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The projective plane is a stabilizer ☆

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ABSTRACT

We prove that every 3-connected GF(q)-representable matroid that contains the projective plane, PG$(2, q)$, as a minor is uniquely representable.

1. Introduction

We prove the following theorem.

Theorem 1.1. *If M is a 3-connected* GF(q)*-representable matroid with a* PG$(2, q)$*-minor, then M is uniquely* GF(q)*-representable.*

We recall that PG(k, q) is the rank-$(k + 1)$ projective geometry over GF(q). Theorem 1.1 was conjectured in [2] where the weaker result with PG$(2, q)$ replaced by PG(q, q) is proved. We hope that Theorem 1.1 will help resolve the following conjecture which was also posed in [2].

Conjecture 1.2. *No excluded minor for the class of* GF(q)*-representable matroids contains* PG$(2, q)$ *as a minor.*

Let $\mathbb{F}$ be a field and let M be a matroid. We use two different notions of equivalence for representations: algebraic equivalence and projective equivalence. Two $\mathbb{F}$-representations of M are *algebraically equivalent* if one can be obtained from the other by elementary row operations, column scaling, and

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field automorphisms. Two $\mathbb{F}$-representations of M are *projectively equivalent* if one can be obtained from the other by elementary row operations and column scaling. We remark that for finite fields of prime order these two notions coincide since such fields have trivial automorphism groups. Two representations that are not projectively equivalent are said to be *projectively inequivalent*.

Let N be a minor of M. We say that N *stabilizes* M *over* $\mathbb{F}$ if no $\mathbb{F}$-representation of N can be extended to two projectively inequivalent $\mathbb{F}$-representations of M. We say that N is a *stabilizer* for $\mathbb{F}$ if N stabilizes each 3-connected $\mathbb{F}$-representable matroid that contains N as a minor. The main step towards a proof of Theorem 1.1 is to prove the following result:

Theorem 1.3. *For each prime power q,* $\mathrm{PG}(2, q)$ *is a stabilizer for* $\mathrm{GF}(q)$.

We will prove a modest strengthening of Theorem 1.3 (see Theorem 3.2) which may be useful with regard to Conjecture 1.2.

A matroid M is *uniquely* $\mathbb{F}$-representable if it is $\mathbb{F}$-representable and any two $\mathbb{F}$-representations of M are algebraically equivalent. The following result is referred to as the Fundamental Theorem of Projective Geometry (see Artin [1, p. 85]).

Theorem 1.4. *For each prime power q and integer $k \geqslant 2$, the projective geometry* $\mathrm{PG}(k, q)$ *is uniquely* $\mathrm{GF}(q)$*-representable.*

Theorem 1.1 is a direct consequence of Theorems 1.3 and 1.4, although for non-prime fields one needs to be a bit careful due to the two different notions of equivalence. Let $\mathrm{Aut}(\mathrm{GF}(q))$ denote the automorphism group of $\mathrm{GF}(q)$ and let M be a 3-connected $\mathrm{GF}(q)$-representable matroid that contains a $\mathrm{PG}(2, q)$-minor. From one representation of M we can construct $|\mathrm{Aut}(\mathrm{GF}(q))|$ projectively inequivalent representations that are algebraically equivalent. Since $\mathrm{PG}(2, q)$ is a stabilizer for M, there are no other projectively inequivalent representations. Thus M is uniquely representable.

2. Preliminaries

We use the notation of Oxley [3]. We refer to flats of rank 1, 2, and 3 as *points*, *lines*, and *planes* respectively.

Let e and f be distinct elements of M. We call e and f *clones* if there is an automorphism of M that swaps e and f and that acts as the identity on all other elements of M; that is, e and f are clones if $r_M(X \cup \{e\}) = r_M(X \cup \{f\})$ for each set $X \subseteq E(M) - \{e, f\}$.

The following lemma is well known but we include the proof for the sake of completeness.

Lemma 2.1. *Let e be an element of a matroid M and let $\mathbb{F}$ be a field. If $M \setminus e$ does not stabilize M over $\mathbb{F}$, then there exists an $\mathbb{F}$-representable matroid M' with $E(M') = E(M) \cup \{f\}$ such that $M = M' \setminus f$, and e and f are independent clones in M'.*

Proof. If $M \setminus e$ does not stabilize M over $\mathbb{F}$, then there is an $\mathbb{F}$-representation, say A, of $M \setminus e$ that extends to two projectively inequivalent $\mathbb{F}$-representations, say $[A, v_1]$ and $[A, v_2]$, of M. Let M' be the $\mathbb{F}$-representable matroid represented by the matrix $[A, v_1, v_2]$ where the last two columns are indexed by e and f respectively. Clearly e and f are clones and, since the representations $[A, v_1]$ and $[A, v_2]$ are projectively inequivalent, $\{e, f\}$ is independent in M'. □

Note that, in Lemma 2.1, if M is 3-connected, then M' is also 3-connected.

The following result is due to Sandra Kingan (private communication).

Lemma 2.2. *Let e and f be elements of a simple rank-3* $\mathrm{GF}(q)$*-representable matroid M. If e and f are clones, then $|E(M) - \mathrm{cl}_M(\{e, f\})| \leqslant q$.*

Proof. Since M is GF(q)-representable, e is on at most $q+1$ lines. Since e and f are clones, none of these lines, other than the line containing both e and f, can contain 3 or more points. Hence $|E(M) - \mathrm{cl}_M(\{e, f\})| \leqslant q$, as required. □

The following result shows that to test whether N is a stabilizer for $\mathbb{F}$ it suffices to check matroids M with $r(M) \leqslant r(N) + 1$ and $r^*(M) \leqslant r^*(N) + 1$.

Theorem 2.3. *(See Whittle [4].) A 3-connected matroid N is a stabilizer for the class of $\mathbb{F}$-representable matroids if and only if N stabilizes each 3-connected $\mathbb{F}$-representable matroid M satisfying one of the following conditions*:

(i) $N = M \setminus e$ *for some* $e \in E(M)$,
(ii) $N = M/e$ *for some* $e \in E(M)$, *or*
(iii) $N = M \setminus e/f$ *for some* $e, f \in E(M)$ *where* $M \setminus e$ *and* M/f *are both 3-connected.*

3. The main results

It remains to prove Theorem 1.3. We will prove a slightly stronger result (see Theorem 3.2). The following lemma is a key step in the proof.

Lemma 3.1. *Let $q \geqslant 3$ be a prime-power and let N be a simple rank-3 GF(q)-representable matroid with $|E(N)| \geqslant q^2 + q - 1$. Let M be a 3-connected GF(q)-representable matroid with elements e and f such that $M/e, f = N$. Then e and f are not clones in M.*

Proof. Assume to the contrary that e and f are clones. Let a and b be distinct elements of N and let $Z = E(N) - \mathrm{cl}_N(\{a, b\})$. Thus $|Z| \geqslant q^2 - 2$. Note that $r(M/a, b) = 3$ and $\{e, f\}$ is an independent clonal pair in $M/a, b$. Then, by Lemma 2.2, there are at most q points of $M/a, b$ that are not spanned by $\{e, f\}$. For each $c \in Z$, the set $\{a, b, c, e, f\}$ is independent in M and, hence, $X \cap \mathrm{cl}_{M/a,b}(\{e, f\}) = \emptyset$. Therefore there is a parallel class of $M/a, b$ that contains at least q elements of Z. So there is a plane P of M containing a and b as well as at least q points of Z. Let $\alpha = |P| - q$; thus $\alpha \geqslant 2$.

Consider M as a restriction of PG$(4, q)$ and let L be the line of PG$(4, q)$ that is spanned by e and f. For each point $z \in L$, let π_z be the set of elements in $E(N) - P$ that are spanned by $P \cup \{z\}$. Thus $(\pi_z\colon\ z \in L)$ is a partition of $E(N) - P$. Since e and f are clones, π_e and π_f are both empty. Since M is 3-connected, $\{e, f\}$ is not a series-pair and, hence, there exist distinct elements x and y in $L - \{e, f\}$ such that π_x and π_y are nonempty. Hence there is a partition (X, Y) of $E(N) - P$ into two nonempty sets such that, for each $x \in X$ and $y \in Y$, the elements x and y are in distinct parts of $(\pi_z\colon\ z \in L)$.

For each $x \in X$ and $y \in Y$, let L_{xy} denote the line of N spanned by x and y. We claim that there exist $x \in X$ and $y \in Y$ such that $|P - L_{xy}| \geqslant q + 1$. Suppose otherwise and consider any two points $x \in X$ and $y \in Y$. Since $|P - L_{xy}| \leqslant q$ and $|P| = q + \alpha$, we have $|L_{xy} \cap P| \geqslant \alpha$. Since $|P - L_{xy}| \leqslant q$, there are at most q/α other lines of N through x that contain points of Y. Hence $|Y| \leqslant (q/\alpha + 1)(q - \alpha)$. Similarly $|X| \leqslant (q/\alpha + 1)(q - \alpha)$. Now

$$\begin{aligned} q^2 + q - 1 &\leqslant \big|E(N)\big| \\ &= |X| + |Y| + |P| \\ &\leqslant 2(q/\alpha + 1)(q - \alpha) + (q + \alpha). \end{aligned}$$

By multiplying through by α and rearranging we get

$$(\alpha - 2)q^2 + \alpha(\alpha - 1) \leqslant 0.$$

This gives the required contradiction since $\alpha \geqslant 2$ and $q \geqslant 0$. Thus there exist $x \in X$ and $y \in Y$ such that $|P - L_{xy}| \geqslant q + 1$, as claimed.

Let $P' = P - L_{xy}$ and let $M' = (M|(P' \cup \{x, y, e, f\}))/x, y$. Note that M' has rank 3 and $\{e, f\}$ is an independent clonal pair in M'. Moreover, $\mathrm{cl}_N(\{x, y\}) \cap P' = \emptyset$ and $r_N(P') = r_M(P')$, so $\mathrm{cl}_M(\{x, y, e, f\}) \cap P' = \emptyset$. Therefore $\mathrm{cl}_{M'}(\{e, f\}) \cap P' = \emptyset$. Thus $|E(M') - \mathrm{cl}_{M'}(\{e, f\})| \geqslant q + 1$, which contradicts Lemma 2.2. □

Since $\mathrm{PG}(2, q)$ has $q^2 + q + 1$ points, the following result generalizes Theorem 1.3.

Theorem 3.2. *Let N be a simple rank-3 $\mathrm{GF}(q)$-representable matroid with $|E(N)| \geqslant q^2 + q - 1$. Then N is a stabilizer for* $\mathrm{GF}(q)$.

Proof. Since binary matroids are uniquely representable we may assume that $q \geqslant 3$. Now it is easy to verify that N is 3-connected. By Theorem 2.3, to prove that N is a stabilizer it suffices to consider 3-connected, $\mathrm{GF}(q)$-representable matroids M of one of the following types:

(i) $N = M \setminus e$ for some $e \in E(M)$.
(ii) $N = M/e$ for some $e \in E(M)$.
(iii) $N = M \setminus e/f$ for some $e, f \in E(M)$ where $M \setminus e$ and M/f are both 3-connected.

Case (i): If N does not stabilize M, then, by Lemma 2.1, there is a 3-connected $\mathrm{GF}(q)$-representable matroid M' obtained by extending M by an element f such that e and f are clones. By Lemma 2.2, $|E(M') - \mathrm{cl}_{M'}(\{e, f\})| \leqslant q$ and, hence, $|E(M')| \leqslant 2q + 1 \leqslant q^2$. However this contradicts the fact that $|E(N)| \geqslant q^2 + q - 1$.

Case (ii): If N does not stabilize M, then, by the dual of Lemma 2.1, there is a 3-connected $\mathrm{GF}(q)$-representable matroid M' obtained by co-extending M by an element f such that e and f are clones. This contradicts Lemma 3.1.

Case (iii): Let $N' = M/f$. By case (i), N stabilizes N'. Applying case (ii) to N' and M, we see that N' stabilizes M. Therefore N stabilizes M.

In each case N stabilizes M over $\mathrm{GF}(q)$. Therefore N is a stabilizer for $\mathrm{GF}(q)$. □

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