JOURNAL OF COMBINATORIAL THEORY, Series B 49, 78–86 (1990)

Quotients of Dilworth Truncations

GEOFFREY WHITTLE

Mathematics Department, University of Tasmania,
G. P. O. Box 252C, Hobart, Tasmania 7001, Australia

Communicated by the Editors

Received October 10, 1986

Sufficient conditions are given for an elementary quotient of the kth Dilworth truncation of a matroid M to be the kth Dilworth truncation of a quotient of M. As special cases, contractions of Dilworth truncations and principal truncations by connected flats of Dilworth truncations are characterised as Dilworth truncations of certain matroids. As an application of the theory it is shown that the degree of the minimal extension field of $GF(q)$ needed to represent the first Dilworth truncation of $PG(r-1, q)$ is greater than $2r-4$.

1. INTRODUCTION

The kth Dilworth truncation, denoted $D_k(M)$, is a canonical construction, first realised in [5] which assigns to any matroid M a new matroid whose ground set is the set of flats of M with rank $k+1$ (in this paper subsets of $E(M)$ with cardinality $k+1$). If M and M' are matroids sharing a common ground set then M' is a quotient of M if every flat of M' is also a flat of M. If the rank of M and M' differ by one then M' is an elementary quotient of M. Elementary quotients are determined by modular cuts of M.

In this paper we show that if $(D_k(M))'$ is an elementary quotient of $D_k(M)$ determined by a modular cut of $D_k(M)$ whose minimal members are connected then $(D_k(M))' = D_k(M')$ where M' is an elementary quotient of M. The modular cut of M determining M' is specified. As special cases we are able to characterise principal truncations by connected flats of $D_k(M)$ and contractions of $D_k(M)$.

As an application of the theory it is shown that the degree of the minimal extension field of $GF(q)$ needed to represent the first Dilworth truncation of $PG(r-1, q)$ is greater than $2r-4$. This improves a bound of Brylawski [2].

0095-8956/90 $3.00

2. Definitions and Preliminary Results

We assume that the reader is familiar with the basic concepts of matroid theory. Matroid terminology used here will in general follow Welsh [9]. The set of elements of a matroid M will be denoted by $E(M)$. If $T \subseteq E(M)$, the restriction of M to $E(M)\backslash T$ will be denoted by $M|(E(M)\backslash T)$ or by $M\backslash T$ and the contraction of M to $E(M)\backslash T$ will be denoted by $M \cdot (E(M)\backslash T)$ or by M/T in either case according to convenience. The closure and rank of T in M will be denoted by $cl_M(T)$ and by $r_M(T)$, respectively, or if no danger of ambiguity exists by $cl(T)$ and $r(T)$, respectively. The simple matroid associated with M will be denoted by $\bar{M}$.

The kth Matroidal Dilworth Truncation

Let M be a matroid. For $1 \leqslant k < r(M)$ the kth *Dilworth truncation* of M, denoted $D_k(M)$, is a matroid on the groundset

$$E(D_k(M)) = \{p : p \subseteq E(M), |p| = k+1\}$$

Whose family $\mathscr{I}$ of independent sets is given by

$$\mathscr{I} = \{I : I \subseteq E(D_k(M)), r_M\left(\bigcup \{p : p \in I'\}\right) \geqslant |I'| + k \text{ for all nonempty subsets } I' \text{ of } I\}.$$

Note that the kth Dilworth truncation as defined in [4, 5, 7] has as groundset the set of rank-$(k+1)$ flats of M and is a matroid isomorphic to the simple matroid associated with $D_k(M)$ defined above as is routinely verified. Our definition generalises that of Brylawski [2]. The statements and proofs of a number of theorems in this paper are simplified by considering matroidal Dilworth truncations rather than Dilworth truncations.

Note that $p \in E(D_k(M))$ is a loop of $D_k(M)$ if and only if p is dependent in M with $|p| = k+1$. It is well known that $r(D_k(M)) = r(M) - k$.

It is worth noting the following geometric interpretation of $\overline{D_k(M)}$. Assume that M is embedded as a restriction of a rank-r projective space P and that F is a rank-$(r-k)$ subspace of P in "general position" relative to M. Then $\overline{D_k(M)}$ is isomorphic to the restriction of P to the set of points of intersections of the subspaces of P spanned by the rank-$(k+1)$ flats of M with F.

Connected Flats

Let F be a flat of the matroid M. Then F is *connected* if whenever x and y are non-loops of M contained in F, there exists a circuit of M contained in F which contains both x and y. Note that, according to this definition, a connected flat may have loops.

For a connected flat F of $D_k(M)$ of positive rank let $\phi(F) = cl_M(\bigcup \{p \in F: p \text{ not a loop of } D_k(M)\})$. It is a straightforward consequence of results in [4, Chap. 7] that ϕ defines a bijection between the connected flats of $D_k(M)$ of positive rank and the flats of M having rank greater than k. We also have $r(\phi(F)) = r(F) + k$ for every connected non-trivial flat F of $D_k(M)$.

We shall use the canonical bijection ϕ frequently in this paper. In particular we have

LEMMA 2.1. *Let F_1 and F_2 be connected flats of $D_k(M)$ then*

(i) *if $r(F_1 \cap F_2) > 0$ then $F_1 \cap F_2$ is a connected flat of $D_k(M)$ and $\phi(F_1 \cap F_2) = \phi(F_1) \cap \phi(F_2)$,*

(ii) *if $r(F_1 \cap F_2) > 0$ then $cl(F_1 \cup F_2)$ is a connected flat of $D_k(M)$ and whenever $cl(F_1 \cup F_2)$ is connected then $\phi(cl(F_1 \cup F_2)) = cl(\phi(F_1) \cup \phi(F_2))$.*

Proof. (i) Routine checking shows that $F_1 \cap F_2 = \phi^{-1}(\phi(F_1) \cap \phi(F_2))$. Now $F_1 \cap F_2$ is certainly connected since it is of the form $\phi^{-1}(F)$ for some flat F of M and we also have $\phi(F_1 \cap F_2) = \phi(F_1) \cap \phi(F_2)$.

(ii) If $r(F_1 \cap F_2) > 0$, then since both F_1 and F_2 are connected it follows from circuit transitivity that $cl(F_1 \cup F_2)$ is connected. Assume that $cl(F_1 \cup F_2)$ is connected. Then $cl(F_1 \cup F_2) = \phi^{-1}(F)$ for some flat F of M.

But F contains $\phi(F_1)$ and $\phi(F_2)$ and therefore $F \supseteq cl(\phi(F_1) \cup \phi(F_2))$. That is, $cl(F_1 \cup F_2) \supseteq \phi^{-1}(cl(\phi(F_1) \cup \phi(F_2)))$. But $\phi^{-1}(cl(\phi(F_1) \cup \phi(F_2)))$ contains F_1 and F_2 so $\phi^{-1}(cl(\phi(F_1) \cup \phi(F_2))) \supseteq cl(F_1 \cup F_2)$. Therefore $cl(F_1 \cup F_2) = \phi^{-1}(cl(\phi(F_1) \cup \phi(F_2)))$ and we have $\phi(cl(F_1 \cup F_2)) = cl(\phi(F_1) \cup \phi(F_2))$.

Modular Cuts and Quotients

Let M'' be a matroid and $M = M''\backslash P$, then M'' is an *extension* of M by P. If P is independent in M'' then M'' is said to be an independent extension of M by P.

Proposition 2.2 is a special case of a result of Higgs [6].

PROPOSITION 2.2. *Let M and M' be matroids with $E(M) = E(M')$, then M' is an elementary quotient of M if and only if there exists an independent single point extension of M by p, say M'', with the property that $M''/p = M'$. The matroid M'' is unique.*

That is, elementary quotients of M are determined by non-trivial single point extensions of M. Such extensions are, in turn, determined by modular cuts of M.

A *modular cut* C of the matroid M is a set of flats of M with the following properties:

(i) if $F_1 \in C$ and $F_2 \supseteq F_1$ then $F_2 \in C$,

(ii) if F_1 and F_2 belong to C and $r(F_1)+r(F_2)= r(F_1 \cap F_2)+r(F_1 \cup F_2)$ (that is, F_1 and F_2 form a *modular pair*), then $F_1 \cap F_2 \in C$.

It is shown in [4] that the modular cut C determines a single point extension M'' of M with ground set $E(M) \cup x$ having the following independent sets. If $I \subseteq E(M)$ then I is independent in M'' if and only if I is independent in M, while $I \cup x$ is independent in M'' if and only if I is independent in M and $cl_M(I)$ does not contain any member of C.

The following definition enables us to bypass the single point extension and go straight from the modular cut to the quotient. If C is a modular cut of the matroid M then M' is the *quotient induced by* C if $M' = M''/x$ where M'' is the single point extension of M determined by C. If C is a proper non-empty modular cut of M (that is, $C \neq \varnothing$ and C does not contain all the flats of M) then the quotient induced by C is an elementary quotient of M. Otherwise the quotient induced by C is just M itself, a case of little interest to us.

One routinely obtains

PROPOSITION 2.3. *Let M be a matroid, C be a proper non-empty modular cut of M and M' the quotient of M induced by C. Then*

(i) *$I \subseteq E(M)$ is independent in M' if and only if I is independent in M and $cl_M(I)$ contains no member of C.*

(ii) *For $S \subseteq E(M)$,*

$$r_{M'}(S) = \begin{cases} r_M(S) & \textit{if } \; cl_M(S) \textit{ contains no member of } C, \\ r_M(S) - 1 & \textit{if } \; cl_M(S) \textit{ contains a member of } C. \end{cases}$$

Certain special cases of quotients are of particular interest to us. Let F be a flat of the matroid M and C be the modular cut consisting of all flats containing F. Then the *first principal truncation of M at F*, denoted $T_{F(1)}(M)$, is the quotient of M induced by C. This definition differs from that given in [1] but is easily seen to be equivalent to it up to associated simple matroids. Geometrically $T_{F(1)}(M)$ is the matroid obtained by placing a point freely on the flat F in M and then contracting the point. It is readily checked that F is a flat of $T_{F(1)}(M)$ and that $T_{F(1)}(T_{F(1)}(M))$ is thus well defined. We therefore define recursively the kth *principal truncation of M at F*, denoted $T_{F(k)}(M)$, by $T_{F(k)}(M) = T_{F(1)}(T_{F(k-1)}(M))$ for $k > 1$. In the case $k = r_M(F) - 1$ one obtains the *complete principal* *truncation of M at F,* denoted $T_F(M)$. This case is of particular interest (see, for example, [1, 3, 10]). In the case $k \geqslant r_M(F)$, $T_{F(k)}(M)\backslash F = M/F$ and since F is the set of loops of $T_{F(k)}(M)$ we have $\overline{T_{F(k)}(M)} \cong \overline{M/F}$.

3. Quotients of Dilworth Truncations

In this section we give a sufficient condition for a quotient of the Dilworth truncation of a matroid M to be the Dilworth truncation of a quotient of M. We need to relate certain modular cuts of $D_k(M)$ to corresponding modular cuts of M.

Lemma 3.1. *If $\phi(F_1)$ and $\phi(F_2)$ are a modular pair of flats of M then F_1 and F_2 are a modular pair of connected flats of $D_k(M)$. On the other hand if F_1 and F_2 are a modular pair of connected flats of $D_k(M)$ with $r(F_1 \cap F_2) > 0$ then $\phi(F_1)$ and $\phi(F_2)$ are a modular pair of flats of M.*

Proof. The flats F_1 and F_2 of $D_k(M)$ are connected with $r(F_1 \cap F_2) > 0$ if and only if $\phi(F_1)$ and $\phi(F_2)$ are flats of M with $r(\phi(F_1) \cap \phi(F_2)) > k$. For such flats we see by Lemma 2.1 that $F_1 \cap F_2$ and $cl(F_1 \cup F_2)$ are connected flats of $D_k(M)$ with $\phi(F_1 \cap F_2) = \phi(F_1) \cap \phi(F_2)$ and $\phi(cl(F_1 \cup F_2)) = cl(\phi(F_1) \cup \phi(F_2))$. Therefore

$$\begin{aligned}
&r(\phi(F_1)) + r(\phi(F_2)) - r(\phi(F_1) \cap \phi(F_2)) - r(\phi(F_1) \cup \phi(F_2)) \\
&\quad = r(\phi(F_1)) + r(\phi(F_2)) - r(\phi(F_1 \cap F_2)) - r(\phi(F_1 \cup F_2)) \\
&\quad = r(F_1) + k + r(F_2) + k - r(F_1 \cap F_2) - k - r(F_1 \cup F_2) - k \\
&\quad = r(F_1) + r(F_2) - r(F_1 \cap F_2) - r(F_1 \cup F_2)
\end{aligned}$$

and hence F_1 and F_2 form a modular pair if and only if $\phi(F_1)$ and $\phi(F_2)$ form a modular pair.

Now assume that $\phi(F_1)$ and $\phi(F_2)$ are a modular pair of flats of M with $r(\phi(F_1) \cap \phi(F_2)) \leqslant k$. In this case $r(F_1 \cap F_2) = 0$. If $cl(F_1 \cup F_2)$ is not connected then F_1 and F_2 certainly form a modular pair so assume that $cl(F_1 \cup F_2)$ is connected. By Lemma 2.1, $\phi(cl(F_1 \cup F_2)) = cl(\phi(F_1) \cup \phi(F_2))$ and therefore we have

$$\begin{aligned}
0 &= r(\phi(F_1)) + r(\phi(F_2)) - r(\phi(F_1) \cap \phi(F_2)) - r(\phi(F_1) \cup \phi(F_2)) \\
&\geqslant r(F_1) + k + r(F_2) + k - k - r(F_1 \cup F_2) - k \\
&= r(F_1) + r(F_2) - r(F_1 \cap F_2) - r(F_1 \cup F_2) \geqslant 0.
\end{aligned}$$

So all inequalities are equalities and therefore F_1 and F_2 form a modular pair.

Associated with each modular cut of a matroid are its minimal members (when ordered by set inclusion). If $\mathscr{F}$ is a set of non-comparable flats of the matroid M then it is easily seen that $\mathscr{F}$ is the set of minimal members of a modular cut C of M if and only if whenever F_1 and F_2 are a modular pair of flats of M, each containing a member of $\mathscr{F}$, then $F_1 \cap F_2$ contains a member of $\mathscr{F}$.

LEMMA 3.2. *If C is a non-trivial modular cut of $D_k(M)$ whose minimal members are connected, then $C' = \{\phi(F)$: F is a connected member of $C\}$ is a modular cut of M.*

Proof. Let $\mathscr{F}' = \{\phi(F)$; F a minimal member of $C\}$; $\mathscr{F}'$ is well defined since the minimal members of C are all connected. Clearly C' consists of all flats of M containing a member of $\mathscr{F}'$. Say $\phi(F_1)$ and $\phi(F_2)$ are a modular pair of flats, each of which belongs to C', then by Lemma 3.1, F_1 and F_2 are a modular pair of connected flats of $D_k(M)$. But both $\phi(F_1)$ and $\phi(F_2)$ contain members of $\mathscr{F}'$ so both F_1 and F_2 contain minimal members of C and therefore $F_1 \cap F_2$ contains a minimal member of C. By Lemma 2.1, $F_1 \cap F_2$ is connected and $\phi(F_1 \cap F_2) = \phi(F_1) \cap \phi(F_2)$ so $\phi(F_1) \cap \phi(F_2)$ contains a member of $\mathscr{F}'$ and the result follows.

We are now in a position to prove our main result.

THEOREM 3.3. *Let M be a matroid, C be a non-trivial modular cut of $D_k(M)$ whose minimal members are connected, and C' be the modular cut of M defined by $C' = \{\phi(F)$: F a connected member of $C\}$. Let $(D_k(M))'$ and M' be the quotients of $D_k(M)$ and M induced by C and C', respectively. Then $D_k(M') = (D_k(M))'$.*

Proof. By Lemma 3.2, C' is indeed a modular cut of M. Assume that $S \subseteq E(M)$ is independent in $[D_k(M)]'$, then by Proposition 2.3, S is independent in $D_k(M)$ and $cl_{D_k(M)}(S)$ contains no member of C. Say the connected components of $cl_{D_k(M)}(S)$ are $F_1, ..., F_k$, then $\phi(F_1)$, $\phi(F_2)$, ..., $\phi(F_k)$ are flats of M none of which belong to C'. Consider $S' \subseteq S$. If $cl_{D_k(M)}(S') \subseteq F_i$ for $i \in \{1, ..., k\}$, then $cl_M(\bigcup \{i; i \in S'\}) \subseteq \phi(F_i)$ and therefore $cl_M(\bigcup \{i; i \in S'\}) \notin C'$. Since S is independent in $D_k(M)$, $r_M(\bigcup \{i; i \in S'\}) \geqslant |S'| + k$, but by Proposition 2.3, $r_{M'}(\bigcup \{i; i \in S'\}) = r_M(\bigcup \{i; i \in S'\})$ and therefore $r_{M'}(\bigcup \{i; i \in S'\}) \geqslant |S'| + k$. If $cl_{D_k(M)}(S') \nsubseteq F_i$ for any $i \in \{1, ..., k\}$ then $cl_{D_k(M)}(S')$ is not connected and therefore $r_M(\bigcup \{i; i \in S'\}) > |S'| + k$. By Proposition 2.3, $r_{M'}(\bigcup \{i; i \in S'\}) \geqslant r_M(\bigcup \{i; i \in S'\}) - 1$ and therefore $r_{M'}(\bigcup \{i; i \in S'\}) \geqslant |S'| + k$. In either case, for $S' \subseteq S$, $r_{M'}(\bigcup \{i; i \in S'\}) \geqslant |S'| + k$ and therefore S is independent in $D_k(M')$.

Assume that S is dependent in $[D_k(M)]'$. Then either S is dependent in

$D_k(M)$ or S is independent in $D_k(M)$ and $cl_{D_k(M)}(S)$ contains a member of C. If S is dependent in $D_k(M)$ then S contains a circuit A of $D_k(M)$. But circuits span connected flats and therefore

$$r_M\left(\bigcup \{a:a\in A\}\right)=r_{D_k(M)}(A)+k=|A|+k-1.$$

But $r_{M'}(\bigcup \{a:a\in A\})\leqslant r_M(\bigcup \{a:a\in A\})$ so

$$r_{M'}\left(\bigcup \{a:a\in A\}\right)<|A|+k$$

and therefore S is dependent in $D_k(M')$. If S is independent in $D_k(M)$ and $cl_{D_k(M)}(S)$ contains a member of C then some minimal member F of C is a subset of $cl_{D_k(M)}(S)$. Any connected component of $cl_{D_k(M)}(S)$ is spanned by a subset of S and at least one connected component contains F (all the minimal members of C are connected). Assume that this component is spanned by $S'\subseteq S$. Then $r_M(\bigcup \{s:s\in S'\})=|S'|+k$ and $cl_M(\bigcup \{s:s\in S'\})\supseteq\phi(F)$. But $\phi(F)\in C'$ so by Proposition 2.3, $r_{M'}(\bigcup \{s:s\in S'\})=r_M(\bigcup \{s:s\in S'\})-1<|S'|+k$ and therefore S' is dependent in $D_k(M')$. That is, if S is dependent in $[D_k(M)]'$, then S is dependent in $D_k(M')$.

Since $[D_k(M)]'$ and $D_k(M')$ share common ground sets the result follows.

We immediately obtain

COROLLARY 3.4. *If F is a connected non-trivial flat of $D_k(M)$, then $D_k(T_{\phi(F)(1)}(M))=T_{F(1)}(D_k(M))$.*

Equivalently, if F is a flat of M with $r(F)>k$, then $D_k(T_{F(1)}(M))=T_{\phi^{-1}(F)(1)}(D_k(M))$.

One routinely shows that if F is connected in M with $r(F)>1$ then F is connected in $T_{F(1)}(M)$ and we therefore have

COROLLARY 3.5. *If F is a connected non-trivial flat of $D_k(M)$, then for $j\leqslant r(F)$, $D_k(T_{\phi(F)(j)}(M))=T_{F(j)}(D_k(M))$.*

Equivalently if F is a flat of M with $r(F)>k$, and $j\leqslant r(F)-k$, then $D_k(T_{F(j)}(M))=T_{\phi^{-1}(F)(j)}(D_k(M))$.

Two special cases are of particular interest.

COROLLARY 3.6. *If F is a rank $j+1$ connected flat of $D_k(M)$ then $D_k(T_{\phi(F)(j)}(M))=T_F(D_k(M))$.*

This characterises complete principal truncations at connected flats of Dilworth truncations. The result is intuitively evident; the complete principal truncation of $D_k(M)$ by F is obtained by putting a set P of j points freely on the flat F and then contracting the set P. The effect is to reduce F to a rank one flat. If $T_F(D_k(M))$ were to be the kth Dilworth truncation of anything, it must be that of a matroid in which $\phi(F)$ has rank $k+1$. The simplest way to do this is to put a set P of j points freely on the flat $\phi(F)$ in M and then contract P. This is exactly what is done and Corollary 3.6 shows that the natural correspondences hold.

Just as evident intuitively is

COROLLARY 3.7. *If F is a rank-j connected flat of $D_k(M)$ then $D_k(T_{\phi(F)(j)}(M))\backslash F = D_k(M)/F$, and since F is a set of loops of $D_k(T_{\phi(F)(j)}(M))\backslash F$ we have, $\overline{D_k(T_{\phi(F)(j)}(M))} \cong \overline{D_k(M)/F}$.*

This characterises contractions by flats of Dilworth truncations. The case when F is not connected is covered by considering each component of F in turn. In the case $k=1$ (the traditional Dilworth truncation on the lines of M), we see that, up to associated simple matroids, the contraction of $D_1(M)$ by a connected flat F is isomorphic to the first Dilworth truncation of the complete principal truncation of M at $\phi(F)$.

As an application of the above theory we turn our attention to a problem of Brylawski. In [2] Brylawski shows that if a matroid M is representable over $GF(q)$, then $D_1(M)$ is representable over some extension field of $GF(q)$. For such a matroid, denote by $d(M, q)$ the degree of the minimal extension field needed to represent $D_1(M)$. Brylawski shows that for $r>1$, $d(PG(r-1, q), q) > r$. We improve on this bound.

PROPOSITION 3.8. $d(PG(r-1, q), q) > 2r-4$.

Proof. Assume $r>2$ (the result is trivial for $r=2$), and let F be a coline of $PG(r-1, q)$. It is straightforward to show that lines of $T_F(PG(r-1, q))$ are either lines of $PG(r-1, q)$ disjoint from F or hyperplanes of $PG(r-1, q)$ containing F. There are q^{2r-4} distinct lines of $PG(r-1, q)$ disjoint from F and there are $q+1$ distinct hyperplanes of $PG(r-1, q)$ containing F (see, for example, Sved [8] for justification of these well known facts). That is, there are $q^{2r-4}+q+1$ distinct lines of $T_F(PG(r-1, q))$ and hence $D_1(T_F(PG(r-1, q))) \cong U_{2, q^{2r-4}+q+1}$. But by Corollary 3.7,

$$\overline{D_1(T_F(PG(r-1, q)))} \cong \overline{D_1(PG(r-1, q))/\phi^{-1}(F)}.$$

That is, $U_{2, q^{2r-4}+q+1}$ is a minor of $D_1(PG(r-1, q))$. But $U_{2, q^{2r-4}+q+1}$ is not representable over $GF(q^{2r-4})$ and therefore $d(PG(r-1, q), q) > 2r-4$.

Finally we note that Theorem 3.3 does not generalise easily to quotients determined by modular cuts of $D_k(M)$ whose minimal members are not connected. For example, let F_4 be the free matroid on 4 points and

$E(F_4) = \{1, 2, 3, 4\}$. Then $F = \{\{1, 2\}, \{3, 4\}\}$ is a disconnected flat of $D_1(F_4)$. Let $(D_1(F_4))'$ be the quotient of $D_1(F_4)$ determined by the modular cut consisting of all flats containing F. Now, apart from the double point $\{\{1, 2\}, \{3, 4\}\}$, $(D_1(F_4))'$ is isomorphic to $U_{2,5}$ and it is readily verified that $(D_1(F_4))'$ is not the Dilworth truncation of any quotient of F_4.

References

1. T. H. Brylawski, Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1–44.
2. T. H. Brylawski, Coordinatizing the Dilworth truncation, *in* "Matroid Theory. Proceedings Janos Bolyai Math. Soc." (L. Lovasz, Ed.), North-Holland, Amsterdam, 1985.
3. T. H. Brylawski and J. G. Oxley, Several identities for the characteristic polynomial of a combinatorial geometry, *Discrete Math.* **31** (1980), 161–170.
4. H. H. Crapo and G.-C. Rota, "On the Foundation of Combinatorial Theory: Combinatorial Geometries," MIT Press, Cambridge, MA, 1970.
5. R. P. Dilworth, Dependence relations in a semi modular lattice, *Duke Math. J.* **11** (1944), 575–587.
6. D. A. Higgs, Strong maps of geometries, *J. Combin. Theory* **5** (1968), 185–191.
7. J. H. Mason, Matroids as the study of geometrical configurations, *in* "Higher Combinatorics" (M. Aigner, Ed.), Reidel, Dordrecht, 1977.
8. M. Sved, Gaussians and Binomials, *Arcs Combin.* **17A** (1984), 325–351.
9. D. J. A. Welsh, "Matroid Theory," London Math. Soc. Monographs, Vol. 8, Academic Press, New York, 1976.
10. G. P. Whittle, Modularity in tangential k-blocks, *J. Combin. Theory Ser. B* **42** (1987), 24–35.