If $f : M \to N$ is transverse to $Q \subseteq N$ and $P = f^{-1}(Q)$ is non-empty then $P \subseteq M$ is a manifold whose codimension equals $\operatorname{codim} Q$.*

In the analysis of singular points of functions, information contained in the first and higher derivatives is crucial. At first order, the corank of the differential is the most significant invariant of a singularity. Given a function $f : M \to N$, we may encode this information up to a given order k by evaluating at each point $x \in M$ the Taylor expansion of f up to kth order terms. The resulting function is called the *k–jet extension* of f and its range is a new manifold constructed from M and N called the *k–jet bundle* and denoted $J^k(M, N)$. Information contained in the function and its differential can therefore be determined from its 1–jet extension.

We are interested in properties of the set $C^\infty(M, N)$ of smooth functions $f : M \to N$. A property is *generic* if it is possessed by almost all functions in $C^\infty(M, N)$. Because this set has a more complicated structure than a Euclidean space, the meaning of 'almost all' has to be adapted and is interpreted topologically rather than in terms of measure. Provided M is compact, for example a sphere, torus, projective space or rotation group, we may take it to mean: *on an open and dense subset*. Thus if a function possesses a generic property, then so do all small perturbations, while if it does not then there is a generic function arbitrarily close to it. The openness condition must be weakened slightly if M is not compact. The technical details may be found in [13].

The key result for generating generic properties is the following:

Theorem 4 (Thom Transversality Theorem) *If Q_i, $i = 1, 2, \dots$ are submanifolds of $J^k(M, N)$, forming a Whitney regular stratification, then the property of $f \in C^\infty(M, N)$, that its k–jet extension is transverse to every Q_i, is generic.*

The condition of Whitney regularity is a technical one requiring that the manifolds fit together in a nice way [12]. In this paper we shall only employ Theorem 4 when $k = 1$.

4 Classification of screw systems

For the classification of screw systems we are indebted to Hunt, whose pioneering work, expounded in [14], provides a natural classification, on the basis of the geometry of the family of the loci of axes of screws of a given pitch—these form, for example in the case of 3–systems, a family of hyperboloids or their degenerations. The classification emphasises the special nature of systems in which a principal pitch, corresponding to a degeneracy, becomes infinite or in which two principal pitches coincide.

A mathematical basis for this classification was developed by Gibson and Hunt [10]. Screws are viewed as points of projective 5–space $\mathbb{P}^5$, whose coordinates are those of Plücker. A screw system is a projective subspace. An orthogonal change of coordinates in $\mathbb{R}^3$ induces a projective automorphism on $\mathbb{P}^5$ and two screw systems are said to be equivalent if there exists such a transformation taking one to the other. The classes of screw systems arise as 'natural' unions of the equivalence classes, which may also be interpreted as orbits of the action of the group of transformations just described. In particular, Gibson and Hunt introduced the pencil of pitch quadrics Q_h where $h \in \mathbb{R} \cup \{\infty\}$, the special case $h = 0$ corresponding to the classical Klein quadric of lines in projective 3–space. Thus the family of sets of screws of a given pitch in a screw system S is just its pencil of intersections with these hypersurfaces—in the case of 3–systems, a pencil of conics, as had already been recognised by Ball [1] in has planar representation of the 3–systems. The criterion for classification is provided by the manner in which a given screw system intersects the pitch quadrics. Indeed Klein had already implicitly introduced the pencil of quadrics in [17] and suggested it as a basis for distinguishing 2–systems.

Further refinement was provided in [7, 8] where screw systems were placed in the context of the Lie group approach to rigid body motions described in Section 2. Given a motion $\lambda : M \to SE(3)$, the screw system representing the infinitesimal capabilities of the rigid body when it is in configuration $x \in M$, is determined by the image of the differential of λ at x. Thus a screw system can be regarded as a linear subspace of the tangent space to the manifold $SE(3)$. In effect the screw system is a subspace of the Lie algebra, $se(3)$, a 6–dimensional vector space. Strictly speaking, the Lie algebra is the tangent space *at the identity* in the group, however there are natural identifications of the tangent spaces elsewhere that correspond to changes of coordinates in either the body or the ambient space. Just as the Euclidean group itself has a product structure, so does the Lie algebra: $se(p) \cong so(p) \times t(p)$. The Lie algebra, $so(p)$, corresponding to the rotations, consists of the skew–symmetric $p \times p$ matrices. For $p = 3$ that means we can write $B \in so(3)$ in the form

$$B = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix},$$

which we identify with the vector $\mathbf{v} = (v_1, v_2, v_3)$. Note that $\mathbf{v}$ spans the kernel of B, so long as $B \neq 0$.

Formally, a *screw* is a one–dimensional subspace of $se(3)$, corresponding to a point of the 5–dimensional projective space $\mathbb{P}se(3)$. For $p = 3$, the representation of a screw as a 6–vector consisting of two 3–vectors corresponds precisely to its representation in $\mathbb{P}^5$ by means of 6 homogeneous coordinates (Plücker coordinates). An *n–system* is an n–dimensional subspace of $se(3)$ or equivalently an $(n-1)$–dimensional projective subspace of $\mathbb{P}se(3)$. The equivalence of Gibson and Hunt can now be interpreted as the action induced on the set of projective subspaces, the Grassmannian space $G(n, 6)$, by the adjoint action of the Lie group on its Lie algebra.

The benefit of this approach is that we can readily interpret the classes of screw systems as subsets of the Grassmannian manifold, measure their dimension and see how they fit together. In effect, we can tell just how special a given class of screw systems is, and the manner in which the classes specialise one to another. The details appear in [8]. We provide here a summary of the classification in the most interesting case, namely 3–systems. Table 1 provides a broad classification of 3–systems on the following basis. Type I systems do not lie wholly in a single pitch quadric and therefore they intersect the pitch quadrics in a pencil of conics as indicated above. Type II systems are contained within a single pitch quadric. The subtypes A, B, C, D distinguish the projective dimension of intersection with Q_∞, subtype A denoting empty intersection, up to subtype D denoting dimension 2. Within the type I systems, further distinction is on the basis of the projective type of the pencil of conics. For each type, a basis of screws in normal form is given—these may be read as Plücker coordinates or Lie algebra coordinates as described above.

type	basis	type	basis
IA_1	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,1;0,0,h_\gamma)$	IIA	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,1;0,0,h)$
IA_2	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,1;0,0,h_\beta)$	IIB	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,0;0,0,1)$
IB_0	$(1,0,0;h,0,0)$ $(0,1,0;0,h,0)$ $(0,0,0;1,0,p)$	IIC	$(1,0,0;h,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;0,0,1)$
IB_3	$(1,0,0;h_\alpha,0,0)$ $(0,1,0;0,h_\beta,0)$ $(0,0,0;0,0,1)$	IID	$(0,0,0;1,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;0,0,1)$
IC	$(1,0,0;0,0,0)$ $(0,0,0;0,1,0)$ $(0,0,0;1,0,p)$		

Table 1: Broad classification of 3–systems.

The intersection of the screw system with Q_0 is also of great significance. The fine classification in Table 2 is based on an extension of the adjoint action to admit similarities as well as isometries [8]. Superscripts refer to the signs of the principal pitches or other invariants.

Each of the classes in the broad and the fine classifications can be shown to be a manifold. The *codimension* of each class is the difference between the dimension of the ambient Grassmannian manifold, $\dim G(3,6) = 9$, and the dimension of the class. It is a measure of the specialness of the class. Figure 2 illustrates the adjacencies between classes, indicated by an arrow, and the codimension of the relevant classes. To avoid overcomplicating the diagram, it shows only the intermediate classes listed in Table 2. Details of adjacency within these classes is given in [8].

We also showed that collectively these manifolds form a Whitney regular strat-

type	intermediate classes	fine classes
IA_1	IA_1	IA_1^{+++}, IA_1^{++-}, IA_1^{+--}, IA_1^{---}
	IA_1^0	IA_1^{++0}, IA_1^{+0-}, IA_1^{0--},
IA_2	IA_2	$IA_2^{(++)+}$, $IA_2^{+(++)}$, $IA_2^{(++)-}$ $IA_2^{+(--)}$, $IA_2^{(--)-}$, $IA_2^{-(--)}$
	IA_2^0	$IA_2^{(++)0}$, $IA_2^{+(00)}$, $IA_2^{(00)-}$, $IA_2^{0(--)}$
IB_0	IB_1	IB_0^+, IB_0^-
	IB_1^0	IB_0^0
	IB_2	$IB_0^{0,+}$, $IB_0^{0,-}$
	IB_2^0	$IB_0^{0,0}$
IB_3	IB_3	IB_3^{++}, IB_3^{+-}, IB_3^{--}
	IB_3^0	IB_3^{+0}, IB_3^{0-}
IC	IC	IC^+
	IC^0	IC^0
IIA	IIA	IIA^+, IIA^-
	IIA^0	IIA^0
IIB	IIB	IIB^+, IIB^-
	IIB^0	IIB^0
IIC	IIC	IIC^+, IIC^-
	IIC^0	IIC^0
IID		no finer subtypes

Table 2: Fine classification of 3–systems.

ification of $G(3,6)$. It follows from Theorem 4 that, in the set $C^\infty(M, SE(3))$ of smooth 3 dof spatial motions, transversality of the 1–jet of a motion to these strata is a generic property. For such a motion we would not encounter any screw systems of codimension ≥ 3. We would also expect, by Theorem 3, to encounter, for example, a smooth surface (codimension 1 manifold) in the 3–dimensional configuration space M where the screw system is of type IA_1^{+0-}, since this stratum has codimension 1.

However, as we shall see in Section 6, if one restricts attention to a specific family of motions within $C^\infty(M, SE(3))$, there is no guarantee that the transversality property is generic within it.

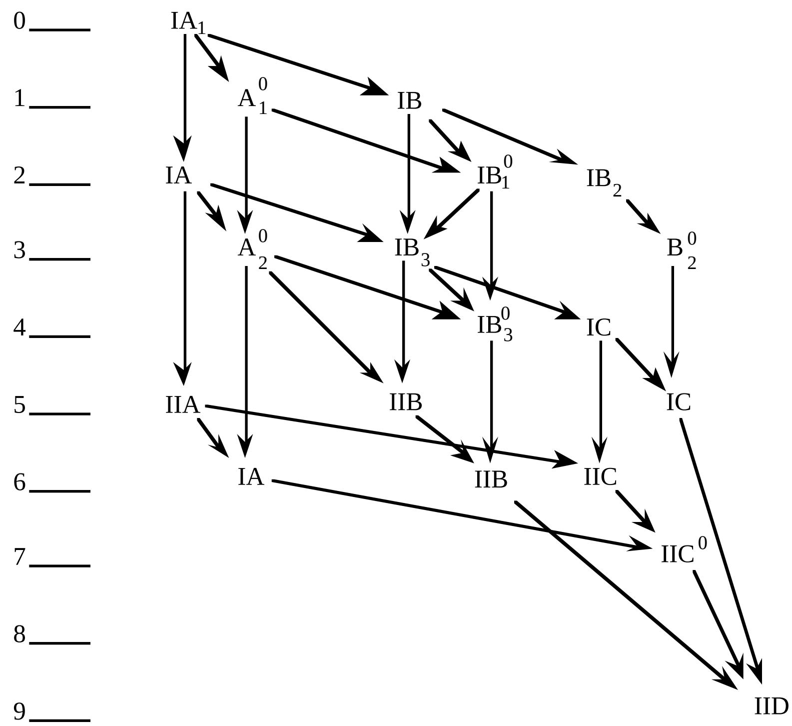

Figure 2: Adjacency diagram for 3–systems.

5 Instantaneous singular sets

We are now in a position to describe the set of points whose trajectories are singular in a given configuration. We restrict attention to the case of 3 dof spatial motions, however more general results appear in [5]. Given a motion $\mu : M \to SE(3)$ and a configuration $x \in M$, define the *instantaneous singular set* to be

$$I(\mu, x) = \{\mathbf{w} \in \mathbb{R}^3 \, : \, x \text{ is a singular point of } \tau_w\} \tag{3}$$

where the trajectory function τ_w associated to μ was defined in (1). Note that τ_w is a composition of the action of the Euclidean group $SE(3)$ on $\mathbb{R}^3$ with the motion μ itself. That is, if $ev_w : SE(3) \to \mathbb{R}^3$ is the map

$$ev_w(A, \mathbf{a}) = A\mathbf{w} + \mathbf{a}, \tag{4}$$

then $\tau_w = ev_w \circ \mu$. It follows from the Chain Rule that $x \in M$ is a singular point of τ_w if and only if the image of the differential of μ intersects the kernel of the differential of ev_w in a vector space of dimension ≥ 1. By definition, the image of the differential is, projectively, the screw system associated to the motion in that configuration, while the kernel of the evaluation map is precisely the α–plane A_w in the Klein quadric Q_0 consisting of lines (screws of pitch zero) through $\mathbf{w}$. Thus we have:

Theorem 5 ([5]) *Let $\mu : M \to SE(3)$ be a 3 dof motion and let S be the screw system at $x \in M$. A tracing point $\mathbf{w}$ belongs to $I(\mu, x)$ if and only if $S \cap A_w$ has projective dimension ≥ 0.*

In fact, this result is implicit in remarks of Hunt (see [14], §12.11). Here are some elementary corollaries of Theorem 5. First, $I(\mu, x)$ depends only on the associated screw system S, not on the particular motion μ used to define it—it is a first–order invariant of the motion. We may therefore write $I(S)$ for the ISS associated in this way to the screw system S. Second, the projective dimension of $S \cap A_w$ plus one is the corank of the singularity of τ_w at x. Third, $I(S)$ is *ruled*, in the sense that for any point $\mathbf{w} \in I(S)$, there is a line through $\mathbf{w}$ contained in $I(S)$. Fourth, a point $\mathbf{w}$ is in $I(S)$ if and only if S contains a screw of pitch zero, and $\mathbf{w}$ lies on its axis. The next theorem provides a fundamental connection between a screw system and its reciprocal (which holds, in fact, for arbitrary screw systems).

Theorem 6 ([5]) *Let S be a 3–system, and let $S^\perp$ be the reciprocal 3–system. Then $I(S) = I(S^\perp)$.*

The possible geometry of $I(S)$ can now be determined from the fine classification of 3–systems in Table 2 together with Theorem 5. An equation for $I(S)$ in terms of the coordinates (x, y, z) of $\mathbf{w}$ is given by the determinant of the 6×6 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -z & y \\ 0 & 1 & 0 & z & 0 & -x \\ 0 & 0 & 1 & -y & x & 0 \\ u_{11} & u_{12} & u_{13} & v_{11} & v_{12} & v_{13} \\ u_{21} & u_{22} & u_{23} & v_{21} & v_{22} & v_{23} \\ u_{31} & u_{32} & u_{33} & v_{31} & v_{32} & v_{33} \end{pmatrix} \tag{5}$$

where the first three rows represent the kernel of the differential of ev_w and the last three are a spanning set of screws for the 3–system. The possible types of

ISS, as determined by the intersection of the screw system with Q_0, together with the maximum corank of any singular trajectory, is given in Table 3.

type of 3–system	intersection	ISS	max corank
IA_1^{+++}, IA_1^{---}	empty	empty	0
IA_1^{++-}, IA_1^{+--}	conic	elliptic 1–sheet hyperboloid	1
IA_1^{++0}, IA_1^{0--}	point	line	1
IA_1^{+0-}	line pair	plane pair	2
$IA_2^{(++)+}$, $IA_2^{+(++)}$	empty	empty	0
$IA_2^{(--)-}$, $IA_2^{-(--)}$	empty	empty	0
$IA_2^{(++)-}$, $IA_2^{+(--)}$	conic	circular 1–sheet hyperboloid	1
$IA_2^{(++)0}$, $IA_2^{0(--)}$	point	line	1
$IA_2^{+(00)}$, $IA_2^{(00)-}$	repeated line	plane	2
IB_0^{+}, IB_0^{-}, $IB_0^{0,+}$, $IB_0^{0,-}$	conic	hyperbolic paraboloid	1
IB_0^{0}, $IB_0^{0,0}$	line pair	plane pair	2
IB_3^{++}, IB_3^{--}	point in Q_∞	empty	0
IB_3^{+-}	line pair	parallel planes	1
IB_3^{+0}, IB_3^{0-}	line	plane	1
IC, IC^0	line	plane	1
IIA^+, IIA^-	empty	empty	0
IIA^0	α–plane	whole space	3
IIB^+, IIB^-	point in Q_∞	empty	0
IIB^0	β–plane	plane	2
IIC^+, IIC^-	line in Q_∞	empty	0
IIC^0	α–plane	whole space	1
IID	Q_∞	empty	0

Table 3: Instantaneous singular sets for 3–systems

It is of interest to note that although a 3–system S of type IA_1^{++-} does not intersect its reciprocal $S^\perp$ of type IA_1^{+--} [10], nevertheless, by Theorem 6, their ISSs coincide. In fact, the lines in $S \cap Q_0$ and $S^\perp \cap Q_0$ correspond to the two reguli of the hyperboloid.

We now turn briefly to the general theory of singularities and consider smooth functions $f : M \to N$, where M and N are manifolds of dimensions m and n respectively. Let $q = \min\{m, n\}$, then for each $r = 0, 1, \ldots, q$, there is a manifold $\Sigma^r \subseteq J^1(M, N)$ such that if $f : M \to N$ has a singularity of corank r then the 1–jet extension of f at x lies in Σ^r. Moreover these manifolds form the *Thom–Boardman stratification* which is Whitney regular. By the Thom Transversality Theorem, it is a generic property of functions in $C^\infty(M, N)$ that their 1–jet extension be transverse to each Σ^r. The following result tells us how large each Σ^r type is.

Theorem 7 ([13]) *The codimension of* Σ^r *in* $J^1(M, N)$ *is*

$$(m - q + r)(n - q + r).$$

So, in the case of trajectories of 3 dof spatial motions where $m = n = 3$, the codimensions of Σ^r for $r = 0, 1, 2, 3$ are $0, 1, 4, 9$ respectively. Applying Theorem 3 to the combined trajectory function τ defined in equation (2), shows that for a typical motion we would expect to find, for each tracing point in $\mathbb{R}^3$, a surface in M where the trajectory has a corank 1 singularity; a surface of tracing points in $\mathbb{R}^3$ whose trajectories possess isolated corank 2 singularities; and no tracing points with corank 3 singularities.

This is summarised in the following version of the Genericity Theorem referred to in the Introduction.

Theorem 8 ([9]) *For almost all motions* $\mu \in C^\infty(M, SE(3))$*, the 1–jet extension of* τ *with respect to* M *is transverse to the Thom–Boardman stratification.*

6 Darboux–Type Motions

The contact of one point of a rigid body with the surface of another constitutes the simplest kind of joint [14]. Although, in the terminology of mechanisms, it is a higher pair, in many cases it is possible to synthesise the resulting motion using lower pairs (joints with contacting surfaces). Each joint of this kind imposes, in general, one constraint, or the loss of one degree of freedom, on the moving body.

In a note to the book of Koenigs [18], Darboux considers the following motion of this kind: a rigid body is constrained so that three non-collinear points are required to lie respectively in each of three planes in general position. The special case in which the planes are the coordinate planes and the points of contact form an acute triangle is analysed by Bottema and Roth [3]. They show that there are two special points in the moving body, such as the one marked **x** in Figure 3, whose trajectories are the interior and boundary of tetrahedra. Boundary points correspond to singularities of the trajectory and the four vertices of the tetrahedron represent quite special singularities, as we shall see below.

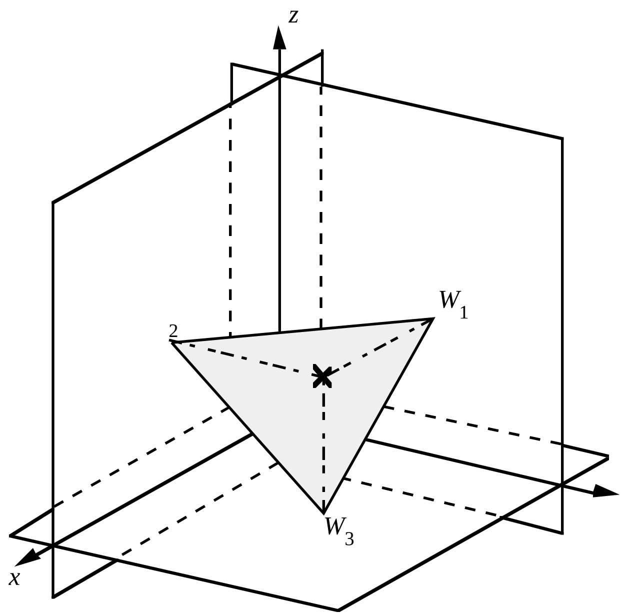

Figure 3: Motion of Darboux.

A modern application of this type of motion is in the *remote centre compliance* device (RCC), invented originally by Watson ([26], quoted in [22]) and Nevins, Whitney *et al* [21, 22]. The device is a non-sensory spring–loaded mechanism designed to facilitate peg–hole insertions by robot manipulators. It is attached between the wrist and gripper of the arm and comprises three rigid links, each joined to the wrist and the gripper by ball joints and symmetrically placed so that in the home position the axes of the links intersect at the tip of the gripper—the remote centre. In effect, the gripper moves so that its three joints lie on spheres whose centres are the connections of the RCC with the wrist. This is illustrated in Figure 4. The remote centre is like the vertices of the tetrahedra in Darboux's example: it represents a special singularity of its trajectory.

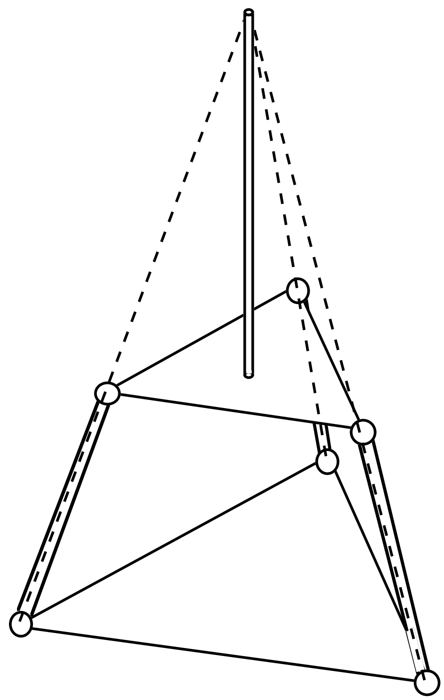

Figure 4: Remote centre compliance device.

Allowing a (considerable) degree of generalisation, we make the following definition:

Definition 2 *A* k*–***point Darboux–type motion** *is a rigid body motion in which a finite set of points* $W_1, \ldots, W_k$ *of the rigid body, satisfying the condition that any subset of four or less points is affinely independent, is constrained to lie, respectively, on a set of* k *smooth surfaces,* $N_1, \ldots, N_k$*. The points* W_i *are the* **contact points** *and the surfaces* N_i *the* **contact surfaces***. If* $k < 3$*, introduce* $3 - k$ *further points* W_j*,* $j = k + 1, \ldots, 3$ *so that* W_1*,* W_2*,* W_3 *are affinely independent and refer to those points as defining the* **coupler triangle***.*

By a smooth surface in the definition we mean a 2–dimensional manifold in $\mathbb{R}^3$. That means that a neighbourhood V of each point of a contact surface N_i can be parametrised by a smooth function $\phi : U \to V$, where U is some open subset of

$\mathbb{R}^2$. However it is also possible that the surfaces are defined implicitly by equations of the form $f(x, y, z) = c$ satisfying the criteria of Theorem 1. The two ways of presenting the surfaces lead to alternative approaches to analysing the motion. To make some statements easier we shall assume from now on that $k \leq 6$.

We do not, at this stage, impose any other constraints on the contact surfaces, than that they be smooth. For example, the surfaces may coincide. Indeed, there may be some value in dropping the independence condition on the contact points. The Mannheim motion described in [3] is the special case of the Darboux motion in which the contact points are collinear. A more interesting modern example is provided by the analysis of the singular configurations of the octahedral manipulator (or Gough–Stewart platform) [15]. When the leg lengths are fixed, the manipulator has the form of a 6–point Darboux–type motion in which the contact surfaces are spheres. However the contact points have coalesced in pairs so fail the independence condition in Definition 2.

We would like to determine general conditions under which the configuration space is a manifold. In the classical case of the N_i being the coordinate planes, Bottema and Roth show that the configuration space for the motion is $SO(3)$. Their approach is, in effect, as follows. Let the vertices W_i, $i = 1, \ldots, k$, of the coupler triangle be given by $\mathbf{w}_i = (w_{i1}, w_{i2}, w_{i3})$ in moving coordinates. Define the contact surfaces implicitly by equations $f_i(x, y, z) = 0$, $i = 1, \ldots, k$. Then the configuration space is defined as a subset of $SE(3)$ by the equation $G(\mu) = 0$ where the components of $G : SE(3) \to \mathbb{R}^k$ in terms of $\mu = (A, \mathbf{a}) \in SE(3)$ are:

$$G_i(A, \mathbf{a}) = f_i(A\mathbf{w}_i + \mathbf{a}), \qquad i = 1, \ldots, k. \tag{6}$$

In the Darboux case (see Figure 3) it is easy to solve these equations (the f_i are simply $x = 0$, $y = 0$, $z = 0$) explicitly for $\mathbf{a}$ in terms of A and hence the configuration space is parametrised by $SO(3)$. In general, we have the following:

Theorem 9 *The configuration space for a k–point Darboux–type motion with contact surfaces defined implicitly by $N_i = f_i^{-1}(0)$, $i = 1, \ldots, k$ is a smooth manifold unless in some configuration, the surface normals at the contact points, thought of as screws of pitch zero, fail to span a k–system.*

Proof. The configuration space is $M = G^{-1}(0)$ as above. We know by Theorem 1 that M is a manifold, unless for some $\mu \in M$, the rank of G at μ is less than k. Suppose we are at such a μ and the fixed contact points are $\mu(\mathbf{w}_i) = \mathbf{x}_i$,

$i = 1, \ldots, k$. By the rank formula of linear algebra, the dimension of the kernel S of the differential $DG(\mu)$ must be $> 6 - k$. Note that S is just a screw system. In fact, since $M = \bigcap_{i=1}^{k} G_i^{-1}(0)$, $S = \bigcap_{i=1}^{k} S_i$, where S_i is the kernel of $DG_i(\mu)$.

Now $g_i = f_i \circ ev_{w_i}$, where ev_{w_i} is defined in equation (4). Since ev_{w_i} is a submersion (i.e. has no singular points) and by assumption 0 is a regular value of f_i, it follows that 0 is a regular value of g_i and hence S_i is a 5–system. Its elements are just those screws for which the instantaneous direction of motion of $\mathbf{w}_i$ lies in the tangent space to N_i at the $\mathbf{x}_i$.

It is clear that S_i must contain the α–plane of pitch zero screws whose axes pass through $\mathbf{x}_i$, since these fix $\mathbf{x}_i$, and also the pitch infinity screws parallel to the tangent plane. Since these five screws are linearly independent, they span S_i. It is now an easy exercise to show that the pitch zero screw with axis normal to the tangent plane is reciprocal to S_i, and hence to $S \subseteq S_i$. Thus the normals all lie in the reciprocal screw system $S^{\perp}$ and in fact span it. Since the dimension of S is greater than $6 - k$ then $S^{\perp}$ must have dimension less than k, as required. □

The fact that the surface normals lie in the reciprocal screw system is the key to the analysis of the singular trajectories of Darboux–type motions in Section 7. But first we clasify the impert of Theorem 9 for 3–point motions.

Corollary 1 *The configuration space of a 3–point Darboux–type motion is a manifold unless for some configuration either*
(a) there is a common normal to two of the contact surfaces at the points of contact or
(b) the surface normals are coplanar and coincident or
(c) the surface normals are coplanar and parallel.

Proof. By Theorem 9 we require the three surface normals in each configuration to span a 3–system. This can only fail if they span a 2–system; a 1–system is not possible as this would require them to coincide, but we have assumed the contact points are affinely independent, so cannot be collinear. We have a 2–system if either two normals coincide and hence we are in case (a), or they are (projectively) collinear as points in Q_0. In that case, three distinct points of a 2–system lie on Q_0 so the 2–system must be of type IIA^0 or IIB^0 [5] and projectively span a line in an α–plane (or β–plane) corresponding to a planar pencil of lines in $\mathbb{R}^3$, with either finite or infinite vertex, giving rise to cases (b) and (c) respectively. □

Note, however, that we cannot rule out the possibility of the configuration space being non-singular even though one of the conditions in Corollary 1 is satisfied. Consider, for example the case when the three surfaces are a single sphere and the coupler triangle is equatorial. Then we are in case (b), but the configuration space is still $SO(3)$. Geometric conditions for non-regularity of the configuration space, similar to those in Corollary 1, can be determined for other values of k. The case $k = 6$ is relevant to the analysis of the octahedral manipulator mentioned above [15].

We now turn to the case where the contact surfaces are given in parametrised form; this requires a different approach. Let the moving coordinates of the coupler triangle and any further contact point W_i, $i = 1, \ldots, l$ (where $l = \min\{3, k\}$) be $\mathbf{w}_i = (w_{i1}, w_{i2}, w_{i3})$, and the parametrisation of the constraint surface N_i, $i = 1, \ldots, k$ be

$$\phi_i(\mathbf{u}_i) = (\phi_{i1}(\mathbf{u}_i), \phi_{i2}(\mathbf{u}_i), \phi_{i3}(\mathbf{u}_i)),$$

where $\mathbf{u}_i = (u_{i1}, u_{i2}) \in U_i \subseteq \mathbb{R}^2$. Then the rigidity of the moving body gives rise to an equation for each pair i, j such that $1 \leq i < j \leq k$:

$$||\phi_i(\mathbf{u}_i) - \phi_j(\mathbf{u}_j)||^2 - ||\mathbf{w}_i - \mathbf{w}_j||^2 = 0. \tag{7}$$

This gives $\frac{1}{2}k(k-1)$ equations in $2k$ variables $u_{11}, u_{12}, \ldots, u_{k1}, u_{k2}$. For $k < 3$ we add further variables x_{i1}, x_{i2}, x_{i3} for $i = k+1, \ldots 3$ denoting the fixed coordinates of the unconstrained vertices of the coupler triangle, and add equations of the form

$$||\phi_i(\mathbf{u}_i) - \mathbf{x}_j||^2 - ||\mathbf{w}_i - \mathbf{w}_j||^2 = 0, \tag{8}$$
$$||\mathbf{x}_i - \mathbf{x}_j||^2 - ||\mathbf{w}_i - \mathbf{w}_j||^2 = 0. \tag{9}$$

In the case $k \geq 5$ there is some dependence between the equations and it suffices to restrict to three equations respecting the distance between the vertices of the coupler triangle and, for each W_j, $j > 3$, the distances from W_j to each vertex of the coupler triangle. Further care is needed in the cases $k \geq 4$ as here the equations do not distinguish between the possible orientations or combinations of orientations of contact tetrahedra. Thus, we must choose the component of the solution set of equations (7) for which the orientation of each contact tetrahedron over the coupler triangle corresponds to that of the moving contact points. In summary, the configuration space has the form $F^{-1}(0)$, with $F : \mathbb{R}^p \to \mathbb{R}^q$, where p, the number of variables, and q, the number of equations, take the values given in Table 4.

Contacts	k	1	2	3	4	5	6
Variables	p	8	7	6	8	10	12
Equations	q	3	3	3	6	9	12

Table 4: Variables and equations for parametric Darboux–type motions.

The following is a parametric version of Corollary 1.

Theorem 10 ([6]) *The configuration space for a 3–point Darboux–type motion, with contact surfaces defined parametrically, is a 3–dimensional manifold in* $\mathbb{R}^6$*, unless any one of the conditions (a), (b) or (c) in Corollary 1 hold.*

Proof. From Table 4, if 0 is a regular value of F defined above, then the configuration pace is a manifold of dimension $6 - 3 = 3$. To check regularity we calculate the Jacobian matrix of F, corresponding to the three equations (7) with $(i, j) = (1, 2), (1, 3), (2, 3)$. Fix a point $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \in F^{-1}(0)$ and let $\phi_i^{(k)}$ denote the tangent vector to the ith contact surface, $(\partial\phi_i/\partial u_{ik})(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, for $k = 1, 2$. Further, let $\mathbf{s}_{ij} = \phi_j(\mathbf{u}_j) - \phi_i(\mathbf{u}_i)$ denote the vector of the ijth side of the coupler triangle in fixed coordinates. Then the Jacobian is

$$\begin{pmatrix} \phi_1^{(1)}.\mathbf{s}_{12} & \phi_1^{(2)}.\mathbf{s}_{12} & -\phi_2^{(1)}.\mathbf{s}_{12} & -\phi_2^{(2)}.\mathbf{s}_{12} & 0 & 0 \\ \phi_1^{(1)}.\mathbf{s}_{13} & \phi_1^{(2)}.\mathbf{s}_{13} & 0 & 0 & -\phi_3^{(1)}.\mathbf{s}_{13} & -\phi_3^{(2)}.\mathbf{s}_{13} \\ 0 & 0 & \phi_2^{(1)}.\mathbf{s}_{23} & \phi_2^{(2)}.\mathbf{s}_{23} & -\phi_3^{(1)}.\mathbf{s}_{23} & -\phi_3^{(2)}.\mathbf{s}_{23} \end{pmatrix}. \qquad (10)$$

Suppose the rank falls below three. Either at least one row vanishes or all rows are non-zero. If one row vanishes, then the corresponding $\mathbf{s}_{ij}$ is normal to the tangent vectors at both its endpoints, and hence the contact surfaces have a common normal. If not, then in any case all 3×3 submatrices must be singular. Choosing the first two columns of the matrix and then each other one in turn shows that the 2×2 submatrix

$$\begin{pmatrix} \phi_1^{(1)}.\mathbf{s}_{12} & \phi_1^{(2)}.\mathbf{s}_{12} \\ \phi_1^{(1)}.\mathbf{s}_{13} & \phi_1^{(2)}.\mathbf{s}_{13} \end{pmatrix}$$

is singular, for otherwise the third row must vanish. Now its determinant is

$$\begin{aligned}(\phi_1^{(1)}.\mathbf{s}_{12})(\phi_1^{(2)}.\mathbf{s}_{13}) - (\phi_1^{(2)}.\mathbf{s}_{12})(\phi_1^{(1)}.\mathbf{s}_{13}) &= (\phi_1^{(1)} \wedge \phi_1^{(2)}).(\mathbf{s}_{12} \wedge \mathbf{s}_{13}) \\ &= \mathbf{n}_1.\mathbf{n}_T\end{aligned}$$

where $\mathbf{n}_1$ is the surface normal at $\phi_1(\mathbf{u}_1)$ and $\mathbf{n}_T$ is normal to the coupler triangle. Since the determinant is zero, it follows that the surface normal lies in the plane of the coupler triangle. The same must hold for the other two surface normals, thus they are all coplanar with the coupler triangle.

Now we must consider 3×3 submatrices corresponding to the choice of one column from each pair. By choosing coordinates so that the coupler triangle lies in the Oxy plane say, it is straightforward to show that there is only one independent condition which may be expressed in the form:

$$\cos\theta_{12}\cos\theta_{23}\cos\theta_{31} = \cos\theta_{13}\cos\theta_{21}\cos\theta_{32}$$

where θ_{ij} is the angle between the tangent line to the ith contact surface in the plane of the coupler triangle and the side $\mathbf{s}_{ij}$ (with the angles measured anticlockwise). This converts to a similar equation in terms of the surface normals, which is equivalent to the condition that the normals intersect at a point (possibly at infinity), by the Theorem of Ceva [16]. The result follows. □

Definition 3 *A Darboux–type motion, with contact surfaces defined implicitly or parametrically, for which the configuration space is the inverse image of a regular value under the defining equations is called* **regular**, *otherwise it is* **singular**.

We may use the parametric formulation to show that having a singular configuration space is the exception.

Theorem 11 *Given k parametric contact surfaces, for almost all choices of k contact points the resulting k–point Darboux–type motion is regular.*

Proof. Treat the coordinates of the contact points as variables in the equations (7). The resulting function can readily be shown to be a submersion, so 0 is a regular value. The result now follows by Theorem 2. □

7 Singularities of Darboux–type motions

For the remainder of the paper we shall assume that our Darboux–type motions are regular. The key to determining the instantaneous singular sets and singular trajectory types lies in the observation, in the proof of Theorem 9, that the normals to the contact surfaces at the points of contact lie in the reciprocal screw system at a given configuration. We have the following result.

Theorem 12 *For a 3–point Darboux–type motion, the contact surface normals lie in the ISS at each configuration.*

Proof. Let S be the 3–system of the motion in a given configuration. Then the reciprocal system $S^{\perp}$ is spanned by the surface normals. However as screws these have pitch zero and so, by Theorem 5, the normals belong to $I(S^{\perp})$. Now apply the invariance of the ISS under reciprocity (Theorem 6) to obtain the result. □

A similar statement is true for k–point motions, $k = 1, 2$, but more care is needed for $k \geq 4$ as a screw belonging to Q_0 does not ensure that its axis is in the ISS; rather we require the k–system to intersect an α–plane in a projective line at least.

The knowledge that any ISS of a regular 3–point Darboux motion contains three distinct lines, together with the information in Table 3, enables us to exclude immediately a number of possible screw types, namely all those for which the ISS is empty or a line. It is also possible to eliminate types $\mathrm{IA}_2^{+(00)}$, $\mathrm{IA}_2^{(00)-}$, IB_3^{+0}, IB_3^{0-} and IC for in those cases the ISS is a plane which would require the surface normals to be coplanar. However then either the normals form a planar pencil, in which case the Darboux–type motion is singular, or the entire 3–system lies in Q_0 and hence is of type II.

The following lemma establishes a simple relationship between the configuration of the surface normals and screw system types.

Lemma 1 *Consider a regular 3–point Darboux–type motion in a given configuration. If the direction vectors of the surface normals:*

- *span* $\mathbb{R}^3$, *then the associated 3–system has type A;*

- *span a plane, then the associated 3–system has type B;*
- *span a line, then the associated 3–system has type C.*

A type D system is not possible.

Proof. The screw system contains a screw of infinite pitch if and only if it corresponds to an infinitesimal translation perpendicular to all the surface normal directions. The result follows. □

Further consideration of the ISSs for each type enables us to establish a precise correspondence between the screw system type and the configuration of the surface normals. This is summarised in Table 5.

type of 3–system	configuration of surface normals
IA_1^{++-}, IA_1^{+--}	3 mutually skew lines
IA_1^{+0-}	2 intersect in finite point, 3rd skew to others
$\mathrm{IA}_2^{(++)-}$, $\mathrm{IA}_2^{+(--)}$	3 mutually skew lines [1]
IIA^0	3 lines intersect in finite point
IB_0^+, IB_0^-	3 mutually skew lines
IB_0^0,	2 intersect in finite point, 3rd in parallel plane
$\mathrm{IB}_0^{0,+}$, $\mathrm{IB}_0^{0,-}$	3 mutually skew lines with common perpendicular
$\mathrm{IB}_0^{0,0}$	2 intersect in finite point, 3rd in parallel plane with common perpendicular through intersection
IB_3^{+-}	2 parallel, 3rd in parallel plane
IIB^0	3 coplanar not meeting in a point
IIC^0	3 parallel but not coplanar

[1] The distances between each pair of lines in the direction parallel to the third are equal [4].

Table 5: 3–system types for Darboux–type motions

The reduced adjacency diagram, showing only those 3–system types possible for Darboux–type motions and their adjacencies is in Figure 5.

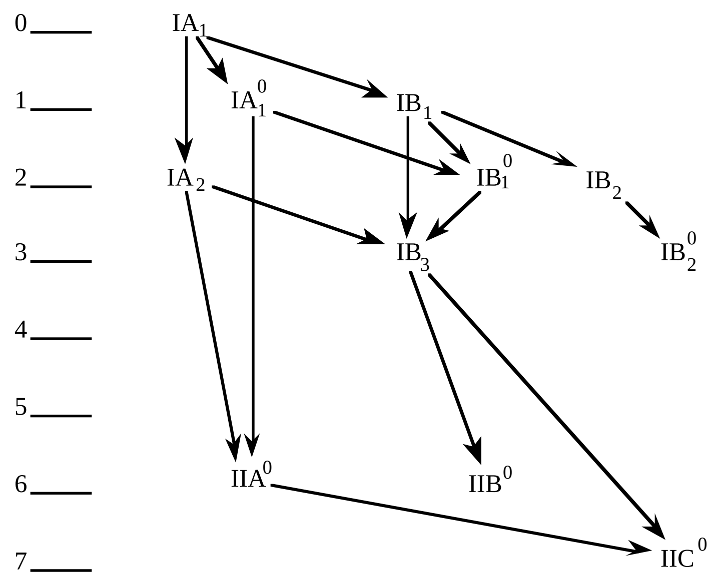

Figure 5: Adjacency diagram for 3–point Darboux–type motions.

It can be noted immediately that the special configurations described at the beginning of Section 6 for the classical Darboux motion and for the remote centre compliance device are ones in which the surface normals are coincident and so the instantaneous screw type is IIA^0. From Table 3, the trajectory of the point in question has corank 3 and indeed every point in the moving body is instantaneously singular. Yet this type has codimension 6 amongst 3–systems and corank 3 singularities have codimension 9 in the Thom–Boardman stratification. These are far from what we expect to see among motions with generic properties and one would expect to avoid them by making arbitrarily small perturbations in the motion.

However this is not the case. While both these special Darboux–type motions have considerable symmetry it is not this which is responsible for the highly degenerate singularities. The class of 3–point Darboux–type motions fails to satisfy a genericity theorem like Theorem 8. For although type IIA^0 3–systems have codimension 6, the condition on the surface normals giving rise to this type, that they mutually intersect, has only codimension 3 among triples of lines in 3–space.

There are two ways of perturbing a Darboux–type motion: either by changing the contact points or by altering the contact surfaces. The latter provides greater leeway and would be more likely to perturb away the degenerate behaviour.

Theorem 13 *Consider a 3–point regular Darboux–type motion for which the contact surfaces are either three planes in general position or three spheres with centres the vertices of a proper triangle, defined either*
(a) implicitly by functions $f_i(x, y, z) = 0$*,* $i = 1, 2, 3$*, or*
(b) parametrically by immersions ϕ_i*,* $i = 1, 2, 3$*.*
Suppose the coupler triangle $\Omega = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \in \mathbb{R}^9$ *is such that in some configuration the motion has type* IIA^0 *screw system. Then*
(i) there is an open neighbourhood of Ω *in* $\mathbb{R}^9$
and
(ii) there is an open neighbourhood of contact surfaces of either (a) (f_1, f_2, f_3) *in the set of triples of smooth functions* $\mathbb{R}^3 \to \mathbb{R}$*, or (b)* (ϕ_1, ϕ_2, ϕ_3) *in the set of triples of smooth immersions* $\mathbb{R}^2 \to \mathbb{R}^3$*,*
such for each choice of coupler triangle or each choice of contact surfaces in the neighbourhoods, the corresponding Darboux–type motion is regular and has a configuration with type IIA^0 *screw system.*

Although we do not present the full details of the proof here (see [4, 6]), since they depend to some extent on the technical definitions of the spaces of smooth functions, the idea of the proof is fairly straightforward. Consider the functions G of Theorem 9, and F, of Theorem 10, which define the configuration space in the implicit and parametric cases respectively. In each case append a function h from the underlying 6–dimensional space into $\mathbb{R}^3$, whose components represent

$$(Q_0(\$_1, \$_2), Q_0(\$_1, \$_3), Q_0(\$_2, \$_3))$$

where $\$_1, \$_2, \$_3$ are the pitch zero screws representing the surface normals. By a well known result in line geometry (see, for example, [10]), $Q_0(\$_1, \$_2) = 0$ if and only if the lines intersect. Therefore h vanishes if and only if the three surface normals intersect in pairs, corresponding to screw types IIA^0, IIB^0 or IIC^0. Thus the extended functions (G, h) and (F, h) are between 6–dimensional spaces and the inverse images of $0 \in \mathbb{R}^6$ are those configurations in which the Darboux–type motion has a screw system of type II. Moreover for the types of contact surfaces in the theorem, it can be shown that the functions have non-singular determinants. Now the required conditions are all preserved under small perturbations: that the motion remain regular, that there exist a point where the extended function vanishes and that the directions of the surface normals are independent (so the 3–system is of type A, not B or C).

8 Conclusion

Using the language and methods of differential topology and singularity theory, we have shown the classification of screw systems can be used to analyse the singular point trajectories of spatial rigid–body motions with several degrees of freedom. Natural concepts of genericity for motions, first in relation to the screw–system classification and, second, in relation to the classification of singular trajectories by corank were described.

The methods were applied to the class of motions, dubbed Darboux–type, generated by a rigid body moving with point–to–surface contacts. It was shown that for two special cases of interest, the classical motion of Darboux in which the contact surfaces are three planes in general position, and the motion employed in remote centre compliance devices in which the surfaces are three spheres with centres at the vertices of an equilateral triangle, an unexpectedly degenerate screw system is encountered giving rise to a corank 3 singularity. In the case of the RCC device, it is this that enables the mechanism to perform its task of facilitating peg–hole insertion. We further showed that this property is robust, in that it cannot be avoided either by perturbing the contact points in the moving body, or by perturbing the contacting surfaces. Thus, while the hypothesis of genericity fails for 3–point Darboux–type motions, nevertheless the outcome confers mechanical advantage in ensuring the stable occurrence of highly singular point trajectories.

9 Acknowledgement

We are indebted to Dr. P. Andreae and Professor C. Wampler for introducing us to the remote centre compliance device.

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