THE EXCLUDED MINORS FOR THE CLASS OF MATROIDS THAT ARE BINARY OR TERNARY

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Abstract. We show that the excluded minors for the class of matroids that are binary or ternary are $U_{2,5}$, $U_{3,5}$, $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, $U_{2,4} \oplus_2 F_7^*$, and the unique matroids obtained by relaxing a circuit-hyperplane in either AG(3, 2) or T_{12}. The proof makes essential use of results obtained by Truemper on the structure of almost-regular matroids.

1. Introduction

In [5], Brylawski considered certain natural operations on minor-closed classes of matroids, and examined how they affect the set of excluded minors for those classes. In particular, he invited the reader to explore the excluded minors for the union of two minor-closed classes. We do so in one special case, and determine the excluded minors for the union of the classes of binary and ternary matroids. This solves Problem 14.1.8 in Oxley's list [19].

Theorem 1.1. *The excluded minors for the class of matroids that are binary or ternary are $U_{2,5}$, $U_{3,5}$, $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, $U_{2,4} \oplus_2 F_7^*$, and the unique matroids obtained by relaxing a circuit-hyperplane in either* AG(3, 2) *or T_{12}.*

Recall that the matroid AG(3, 2) is a binary affine space and is produced by deleting a hyperplane from PG(3, 2). Up to isomorphism, there is a unique matroid produced by relaxing a circuit-hyperplane in AG(3, 2). We shall use $\text{AG}(3, 2)'$ to denote this unique matroid.

The matroid T_{12} was introduced by Kingan [13]. It is represented over GF(2) by the matrix displayed in Figure 1. It is clear that T_{12} is self-dual. Moreover, T_{12} has a transitive automorphism group and a unique pair of circuit-hyperplanes. These two circuit-hyperplanes are disjoint. Up to isomorphism, there is a unique matroid produced by relaxing a circuit-hyperplane in T_{12}. We denote this matroid by T'_{12}.

A result due to Semple and Whittle [22] can be interpreted as showing that $U_{2,5}$ and $U_{3,5}$ are the only 3-connected excluded minors for the class in Theorem 1.1 that are representable over at least one field. We complete the characterization by finding the non-representable excluded minors and the excluded minors that are not 3-connected.

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$$\left[\begin{array}{c|cccccc} & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \\ I_6 & 0 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

FIGURE 1. A representation of T_{12}.

The binary matroids and the ternary matroids are well known to have, respectively, one excluded minor and four excluded minors. In this case, the union of two classes with finitely many excluded minors itself has only finitely many excluded minors. Brylawski [5] asked whether this is always true in the case that the two classes have a single excluded minor each. In unpublished work, Vertigan answered this question in the negative (see [7, Section 5]).

Vertigan's examples indicate that Brylawski's project of finding the excluded minors for the union of minor-closed classes is a difficult one. However, in some special cases it may be more tractable. Matroids that are representable over a fixed finite field have received considerable research attention. Indeed, the most famous unsolved problem in matroid theory is Rota's conjecture that there is only a finite number of excluded minors for representability over any fixed finite field [21]. This would stand in contrast to general minor-closed classes. Rota's conjecture is currently known to hold for the fields GF(2), GF(3), and GF(4) [3, 9, 24, 29].

For a collection, $\mathcal{F}$, of fields, let $\mathcal{M}_{\cup}(\mathcal{F})$ be the set of matroids that are representable over at least one field in $\mathcal{F}$. We believe that the following is true.

Conjecture 1.2. *Let $\mathcal{F}$ be a finite family of finite fields. There is only a finite number of excluded minors for $\mathcal{M}_{\cup}(\mathcal{F})$.*

Until now, Conjecture 1.2 was known to hold for only four families, namely $\{\mathrm{GF}(2)\}$, $\{\mathrm{GF}(3)\}$, $\{\mathrm{GF}(4)\}$, and $\{\mathrm{GF}(2), \mathrm{GF}(4)\}$. Thus Theorem 1.1 proves the first case of Conjecture 1.2 that does not reduce to a case of Rota's conjecture.

We note that if we relax the constraint that $\mathcal{F}$ is a finite collection, then $\mathcal{M}_{\cup}(\mathcal{F})$ may have infinitely many excluded minors: the authors of [14] construct an infinite number of excluded minors for real-representability that are not representable over any field. Rado [20] shows that any real-representable matroid is representable over at least one finite field. Thus, if $\mathcal{F}$ is the collection of all finite fields, then $\mathcal{M}_{\cup}(\mathcal{F})$ has an infinite number of excluded minors.

We remark also that although an affirmative answer to Conjecture 1.2 would imply that Rota's conjecture is true, it is conceivable that Conjecture 1.2 fails while Rota's conjecture holds.

Next we note a conjecture of Kelly and Rota [12] that is a natural companion to Conjecture 1.2. Suppose that $\mathcal{F}$ is a family of fields. Let $\mathcal{M}_{\cap}(\mathcal{F})$ be the class of matroids that are representable over every field in $\mathcal{F}$.

Conjecture 1.3. *Let $\mathcal{F}$ be a family of finite fields. There is only a finite number of excluded minors for $\mathcal{M}_{\cap}(\mathcal{F})$.*

It is easy to see that this conjecture holds when $\mathcal{F}$ is finite and contains only fields for which Rota's conjecture holds. Thus Conjecture 1.3 is known to hold if $\mathcal{F}$ contains no field other than GF(2), GF(3), or GF(4). Moreover, the conjecture holds if $\mathcal{F} = \{\mathrm{GF}(3), \mathrm{GF}(4), \mathrm{GF}(5)\}$, in which case $\mathcal{M}_{\cap}(\mathcal{F})$ is Whittle's class of near-regular matroids (see [10, 30, 31]).

It seems likely that the Matroid Minors Project of Geelen, Gerards, and Whittle will affirm both Rota's conjecture and Conjecture 1.3 (see [8]).

The proof of Theorem 1.1 relies heavily upon results due to Truemper [27]. If a matrix is not totally unimodular, but each of its proper submatrices is totally unimodular, then it is called a *minimal violation matrix* for total unimodularity. Truemper studied such matrices and related them to a class of binary matroids which he called "almost-regular". An almost-regular matroid is not regular, but every element has the property that either its deletion or its contraction produces a regular matroid. Truemper gives a characterization of almost-regular matroids, by showing that they can all be produced from the Fano plane or an eleven-element matroid called N_{11}, using only Δ-Y and Y-Δ operations, along with series and parallel extensions.

Truemper's characterization of almost-regular matroids is deep, and perhaps not sufficiently appreciated within the matroid theory community. He does much more than simply provide a Δ-Y reduction theorem. In the process of obtaining this characterization, he obtains specific detailed information about the structure of almost-regular matroids. Without access to these structural insights, we would not have been able to obtain Theorem 1.1. We define almost-regular matroids and discuss Truemper's result in Section 2.6.

In the first half of our proof, we establish that every excluded minor for the class of binary or ternary matroids is a relaxation of an excluded minor for the class of almost-regular matroids, or more precisely the class consisting of the almost-regular matroids and their minors. (Here we are assuming certain conditions on the rank, corank, and connectivity of the excluded minor.) Having done this, we perform a case analysis that bounds the size of the excluded minor.

Now we give a more detailed description of the article. Section 2 establishes some fundamental notions and results that we use throughout the rest of the proof. In Section 2.9 we prove that each of the matroids listed in Theorem 1.1 is indeed an excluded minor for the class of matroids that are binary or ternary. Section 3 contains a discussion of the excluded minors that have low rank, corank, or connectivity. Specifically, we show that any excluded minor that has rank or corank at most three, or that fails to be 3-connected, must be one of those listed in Theorem 1.1. In Section 4 we examine the excluded minors on eight or nine elements, and we show that there is precisely one such matroid: $\mathrm{AG}(3,2)'$.

The results of Sections 3 and 4 show that we can restrict our attention to 3-connected excluded minors with rank and corank at least four and with at least ten elements. We do so in Section 5 where Theorem 5.1 shows that if M is such an excluded minor, then M can be produced by relaxing a circuit-hyperplane in a binary matroid, which we call M_B. Section 6 shows that every proper minor of M_B is either regular, or belongs to Truemper's class of almost-regular matroids.

In Section 7 we use Truemper's structural results on almost-regular matroids and perform a case analysis that reduces the problem of finding the remaining excluded minors to a finite task. We consider three cases: M_B has an R_{10}-minor; M_B has an R_{12}-minor; and M_B has neither an R_{10}- nor an R_{12}-minor. In the first case we show that $|E(M_B)| = 12$. Next we show that the second case cannot arise, and finally we show that if M_B has no minor isomorphic to R_{10} or R_{12}, then $|E(M_B)| \leq 16$. Having reduced the problem to a finite case-check, we complete the proof of Theorem 1.1 in Section 8.

2. Preliminaries

Throughout the article, $\mathcal{M}$ will denote the class of matroids that are either binary or ternary; that is, $\mathcal{M} = \mathcal{M}_{\cup}(\{\mathrm{GF}(2), \mathrm{GF}(3)\})$. The matroid terminology used throughout will follow Oxley [19], except that $\mathrm{si}(M)$ and $\mathrm{co}(M)$ respectively are used to denote the simple and cosimple matroids associated with the matroid M. A *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. We shall occasionally refer to a rank-2 flat as a *line*. Suppose that a binary matroid is represented over GF(2) by $[I_r|A]$. We shall say that A is a *reduced representation* of M.

We start by stating the well-known excluded-minor characterizations of binary and ternary matroids.

Theorem 2.1. (Tutte [29]). *A matroid is binary if and only if it has no minor isomorphic to* $U_{2,4}$.

Theorem 2.2. (Reid, Bixby [3], Seymour [24]). *A matroid is ternary if and only if it has no minor isomorphic to* $U_{2,5}$, $U_{3,5}$, F_7, *or* F_7^*.

2.1. **Connectivity.** Suppose that M is a matroid on the ground set E. If $X \subseteq E$, then $\lambda_M(X)$ (or just $\lambda(X)$) is defined to be $r_M(X) + r_M(E - X) - r(M)$. Note that $\lambda(X) = \lambda(E - X)$ and $\lambda_M(X) = \lambda_{M^*}(X)$ for all subsets $X \subseteq E$. A *k-separation* of M is a partition (X_1, X_2) of E such that $|X_1|, |X_2| \geq k$, and $\lambda_M(X_1) < k$. A k-separation (X_1, X_2) is *exact* if

$\lambda_M(X_1) = k - 1$. We say that M is *n-connected* if it has no k-separations where $k < n$. A 2-connected matroid is often said to be *connected*. We say that M is *internally 4-connected* if M is 3-connected, and, whenever (X_1, X_2) is a 3-separation, $\min\{|X_1|, |X_2|\} = 3$.

Proposition 2.3. *Suppose that N is a minor of a matroid M, and that X is a subset of $E(N)$. Then $\lambda_N(X) \leq \lambda_M(X)$.*

Suppose that M_1 and M_2 are matroids on the ground sets E_1 and E_2 respectively, and that $\mathcal{C}_i$ is the collection of circuits of M_i for $i = 1, 2$. If $E_1 \cap E_2 = \varnothing$, then the *1-sum* of M_1 and M_2, denoted by $M_1 \oplus M_2$, is defined to be the matroid with $E_1 \cup E_2$ as its ground set and $\mathcal{C}_1 \cup \mathcal{C}_2$ as its collection of circuits.

If $E_1 \cap E_2 = \{p\}$ and neither M_1 nor M_2 has p as a loop or a coloop, then we can define the *2-sum* of M_1 and M_2, denoted by $M_1 \oplus_2 M_2$. The ground set of $M_1 \oplus_2 M_2$ is $(E_1 \cup E_2) - p$, and its circuits are the members of

$$\{C \in \mathcal{C}_1 \mid p \notin C\} \cup \{C \in \mathcal{C}_2 \mid p \notin C\} \cup \{(C_1 \cup C_2) - p \mid C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2, p \in C_1 \cap C_2\}.$$

We say that p is the *basepoint* of the 2-sum.

The next results follow from [25, (2.6)] and [19, Proposition 7.1.15 (v)] respectively.

Proposition 2.4. *If (X_1, X_2) is an exact 2-separation of a matroid M, then there are matroids M_1 and M_2 on the ground sets $X_1 \cup p$ and $X_2 \cup p$ respectively, where p is in neither X_1 nor X_2, such that M is equal to $M_1 \oplus_2 M_2$. Moreover, M has proper minors isomorphic to both M_1 and M_2.*

Proposition 2.5. *Suppose that M_1 and M_2 are matroids and that the 2-sum of M_1 and M_2 along the basepoint p is defined. If N_i is a minor of M_i such that $p \in E(N_i)$ for $i = 1, 2$, and p is a loop or coloop in neither N_1 nor N_2, then $N_1 \oplus_2 N_2$ is a minor of $M_1 \oplus_2 M_2$.*

2.2. **Relaxations.** Suppose that M_1 and M_2 are matroids sharing a common ground set, and that the collections of bases of M_1 and M_2 agree with the exception of a single set Z that is a circuit-hyperplane in M_1 and a basis in M_2. In this case we say that M_2 is obtained from M_1 by *relaxing* the circuit-hyperplane Z.

Next we list some well-known properties of relaxation.

Proposition 2.6. *Suppose that M_2 is obtained from M_1 by relaxing the circuit-hyperplane Z. If $e \in Z$ then $M_1\backslash e = M_2\backslash e$. Moreover, $Z - e$ is a hyperplane of M_1/e, and M_2/e is obtained from M_1/e by relaxing $Z - e$. Similarly, if $e \notin Z$ then $M_1/e = M_2/e$ and Z is a hyperplane of $M_1\backslash e$. Then $M_2\backslash e$ is obtained from $M_1\backslash e$ by relaxing Z.*

If M_1 and M_2 are matroids on the same set such that $M_1 \neq M_2$, then there is a some set that is independent in exactly one of M_1 and M_2. We shall call such a set a *distinguishing set*. The next result is obvious.

Proposition 2.7. *Suppose that M_1 and M_2 are two matroids on the same ground set and that Z is a minimal distinguishing set for M_1 and M_2. Then Z is a circuit in one of M_1 and M_2, and independent in the other.*

Proposition 2.8. *Let M_1 and M_2 be loopless matroids such that $E(M_1) = E(M_2)$ and $r(M_1) = r(M_2)$. Suppose that M_1 and M_2 have a unique distinguishing set Z, and that Z is independent in M_2. Then Z is a circuit-hyperplane of M_1 and a basis of M_2, and M_2 is obtained from M_1 by relaxing Z. Furthermore, $Z \cup e$ is a circuit of M_2 for all $e \in E(M_2) - Z$.*

Proof. As Z is the unique distinguishing set, it is also a minimal distinguishing set. Therefore Z is a circuit of M_1 by Proposition 2.7. If Z is not a basis of M_2, then Z is properly contained in a basis B of M_2. Since $Z \subset B$, we deduce that B is dependent in M_1, and we have a contradiction to the uniqueness of Z. Thus Z is a basis of M_2.

Suppose that there is an element y in $\mathrm{cl}_{M_1}(Z) - Z$. Then there is a circuit C of M_1 such that $y \in C$ and $C \subseteq Z \cup \{y\}$. Since $C \neq Z$ and C is dependent in M_1, it follows that C is dependent in M_2. But $Z \cup \{y\}$ contains a unique circuit $C_{M_2}(y, Z)$ of M_2. Therefore $C_{M_2}(y, Z) \subseteq C$. As y is not a loop, it follows that there is an element e in $C_{M_2}(y, Z) - \{y\}$. By circuit elimination in M_1 using the circuits C and Z and the common element e, we deduce that there is a circuit C' of M_1 such that $y \in C'$ and $C' \subseteq (Z \cup \{y\}) - \{e\}$. Now $C' \neq Z$, so C' is dependent in M_2. We can again conclude that $C_{M_2}(y, Z) \subseteq C'$. But this is a contradiction as $e \notin C'$. Therefore Z is a flat of M_1. As $|Z| = r(M_2) = r(M_1)$, it follows that Z is a circuit-hyperplane of M_1.

The independent sets of the matroid obtained from M_1 by relaxing Z are precisely the independent sets of M_1, along with Z. This is exactly the collection of independent sets of M_2, so M_2 is obtained from M_1 by relaxing Z. Suppose that $e \in E(M_2) - Z$. As Z is a basis of M_2, there is a circuit C of M_2 such that $e \in C$ and $C \subseteq Z \cup e$. Since $C \neq Z$, the set C cannot be distinguishing. Therefore C is dependent in M_1. But the only circuit of M_1 that is contained in $Z \cup e$ is Z itself. Therefore C contains Z, so $C = Z \cup e$. This completes the proof. □

Recall that $\mathcal{W}_n$ is the graph obtained from the cycle on n vertices by adding a new vertex adjacent to all other vertices. The edges adjacent to the new vertex are known as *spoke* edges, and all other edges are known as *rim* edges. We refer to $M(\mathcal{W}_n)$ as the *rank-n wheel*. The rim edges form a circuit-hyperplane of the rank-n wheel. The matroid produced by relaxing this circuit-hyperplane is the *rank-n whirl*, denoted by $\mathcal{W}^n$.

An *enlarged wheel* is obtained by adding parallel elements to spoke edges and adding series elements to rim edges by subdividing them. The rim edges of the original graph, along with all the added series elements, form a circuit-hyperplane of the enlarged wheel, this set of edges is called the *rim* of the enlarged wheel.

The following result of Oxley and Whittle characterizes when a relaxation of a ternary matroid is ternary.

Lemma 2.9. [16, Theorem 5.3]. *Suppose that M is a ternary matroid and that Z is a circuit-hyperplane of M. Let M' be the matroid obtained from M by relaxing Z in M. If M' is ternary, then there is an enlarged wheel G such that $M = M(G)$ and Z is the rim of G.*

2.3. **The Splitter Theorem.** Suppose that $\mathcal{N}$ is a class of matroids that is closed under taking minors. A *splitter* of $\mathcal{N}$ is a matroid $N \in \mathcal{N}$ such that if N' is a 3-connected member of $\mathcal{N}$ and N' has an N-minor, then N' is isomorphic to N.

Seymour's Splitter Theorem [25] reduces the problem of identifying splitters to a finite case check (see [19, Theorem 11.1.2]).

Theorem 2.10. *Let N be a 3-connected proper minor of a 3-connected matroid M and suppose that $|E(N)| \geq 4$. Assume also that if N is a wheel, then M has no larger wheel as a minor, while if N is a whirl, then M has no larger whirl as a minor. Then M has an element e such that $M\backslash e$ or M/e is 3-connected and has an N-minor.*

2.4. **The Δ-Y operation.** Suppose that M is a matroid and that T is a coindependent triangle of M. Let N be an isomorphic copy of $M(K_4)$, where $E(N) \cap E(M) = T$ and T is a triangle of N. Then $P_T(N, M)$, the *generalized parallel connection* of N and M, is defined [4]. It is the matroid on the ground set $E(M) \cup E(N)$ with flats being all sets F such that $F \cap E(M)$ and $F \cap E(N)$ are flats of M and N respectively. Then $P_T(N, M)\backslash T$ is said to be obtained from M by performing a *Δ-Y operation* upon M. We denote this matroid by $\Delta_T(M)$. If T is an independent triad of M, then $(\Delta_T(M^*))^*$ is defined and is said to be obtained from M by a *Y-Δ operation*. The resulting matroid is denoted by $\nabla_T(M)$.

2.5. **Regular decomposition.** We shall make use of some of the intermediate results proved by Seymour [25] as part of his decomposition theorem for regular matroids.

Theorem 2.11. *Every regular matroid can be constructed using 1-, 2-, and 3-sums, starting from matroids that are graphic, cographic, or isomorphic copies of R_{10}.*

The following matrix is a reduced representation of R_{10}.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Any single-element deletion of R_{10} is isomorphic to $M(K_{3,3})$ and any single-element contraction is isomorphic to $M^*(K_{3,3})$. Moreover, the automorphism group of R_{10} acts transitively upon pairs of elements, and R_{10} is isomorphic to its dual [25, p. 328].

Proposition 2.12. [25, (7.4)]. *The matroid R_{10} is a splitter for the class of regular matroids.*

The proof of the decomposition theorem features another important binary matroid, R_{12}. The following matrix, A, is a reduced representation of R_{12}.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Clearly R_{12} is self-dual. Suppose that the columns of $[I_6|A]$ are labeled $1, \ldots 12$. Then $X_1 = \{1, 2, 5, 6, 9, 10\}$ is a union of two triangles. If we let X_2 be the complement of X_1, then (X_1, X_2) is a 3-separation of R_{12}. Moreover, if M is a regular matroid and R_{12} is a minor of M, then there is a 3-separation (Y_1, Y_2) of M such that $X_i \subseteq Y_i$ for $i = 1, 2$ (see [25, (9.2)]).

One of the important steps in the decomposition theorem is to prove the following result.

Lemma 2.13. [25, (14.2)]. *If a 3-connected regular matroid has no minor isomorphic to R_{10} or R_{12}, then it is either graphic or cographic.*

2.6. **Almost-regular matroids.** Next we discuss Truemper's class of almost-regular matroids [27]. Recall that a matroid is regular if and only if it can be represented by a matrix over the real numbers with the property that every subdeterminant belongs to $\{0, 1, -1\}$. Such a matrix is said to be *totally unimodular*. If a matrix is not totally unimodular, but removing any row or column produces a totally unimodular matrix, then it is said to be a *minimal violation matrix* for total unimodularity. The study of this class of matrices motivated Truemper to make the following definition.

Definition 2.14. A matroid M is *almost-regular* if it is binary but not regular, and $E(M)$ can be partitioned into non-empty sets *del* and *con*, such that

(i) if $e \in del$ then $M \backslash e$ is regular;
(ii) if $e \in con$ then M/e is regular;
(iii) the intersection of any circuit with *con* has even cardinality; and
(iv) the intersection of any cocircuit with *del* has even cardinality.

Truemper shows that the study of minimal violation matrices for total unimodularity is essentially reduced to the study of almost-regular matroids (see [28, Section 12.4]). Any such matrix that does not represent an almost-regular matroid (over GF(2)) belongs to one of two simple classes.

Proposition 2.15. [27, Theorem 21.4 (ii)]. *The class of almost-regular matroids is closed under duality.*

Proposition 2.16. [27, Theorem 21.4 (iii)]. *Suppose that M is an almost-regular matroid. Then every minor of M is either regular or almost-regular.*

The focus of Truemper's investigation into almost-regular matroids is the class of almost-regular matroids that are irreducible. An almost-regular matroid M is *irreducible* if M cannot be reduced in size by performing a sequence of the following operations: (i) Δ-Y and Y-Δ operations; and (ii) replacing a parallel (series) class with a non-empty parallel (series) class of a different size. (Note that certain restrictions are placed upon these operations. The restrictions depend upon the partition of the ground set into *del* and *con*.) An irreducible almost-regular matroid is necessarily internally 4-connected [27, Theorem 22.1].

The main result of [27] shows that every almost-regular matroid can be constructed using a sequence of the operations listed above, starting from one of two matroids: F_7 and N_{11}. The second of these matroids is defined in Section 7.1.

2.7. **Grafts.** Suppose that G is a graph and that D is a set of vertices of G. We say that the pair (G, D) is a *graft*. Let A be the vertex-edge incidence matrix describing G, so that the rows of A correspond to vertices of G, and columns of A correspond to edges. Then $M(G) = M[A]$, where A is considered as a matrix over GF(2). Let A' be the matrix obtained from A by adding a column with entries from GF(2), so that an entry in the new column is non-zero if and only if it appears in a row corresponding to a vertex in D. Let $M(G, D)$ be the binary matroid $M[A']$. We abuse terminology slightly by calling any binary matroid of the form $M(G, D)$ a *graft*. We shall call the element of $M(G, D)$ that corresponds to the new column of A' the *graft element*. Clearly a binary matroid is a graft if and only if it is a single-element extension of a graphic matroid.

The next result is easy to verify.

Proposition 2.17. *Suppose that (G, D) is a graft. Let e be an edge of G with end-vertices u and v. Then $M(G, D)\backslash e = M(G\backslash e, D)$. Furthermore, suppose that w is the vertex of G/e produced by identifying u and v. Then $M(G, D)/e = M(G/e, D')$, where:*

(i) $D' = D$ if $|\{u, v\} \cap D| = 0$;
(ii) $D' = (D - \{u, v\}) \cup w$ if $|\{u, v\} \cap D| = 1$; and
(iii) $D' = D - \{u, v\}$ if $|\{u, v\} \cap D| = 2$.

Let (G, D) be a graft. Suppose that v is a vertex of degree two in G and that $v \in D$. Suppose that v is adjacent to the two vertices u and w. Let a be the edge between v and u, and let b be the edge between v and w. Consider the graph G' with the following properties: G' has the same edge set as G, and a joins v to w in G', while b joins v to u. All other edges have the same incidences as they do in G. Let D' be the symmetric difference of D and $\{u, w\}$. Then $M(G', D') = M(G, D)$. We say that (G', D') is obtained from (G, D) by *switching*.

2.8. **Truemper graphs.** In this section we introduce a family of graphs that provide an important tool for studying almost-regular matroids.

Definition 2.18. A graph G is a *Truemper graph* if it contains two vertex-disjoint paths R and S, such that every vertex of G is in either R or S, and any edge not in either R or S joins a vertex of R to a vertex of S.

We shall use the notation $G = (R, S)$ to indicate that G is a Truemper graph, and that R and S are the vertex-disjoint paths described in Definition 2.18. In this case we shall say that an edge in either R or S is a *path edge*, and any other edge is a *cross edge*. We shall say that the end-vertices of R and S are *terminal vertices*. All other vertices will be known as *internal vertices*. Often we are interested in a graft (G, D), where G is a Truemper graph, and D consists of the four terminal vertices of G. However, much of our argument will focus on structure in the underlying Truemper graph.

Let $G = (R, S)$ be a Truemper graph. We say that G has an XX-minor if we can obtain the graph shown in Figure 2 by contracting path edges and deleting cross edges from G. The remaining path edges of G are the horizontal edges in the diagram.

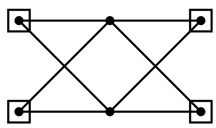


FIGURE 2. An XX-minor.

Proposition 2.19. [27, 23.50]. *Suppose that $G = (R, S)$ is a Truemper graph. Let D be the set of terminal vertices of G. If the graft $M(G, D)$ is almost-regular, then G does not have an* XX*-minor.*

Proof. Assume that G does have an XX-minor. Proposition 2.17 implies that $M(G, D)$ has $M(G', D)$ as a minor, where G' is the graph shown in Figure 2, and D is the set of vertices marked by squares. But $M(G', D)$ has a minor isomorphic to $\mathrm{AG}(3, 2)$. Certainly $\mathrm{AG}(3, 2)$ is not regular and every single-element deletion or contraction of $\mathrm{AG}(3, 2)$ is isomorphic to F_7^* or F_7 respectively. Therefore AG(3, 2) is not almost-regular. Proposition 2.16 implies that $M(G, D)$ cannot be almost-regular. □

The next result is easy to prove.

Proposition 2.20. *Let $G = (R, S)$ be a Truemper graph with no* XX*-minor such that both R and S contain at least two vertices. Suppose that F is a set* *of four cross edges such that every terminal vertex of G is incident with at least one edge in F. Then at least one edge in F joins two terminal vertices.*

Corollary 2.21. *Let $G = (R, S)$ be a Truemper graph with no* XX*-minor such that both R and S have at least two vertices. Suppose that the cross edges of G form a spanning cycle. Then one of the following holds:*

(i) one of the end-vertices of R is adjacent to both of the end-vertices of S.
(ii) one of the end-vertices of S is adjacent to both of the end-vertices of R.

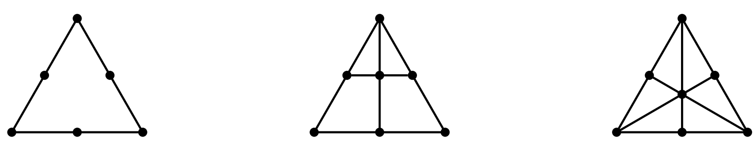
Proof. Suppose that the result fails. Since every vertex in G is incident with exactly two cross edges, this means that for each terminal vertex v, we can find a cross edge which joins v to an internal vertex. This provides a contradiction to Proposition 2.20. □

2.9. **Excluded minors.** We end this preliminary section by proving one direction of our main theorem.

Lemma 2.22. *The matroids $U_{2,5}$, $U_{3,5}$, $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, $U_{2,4} \oplus_2 F_7^*$, $\mathrm{AG}(3,2)'$, and T_{12}' are all excluded minors for $\mathcal{M}$.*

Proof. The only matroids listed here for which the result is not obvious are $\mathrm{AG}(3,2)'$ and T_{12}'. Let M_1 be a matroid such that $M_1 \cong \mathrm{AG}(3,2)$ and let Z be a circuit-hyperplane of M_1. Let M_2 be the matroid obtained from M_1 by relaxing Z. Suppose that $e \in Z$. By Proposition 2.6, we see that $M_2 \backslash e \cong \mathrm{AG}(3,2) \backslash e \cong F_7^*$ and that M_2/e can be obtained from $\mathrm{AG}(3,2)/e \cong F_7$ by relaxing a circuit-hyperplane. Therefore $M_2/e \cong F_7^-$, where F_7^- is illustrated in Figure 3. Since F_7^- is non-binary, these facts show that M_2 is neither binary nor ternary.

On the other hand, if $e \notin Z$ then $M_2/e = M_1/e \cong F_7$, and $M_2 \backslash e$ is isomorphic to the matroid obtained from $\mathrm{AG}(3,2)\backslash e \cong F_7^*$ by relaxing a circuit-hyperplane. Thus $M_2 \backslash e \cong (F_7^-)^*$, so every single-element deletion or contraction of M_2 is either binary or ternary, and we are done.



$\mathcal{W}_3$ P_7 F_7^-

FIGURE 3. $\mathcal{W}^3$, P_7, and F_7^-.

Now we will assume that M_1 is isomorphic to T_{12}. Assume that the columns of the matrix in Figure 1 are labeled $\{1, \ldots, 12\}$. Then $Z = \{2, 4, 6, 8, 10, 12\}$ is a circuit-hyperplane. Let M_2 be the matroid obtained by relaxing Z. Note that $Z \cup \{1\}$ and $Z \cup \{3\}$ are circuits of M_2. If M_2 were binary, then the symmetric difference of these sets, that is $\{1, 3\}$, would be a union of circuits. Therefore M_2 is non-binary.

By pivoting on the entry in column 7 and row 2, we see that $M_1/\{1, 3, 7\}\backslash\{2, 12\}$ is isomorphic to F_7, so $M_1\backslash 12$ has an F_7-minor. Therefore $M_2\backslash 12$ has an F_7-minor, by Proposition 2.6, so M_2 is not ternary.

Proposition 2.6 implies that $M_2\backslash 6 = M_1\backslash 6$, so $M_2\backslash 6$ is binary. Consider $M_2/6$. It is not difficult to show that this matroid is represented over GF(3) by the matrix produced by deleting row 6 from the matrix in Figure 1. Thus $M_2/6$ is ternary. Now suppose that e is any element in $\{1, \ldots, 12\}$. Since the automorphism group of T_{12} is transitive, there is an automorphism which takes e to 6. Thus $M_2\backslash e$ and M_2/e are isomorphic to $M_2\backslash 6$ and $M_2/6$, and are therefore binary and ternary respectively. It follows that M_2 is an excluded minor for $\mathcal{M}$, as desired. □

3. Excluded minors with low rank, corank, or connectivity

In this section we find all the excluded minors for $\mathcal{M}$ that have rank or corank at most three, or that fail to be 3-connected.

Proposition 3.1. *If M is an excluded minor for $\mathcal{M}$, then M cannot have as a minor either a simple connected single-element extension of F_7 or a cosimple connected single-element coextension of F_7^*.*

Proof. It follows from the fact that F_7 is a projective plane that it has exactly two simple connected single-element extensions; one is obtained by adding an element freely to F_7, and the other is obtained by adding an element freely on a line of F_7. In either case, on contracting the newly added element, we obtain a matroid with a $U_{2,5}$-restriction, a contradiction as $U_{2,5}$ is an excluded minor for $\mathcal{M}$. Hence M has no simple connected single-element extension of F_7 as a minor. The second part of the result follows by duality. □

Lemma 3.2. *The only excluded minors for $\mathcal{M}$ that have rank or corank less than four are $U_{2,5}$ and $U_{3,5}$.*

Proof. It is clear that $U_{2,5}$ is the only rank-2 excluded minor for $\mathcal{M}$. By duality, $U_{3,5}$ is the unique excluded minor for $\mathcal{M}$ with corank two. Now let M be a rank-3 excluded minor for $\mathcal{M}$ that is not isomorphic to $U_{3,5}$. Since M is non-ternary and has rank three, it follows from Theorem 2.2 that M has F_7 as a minor. But M is non-binary and simple, and therefore has a simple connected single-element extension of F_7 as a restriction. This contradiction to Proposition 3.1 implies that $U_{3,5}$ is the unique rank-3 excluded minor for $\mathcal{M}$ and, by duality, $U_{2,5}$ is the unique excluded minor for $\mathcal{M}$ with corank three. □

Lemma 3.3. *The only excluded minors for $\mathcal{M}$ that are not 3-connected are $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, and $U_{2,4} \oplus_2 F_7^*$.*

Proof. We shall show that the excluded minors for $\mathcal{M}$ that are connected but not 3-connected are $U_{2,4} \oplus_2 F_7$ and $U_{2,4} \oplus_2 F_7^*$. A similar, but simpler, argument shows that the disconnected excluded minors for $\mathcal{M}$ are precisely $U_{2,4} \oplus F_7$ and $U_{2,4} \oplus F_7^*$.

Let M be an excluded minor for $\mathcal{M}$ that is connected but not 3-connected. It follows from Proposition 2.4 that M is the 2-sum of matroids M_1 and M_2 along the basepoint p. Then M_1 and M_2 are connected, for otherwise M is not connected. Each of M_1 and M_2, being isomorphic to a proper minor of M, is either binary or ternary. Moreover, since the property of being representable over a particular field is closed under 2-sums, it follows that at least one of M_1 and M_2 is non-binary, and at least one is non-ternary. Thus we may assume that M_1 is ternary but non-binary, and that M_2 is binary but non-ternary. Thus M_1 has a $U_{2,4}$-minor, and M_2 has a minor isomorphic to one of $U_{2,5}$, $U_{3,5}$, F_7, or F_7^*. Both $U_{2,5}$ and $U_{3,5}$ are excluded minors for $\mathcal{M}$. Thus neither is a minor of M_2. Hence M_2 has a minor isomorphic to one of F_7 and F_7^*. It follows from roundedness results of Seymour [23] and Bixby [2] (or see [19, p. 374]), that M_2 has an F_7- or F_7^*-minor using p, and M_1 has a $U_{2,4}$-minor using p. Thus M has a minor isomorphic to one of $U_{2,4} \oplus_2 F_7$ or $U_{2,4} \oplus_2 F_7^*$ by Proposition 2.5. Since these two matroids are excluded minors for $\mathcal{M}$, it follows that M is isomorphic to either $U_{2,4} \oplus_2 F_7$ or $U_{2,4} \oplus_2 F_7^*$. This completes the proof. □

4. Excluded minors with at most nine elements

In this section we find those excluded minors for $\mathcal{M}$ that have at most nine elements.

Lemma 4.1. *There is a unique* 8*-element excluded minor for* $\mathcal{M}$*, namely* $\mathrm{AG}(3,2)'$.

Proof. Let M be an 8-element excluded minor for $\mathcal{M}$. Thus M has no $U_{2,5}$-minor and no $U_{3,5}$-minor. Moreover, M must be 3-connected by Lemma 3.3. It follows from Lemma 3.2 that $r(M) \geq 4$ and $r^*(M) \geq 4$, so in fact $r(M) = r^*(M) = 4$. We shall show next that M has no triangles and no triads. By duality, it suffices to show that M has no triads.

Assume that M has a triad, T. Certainly T is independent, for M is 3-connected. Suppose that $T = \{a, b, c\}$. Note that T is a triangle in $\nabla_T(M)$. Now $\nabla_T(M)$ has rank three (see [15, Lemma 2.6]). Moreover, since M is neither binary nor ternary, it follows by the proof of Theorem 6.1 in [1] that $\nabla_T(M)$ is neither binary nor ternary. Lemma 3.2 implies that $\nabla_T(M)$ has a minor isomorphic to either $U_{2,5}$ or $U_{3,5}$. If $\nabla_T(M)$ has a $U_{3,5}$-minor, then, as $U_{3,5}$ has no triangles, we can assume by relabeling if necessary that $\nabla_T(M)\backslash a$ has a $U_{3,5}$-minor. It follows that M/a has a $U_{3,5}$-minor ([15, Corollary 2.14]). This is a contradiction, so $\nabla_T(M)$ has no $U_{3,5}$-minor but it does have a $U_{2,5}$-minor. Note that $\mathrm{si}(\nabla_T(M))$ has rank three. Suppose that the corank of $\mathrm{si}(\nabla_T(M))$ is at most two. Then $\mathrm{si}(\nabla_T(M))$ contains at most five elements. Since we can assume that T is a triangle of $\mathrm{si}(\nabla_T(M))$, it follows that $\mathrm{si}(\nabla_T(M))$ is not 3-connected. If $\mathrm{si}(\nabla_T(M))$ has corank at least three, then it follows from [19, Proposition 11.2.16] that $\mathrm{si}(\nabla_T(M))$ is not 3-connected, so $\mathrm{si}(\nabla_T(M))$ is not 3-connected in either case. Thus $\nabla_T(M)$ is the union of two rank-2 flats, one of which contains $\{a, b, c\}$. Since $M = \Delta_T(\nabla_T(M))$ [15, Corollary 2.12], it is easy to see that M also fails to be 3-connected, and this is a contradiction. We conclude that M has no triads (and by duality, M has no triangles).

Now M is non-ternary but has no $U_{2,5}$- or $U_{3,5}$-minor. Thus M has F_7 or F_7^* as a minor. By duality, we may assume that M has an F_7^*-minor. Let us assume that $E(M) = \{1, \ldots, 8\}$ and that $M\backslash 8 \cong F_7^*$. Consider $M/8$. Since M is non-binary and 3-connected, and $M\backslash 8$ is binary, it follows from [18, Corollary 3.9] that if $M/8$ is binary, then $M \cong U_{2,4}$, which is impossible. Therefore $M/8$ is non-binary and hence ternary. Since M has no triangles and no $U_{3,5}$-minors, we see that $M/8$ is simple and has no rank-2 flat containing more than three points. This implies that $M/8$ is 3-connected. Since $M/8$ has $\mathcal{W}^2$ (that is, $U_{2,4}$) as a minor but has no $U_{2,5}$- or $U_{3,5}$-minor, we deduce from Theorem 2.10 that $M/8$ has a $\mathcal{W}^3$-minor. Thus $M/8$ is a 3-connected and ternary single-element extension of $\mathcal{W}^3$ and $M/8$ has no lines with more than three points. We will show that $M/8$ is isomorphic to either P_7 or F_7^-, where these matroids are illustrated in Figure 3.

Let us suppose that $M/8\backslash 7 \cong \mathcal{W}^3$. Since matroid representations over GF(3) are unique [6], we can assume that $M/8\backslash 7$ has the following reduced representation.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

By adjoining a single column to this matrix, we can obtain a representation over GF(3) of $M/8$. This new column must contain three non-zero elements, for $M/8$ is 3-connected and has no four-element lines. By scaling we may assume that the first entry is 1. If the new column is $[1\ 1\ 1]^T$, then $M/8$ is isomorphic to F_7^-. In all other cases, $M/8 \cong P_7$.

Suppose that $M/8 \cong P_7$. Since $M/8$ has two disjoint triangles, M has two 4-element circuits meeting in $\{8\}$. These circuits must also be hyperplanes of M, as M has no triangle and no $U_{3,5}$-minor. Deleting 8 from each of these two circuit-hyperplanes produces two disjoint hyperplanes of F_7^* of size three. Thus we can find two 4-element circuits of F_7 whose union is equal to the ground set. This is easily seen to be impossible, so $M/8 \cong F_7^-$.

Since F_7^- has exactly six non-trivial lines, there are exactly six 4-element circuits of M that contain 8. Each of these must also be a hyperplane of M. Thus M has exactly six 4-element cocircuits that avoid 8. Each of these cocircuits is also a 4-element cocircuit of $M\backslash 8 \cong F_7^*$. But F_7^* has exactly seven 4-element cocircuits. Thus precisely one of the 4-element cocircuits of $M\backslash 8$ arises by deleting 8 from a 5-element cocircuit of M. We may assume, without loss of generality, that $\{4,5,6,7,8\}$ is a cocircuit of M. Therefore $\{1,2,3\}$ is an independent hyperplane of M and $\{1,2,3,e\}$ is a basis of M, for any $e \in \{4,5,6,7,8\}$.

When B is a basis of a matroid N, consider a matrix $[I_{r(N)}|A]$, where the columns of $I_{r(N)}$ and of A are labeled by the elements of B and by the elements of $E(N) - B$, respectively. Therefore there is a natural correspondence between the elements of B and the rows of A. We call $[I_{r(M)}|A]$ a *partial representation* of N with respect to B if, for each x in B and each y in $E(N) - B$, the entry in row x and column y of A is a one if $(B - x) \cup y$ is a basis of N, and a zero otherwise.

Let $[I_4|A]$ be a partial representation of M with respect to $\{1,2,3,4\}$. The fact that $\{1,2,3,e\}$ is a basis of M for all $e \in \{4,5,6,7,8\}$ means that each of the entries in A in the row associated with 4 must be one. Note that, as $M\backslash 8$ is binary, the matrix produced by deleting the column labeled by 8 from $[I_4|A]$ actually represents $M\backslash 8$ over GF(2). Each column labeled by 5, 6, or 7 must contain at least three ones, as $M\backslash 8 \cong F_7^*$ has no triangles. However, $M\backslash 8$ has no circuits of size five, so each of these columns contains exactly three ones. Now we can assume that $[I_4|A]$ is the matrix shown in Figure 4. As M has no triads, each of x_1, x_2, and x_3 must be equal to one.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\left[\begin{array}{c|cccc} & 0 & 1 & 1 & x_1 \\ I_4 & 1 & 0 & 1 & x_2 \\ & 1 & 1 & 0 & x_3 \\ & 1 & 1 & 1 & x_4 \end{array}\right]$$

Figure 4. A partial representation of M.

For each e in $\{1,2,3\}$, the matroid $M\backslash 8/e \cong M(K_4)$. Thus M/e is a binary or ternary extension of $M(K_4)$ with no 4-element lines, so M/e is isomorphic to F_7 or F_7^-. Because $\{1,2,3,8\}$ is not a circuit of M, it follows that $M/e \cong F_7^-$ for each $e \in \{1,2,3\}$. Using this, one easily checks that the following six sets must be circuits of M:

$$\{1,4,5,8\}, \{1,6,7,8\}, \{2,4,6,8\}, \{2,5,7,8\}, \{3,4,7,8\}, \{3,5,6,8\}.$$

In addition, all seven 4-element circuits of $M\backslash 8$ are also circuits of M. We have now described thirteen 4-element circuits of M. If this is the complete list of 4-element circuits of M, then it is easy to see that $M \cong \mathrm{AG}(3,2)'$. Therefore assume that C is a 4-element circuit of M that is not one of the thirteen circuits we have described. Obviously $8 \in C$. We have already stated that $\{1,2,3,8\}$ is a basis, so $C \neq \{1,2,3,8\}$. Now $|C \cap \{1,2,3,8\}| \neq 3$, for otherwise M restricted to $C \cup \{1,2,3,8\}$ is isomorphic to $U_{3,5}$. Similarly, $|C \cap \{4,5,6,7\}| \neq 3$. Thus C contains 8, a single element from $\{1,2,3\}$, and two elements from $\{4,5,6,7\}$. We can again find a 4-element circuit that meets C in three elements, and deduce the presence of a $U_{3,5}$-minor. This contradiction completes the proof. □

Our next task is to prove that there are no excluded minors with nine elements. We need some preliminary facts.

Proposition 4.2. *Suppose that M is a 3-connected excluded minor for $\mathcal{M}$. For every element $e \in E(M)$, either $M\backslash e$ or M/e is ternary.*

Proof. Suppose that, for some element e of M, neither $M\backslash e$ nor M/e is ternary. Then both $M\backslash e$ and M/e are binary. Thus M is isomorphic to $U_{2,4}$ by a result of Oxley's [18, Corollary 3.9]. This contradiction completes the proof. □

Proposition 4.3. *Suppose that M is a 3-connected excluded minor for $\mathcal{M}$. Then M has no minor isomorphic to* $\mathrm{AG}(3,2)$.

Proof. For every element e of $\mathrm{AG}(3,2)$, the matroids $\mathrm{AG}(3,2)\backslash e$ and $\mathrm{AG}(3,2)/e$ are isomorphic to F_7^* and F_7, respectively. As neither of the last two matroids is ternary, the result follows by Proposition 4.2. □

The binary matroid S_8 is represented over $\mathrm{GF}(2)$ by the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Clearly S_8 is self dual. Seymour [26] proved the following result.

Proposition 4.4. *The only 3-connected binary single-element coextensions of F_7 are* $\mathrm{AG}(3,2)$ *and S_8.*

Proposition 4.5. *Suppose that M is a 3-connected excluded minor for $\mathcal{M}$ and that $|E(M)| \geq 9$. Then M has S_8 as a minor.*

Proof. The hypotheses imply that M has no minor isomorphic to $U_{2,5}$ or $U_{3,5}$. As M is non-ternary it must have either an F_7-minor or a F_7^*-minor. The Splitter Theorem (2.10) implies that M has a minor M_1 such that M_1 is a 3-connected single-element extension or coextension of either F_7 or F_7^*. Proposition 3.1 implies that M_1 is an extension of F_7^* or a coextension of F_7. If M_1 is non-binary, then M_1 is both non-binary and non-ternary, so $M_1 = M$ and hence $|E(M)| = 8$, contradicting our assumption. Therefore M_1 is binary and so, by Propositions 4.4 and 4.3, M_1 is isomorphic to S_8. □

Lemma 4.6. *Suppose that M is a 3-connected excluded minor for $\mathcal{M}$. Then $|E(M)| \neq 9$.*

Proof. Assume that $E(M) = \{1, \ldots, 9\}$. Lemma 3.2 implies that the rank and corank of M both exceed three. By duality we may assume that $r(M) = 4$. Proposition 4.5 implies that M has an S_8-minor, so assume that $M\backslash 9 \cong S_8$. Thus M has the partial representation shown in Figure 5.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & x_1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & x_2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & x_3 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & x_4 \end{bmatrix}$$

FIGURE 5. A partial representation for M.

Let M_B be the binary matroid for which this partial representation is a GF(2)-representation. Clearly $M\backslash 9 = M_B\backslash 9$. Furthermore, $M\backslash 8\backslash 9 = M_B\backslash 8\backslash 9 \cong F_7^*$, so $M\backslash 8$ is non-ternary. Thus $M\backslash 8$ is binary, so $M\backslash 8 = M_B\backslash 8$. Moreover, $M\backslash 9/1 = M_B\backslash 9/1 \cong F_7$. Therefore $M/1$ is non-ternary, and hence binary, so $M/1 = M_B/1$.

Recall that a distinguishing set for M and M_B is some set $Z \subseteq \{1, \dots, 9\}$ such that Z is independent in one of M and M_B and dependent in the other. Let Z be such a distinguishing set. The arguments above show that

$$\{8,9\} \subseteq Z \subseteq E(M) - \{1\}. \tag{4.1}$$

Suppose that M_B is not simple. As 9 is not a loop of M, it follows that 9 is in a parallel pair P in M_B. As M contains no parallel pairs, we deduce that P is a distinguishing set for M and M_B, so (4.1) implies that $P = \{8,9\}$. Thus $(x_1, x_2, x_3, x_4) = (1,1,1,1)$. Now $\{2,7,9\}$ and $\{3,6,9\}$ are triangles of $M_B\backslash 8 = M\backslash 8$. Moreover, $\{2,7,8\}$ and $\{3,6,8\}$ are triangles of $M_B\backslash 9 = M\backslash 9$. Let $A = \{2,7,8,9\}$ and let $B = \{3,6,8,9\}$. Then $r_M(A) = r_M(B) = 2$. Moreover, $r_M(A \cup B) > 2$, otherwise $M|(A \cup B) \cong U_{2,6}$. Now

$$r_M(\{8,9\}) = r_M(A \cap B) \leq r_M(A) + r_M(B) - r_M(A \cup B) \leq 1,$$

so M contains a parallel pair, a contradiction.

We may now assume that M_B is simple. Let Z be a minimal distinguishing set for M and M_B. By symmetry, there are three possibilities for (x_1, x_2, x_3, x_4):

(i) $(0,1,1,1)$;
(ii) $(0,0,1,1)$; and
(iii) $(1,1,0,0)$.

In the first case, $M_B\backslash 8 = M\backslash 8$ is isomorphic to AG$(3,2)$, contradicting Proposition 4.3. Suppose that case (ii) holds. Note that $\{2,7,8\}$ is a circuit of M_B, and as it avoids 9, it is also a circuit of M. Hence

$$M/2\backslash 7 \cong M/2\backslash 8 = M_B/2\backslash 8 \cong F_7.$$

Thus $M/2\backslash 7$ is non-ternary, so $M/2$ and $M\backslash 7$ are non-ternary and hence binary. Therefore $M/2 = M_B/2$ and $M\backslash 7 = M_B\backslash 7$. Hence $7 \in Z$ but $2 \notin Z$, so $\{7,8,9\} \subseteq Z \subseteq \{3, \dots, 9\}$. Suppose that $|Z| = 3$. Then $Z = \{7,8,9\}$. As Z is not a triangle of M_B, it follows that Z is independent in M_B and a triangle in M. As $\{2,7,8\}$ is a triangle in $M_B\backslash 9 = M\backslash 9$, we see that

$\{2,7,8,9\}$ is a rank-2 flat of M. Thus $M/2$ contains a parallel class of size three. But we concluded above that $M/2\backslash 7 \cong F_7$, so we have a contradiction. Therefore $|Z| = 4$. There is no 4-element dependent set in M_B that contains $\{7,8,9\}$, so Z is a basis of M_B. Proposition 2.7 implies that Z is a 4-element circuit of M. Now $\{2,7,8\}$ is a circuit of M_B and of M, and $Z = \{7,8,9,x\}$ for some element $x \in \{3,4,5,6\}$. By circuit elimination in M, there is a circuit C of M contained in $\{2,7,9,x\}$. Since this circuit does not contain 8, it is also a circuit of M_B. But there is no 3- or 4-element circuit in M_B containing $\{2,7,9\}$. The only 3-element circuits of M_B containing two of 2, 7, and 9 are $\{2,7,8\}$ and $\{1,7,9\}$. But $x \notin \{1,8\}$, so we have a contradiction.

Now we suppose that case (iii) holds. We note that $\{1,4,6,7\}$ is a basis of M_B and hence of M, and the fundamental circuits of M and M_B with respect to this basis are the same since no such circuit can contain $\{8,9\}$. Thus the matrix in Figure 6 is a representation for M_B and a partial representation for M.

$$\begin{array}{ccccccccc} 1 & 6 & 7 & 4 & 5 & 2 & 3 & 8 & 9 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

FIGURE 6. A partial representation for M.

Since $M_B/7$ has $\{2,8\}$ as a circuit, so does $M/7$. Thus

$$M/7\backslash 2 \cong M/7\backslash 8 = M_B/7\backslash 8 \cong F_7.$$

Hence $M/7\backslash 2$ is non-ternary. Therefore $M/7$ and $M\backslash 2$ are binary, so $M/7 = M_B/7$ and $M\backslash 2 = M_B\backslash 2$. It follows that $2 \in Z$ and that $Z \subseteq \{2,3,4,5,6,8,9\}$.

Suppose that $|Z| = 3$, so that $Z = \{2,8,9\}$. As $\{2,8,9\}$ is independent in M_B, we see that Z is a triangle in M. As $\{2,7,8\}$ is also a triangle of M, it follows that $M/7\backslash 2$ cannot be isomorphic to F_7, a contradiction.

We know now that $|Z| = \{2,8,9,x\}$ for some $x \in \{3,4,5,6\}$. By circuit exchange in M between Z and $\{2,7,8\}$, we conclude that $\{2,7,9,x\}$ contains a circuit of $M\backslash 8 = M_B\backslash 8$. But the only circuits of M_B that meet $\{2,7,9\}$ in more than one element are $\{1,2,9\}$ and $\{2,7,8\}$. As $x \notin \{1,8\}$, we have arrived at a contradiction that completes the proof. □

In the light of Lemmas 3.2, 3.3, 4.1, and 4.6, we need now only characterize the excluded minors for $\mathcal{M}$ that are 3-connected with rank and corank at least four, and which have a ground set containing at least ten elements. In the next section we begin to move towards this goal.

5. A structure theorem for excluded minors

The following theorem is the main result of this section.

Theorem 5.1. *Let M be a 3-connected excluded minor for $\mathcal{M}$ such that $|E(M)| \geq 10$ and both the rank and corank of M exceed three. Then there is a 3-connected binary matroid M_B such that $E(M_B) = E(M)$ and:*

(i) there are disjoint circuit-hyperplanes J and K in M_B such that $E(M_B) = J \cup K$;
(ii) M is obtained from M_B by relaxing J; and
(iii) the matroid M_T that is obtained from M_B by relaxing J and K is ternary.

Before we prove Theorem 5.1, we discuss some preliminary facts. The binary matroid P_9 is a 3-connected extension of S_8, and is represented over GF(2) by the matrix in Figure 7.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 7. A representation of P_9.

The next result follows from [17, Lemma (2.6)].

Proposition 5.2. *Every binary 3-connected single-element extension of S_8 is either isomorphic to P_9 or has an AG(3, 2)-minor.*

Proposition 5.3. *Suppose that M is a 3-connected excluded minor for $\mathcal{M}$ and that $|E(M)| \geq 10$. Then M has either P_9 or P_9^* as a minor.*

Proof. Proposition 4.5 implies that M has a minor M_1 isomorphic to S_8. Now the Splitter Theorem implies that M has a minor M_2 that is a 3-connected extension or coextension of S_8. If M_2 is non-binary, then M_2 is both non-binary and non-ternary, so $M_2 = M$ and hence $|E(M)| = 9$. This is a contradiction, so M_2 is binary. Thus, by Propositions 5.2 and 4.3 and duality, we see that M_2 is isomorphic to either P_9 or P_9^*. □

Proof of Theorem 5.1. Let M be a 3-connected excluded minor for $\mathcal{M}$ such that $r(M), r(M^*) \geq 4$ and $|E(M)| \geq 10$. By duality, we may assume that $r(M) \leq r^*(M)$. By Proposition 5.3, M has a minor N that is isomorphic to P_9 or P_9^*. Suppose that $N = M\backslash X/Y$, where we may assume that Y is independent and that X is coindependent in M. We assume that N has ground set $\{1, 2, \ldots, 9\}$ and that if N is P_9, then N is represented over GF(2) by the matrix in Figure 7. Similarly, we assume that if N is P_9^*, then N is the dual of the matroid represented in Figure 7.

As $|E(M)| \geq 10$, it follows that $X \cup Y$ is non-empty. We note, for future reference, that $P_9/1\backslash 7$, $P_9/1\backslash 9$, $P_9/2\backslash 7$, and $P_9/2\backslash 8$ are all isomorphic to F_7. Thus $P_9^*\backslash 1/7$, $P_9^*\backslash 1/9$, $P_9^*\backslash 2/7$, and $P_9^*\backslash 2/8$ are all isomorphic to F_7^*.

We wish to fix a basis B and a cobasis B' of N. If $N = P_9$, we choose $B = \{1, 2, 3, 4\}$ and $B' = \{5, 6, 7, 8, 9\}$, while if $N = P_9^*$, we choose $B = \{5, 6, 7, 8, 9\}$ and $B' = \{1, 2, 3, 4\}$. Now $Y \cup B$ is a basis of M. Suppose that $[I_r|A]$ is a partial representation of M with respect to the basis $Y \cup B$. Let M_B be the binary matroid represented over GF(2) by $[I_r|A]$. The rest of the proof involves showing that M_B has the properties specified in the theorem. Note that there must be at least one subset of $E(M)$ that is independent in one of M and M_B and dependent in the other. Recall that we call any such set a distinguishing set.

Lemma 5.4. *If $x \in X \cup B'$ and $M\backslash x$ is binary, then $M\backslash x = M_B\backslash x$. If $y \in Y \cup B$ and M/y is binary, then $M/y = M_B/y$.*

Proof. Suppose $x \in X \cup B'$ and that $M\backslash x$ is binary. Then, by deleting the column of $[I_r|A]$ labeled by x, we obtain a partial representation for $M\backslash x$. Since $M\backslash x$ is binary, this matrix in fact represents $M\backslash x$ over GF(2). It also represents $M_B\backslash x$ over GF(2), so $M\backslash x = M_B\backslash x$. The second statement follows by a similar argument. □

Lemma 5.5. *There are subsets X' and Y' of $E(M)$ with the following properties:*

(i) $X \subseteq X' \subseteq X \cup B'$ *and* $Y \subseteq Y' \subseteq Y \cup B$;
(ii) $E(M) - (X' \cup Y') = \{3, 4, 5, 6\}$;
(iii) *if* $x \in X'$, *then* $M\backslash x$ *is non-ternary and* $M\backslash x = M_B\backslash x$;
(iv) *if* $y \in Y'$, *then* M/y *is non-ternary and* $M/y = M_B/y$;
(v) $|X'| = r^*(M) - 2$;
(vi) $|Y'| = r(M) - 2$;
(vii) *if* $e \in E(M) - (X' \cup Y')$, *then* $X' \cup Y' \cup \{e\}$ *spans both* M *and* M_B; *and*
(viii) *if* Z *is a distinguishing set for* M *and* M_B *then* $X' \subseteq Z \subseteq E(M) - Y'$.

Proof. We first consider the case that $N \cong P_9$. Let $X' = X \cup \{7, 8, 9\}$ and let $Y' = Y \cup \{1, 2\}$. Then (i) and (ii) are certainly true. Suppose that $x \in X$. Then $M\backslash x$ has a P_9-minor, and as P_9 has an F_7-minor, it follows that $M\backslash x$ is non-ternary, and therefore binary. The fact that $M\backslash x = M_B\backslash x$ follows from Lemma 5.4. Moreover, if $x \in \{7, 8, 9\}$, then $N\backslash x$ has an F_7-minor, so $M\backslash x$ is non-ternary, and hence binary. Therefore (iii) holds. A similar argument shows that (iv) holds. Statement (viii) follows immediately from (iii) and (iv). As N has rank four and corank five, it follows that $|X| = r^*(M) - 5$ and $|Y| = r(M) - 4$. Thus (v) and (vi) are immediate. To see that (vii) is true, note that $M\backslash X/Y = M_B\backslash X/Y = P_9$. Since $\{3, 4, 5, 6\}$ is a cocircuit of P_9, it follows that if $e \in \{3, 4, 5, 6\}$, then $\{1, 2, 7, 8, 9, e\}$ contains a basis of N. Thus $\{1, 2, 7, 8, 9, e\} \cup Y$ contains a basis B_0 of M. Suppose that B_0 is not a basis of M_B. Then there is a minimal distinguishing set Z

for M and M_B such that $Z \subseteq B_0$ and Z is independent in M and dependent in M_B. Part (viii) shows that $\{7,8,9\} \subseteq Z$, but Z does not contain any element in $\{1,2\} \cup Y$. It follows that there is a circuit of $M_B \backslash X/Y = N$ that is contained in $\{7,8,9,e\}$. But no such circuit exists, so B_0 is a basis of both M and M_B. Therefore (vii) holds.

In the case that $N \cong P_9^*$, we set X' to be $X \cup \{1,2\}$ and Y' to be $Y \cup \{7,8,9\}$. If x is any element in X', then $M \backslash x$ has a F_7^*-minor, and is therefore not ternary. Similarly, if $y \in Y'$, then M/y is not ternary. Therefore the proofs of statements (i), (ii), (iii), (iv), (v), (vi), and (viii) are identical.

To prove (vii), we observe that $\{3,4,5,6\}$ is also a cocircuit of P_9^*. Therefore $\{1,2,7,8,9,e\}$ contains a basis of N, for any element $e \in \{3,4,5,6\}$. Hence $\{1,2,7,8,9,e\} \cup Y$ contains a basis B_0 of M. If B_0 is not a basis of M_B, we can again find a minimal set $Z \subseteq B_0$ such that Z is independent in M and dependent in M_B. Then $\{1,2\} \subseteq Z \subseteq E(M) - \{7,8,9\}$, so there is a circuit of $M_B \backslash X/Y = N$ that is contained in $\{1,2,e\}$. As no such circuit exists, B_0 is a basis of both M and M_B. □

For the rest of the proof, X' and Y' refer to the sets described in Lemma 5.5. We will make frequent use of the following fact.

Proposition 5.6. *Suppose that $\{M_1, M_2\} = \{M, M_B\}$ and that C is a circuit of M_1. If C does not contain X', then it is also a circuit of M_2.*

Proof. Suppose that C does not contain X'. Then C cannot be a distinguishing set by Lemma 5.5 (viii). Since C is dependent in M_1, it must therefore be dependent in M_2. If C is not a circuit of M_2, it properly contains a circuit C' of M_2. Now C' is independent in M_1, so it is a distinguishing set of M_1 and M_2. However, C' does not contain X', so we have a contradiction to Lemma 5.5 (viii). Therefore C is a circuit of M_2. □

Lemma 5.7. *Suppose that Z is a distinguishing set for M and M_B. Then $|Z| = r(M) = r(M_B)$.*

Proof. Let r be the common rank of M and M_B and let $r^* = r^*(M) = r^*(M_B)$. Suppose that Z is a distinguishing set for M and M_B and that $|Z| \neq r$. Obviously $|Z| \leq r$, so $|Z| \leq r-1$. Let $\{M_1, M_2\} = \{M, M_B\}$, where we assume that Z is dependent in M_1 and independent in M_2. We can assume that Z is a minimal distinguishing set, so Z is in fact a circuit of M_1.

Lemma 5.5 (v) and (viii) imply that $X' \subseteq Z$, and $|X'| = r^* - 2$. Therefore $r^* - 2 \leq r - 1$. But we have assumed that $r \leq r^*$, so $r^* \in \{r, r+1\}$. Hence $|X'| \in \{r-2, r-1\}$.

Note that $Z \cap Y' = \varnothing$ by Lemma 5.5 (viii). Suppose that $y \in Y'$. Then $M_1/y = M_2/y$ by Lemma 5.5 (iv). As Z is dependent in M_1/y, it follows that Z is dependent in M_2/y, so $Z \cup y$ is dependent in M_2. As Z is independent in M_2, this means that $y \in \mathrm{cl}_{M_2}(Z)$. Therefore $Y' \subseteq \mathrm{cl}_{M_2}(Z)$.

Suppose that $Z \neq X'$. Then $|X'| = r - 2$ and $|Z| = r - 1$, so $Z - X'$ contains a unique element z. Moreover, $z \notin X' \cup Y'$ by Lemma 5.5 (viii). We have already shown that $Y' \subseteq \text{cl}_{M_2}(Z)$, so Lemma 5.5 (vii) implies that Z is spanning in M_2. This is a contradiction since $|Z| < r(M_2)$. We conclude that $Z = X'$.

Now suppose that $y \in Y'$ and that $y \in \text{cl}_{M_1}(Z)$. Then there is a circuit $C \subseteq Z \cup y$ such that $y \in C$. Note that C does not contain $X' = Z$, as Z is a circuit of M_1. Proposition 5.6 implies that C is a circuit of M_2. The fact that M and M_B are loopless means that $C \neq \{y\}$, so there is an element $e \in X' \cap C$.

By circuit elimination between Z and C in M_1, there is a circuit C' of M_1 such that $y \in C'$ and $C' \subseteq (Z - e) \cup y$. As C' does not contain e, Proposition 5.6 implies that C' is a circuit of M_2. Now C and C' are circuits of M_2 contained in $Z \cup y$, and $C \neq C'$ as $e \notin C'$. But Z is independent in M_2, so this leads to a contradiction. This shows that $\text{cl}_{M_1}(Z) \cap Y' = \varnothing$.

We have shown that if $y \in Y'$ then $y \in \text{cl}_{M_2}(Z)$. In fact, we can prove something stronger: that $Z \cup y$ is a circuit of M_2. Suppose that this is not the case. Then there is a circuit C that is properly contained in $Z \cup y$, such that $y \in C$. Certainly C does not contain $X' = Z$, so C is a circuit of M_1. Therefore $y \in \text{cl}_{M_1}(Z)$, contrary to our earlier conclusion. Thus $Z \cup y$ is indeed a circuit of M_2.

We know that $|Y'| \geq 2$, so let y and y' be distinct members of Y'. Then $Z \cup y$ and $Z \cup y'$ are circuits of M_2. If M_2 is binary, then $\{y, y'\}$ contains a circuit of M_2. But Y' is contained in the common basis $B \cup Y$ so this leads to a contradiction. Therefore $M_2 \neq M_B$, so $M_1 = M_B$ and $M_2 = M$.

Suppose that $r^* = r$. Then $|Z| = |X'| = r^* - 2 = r - 2$. As Z is independent in M_2, it follows that $r_{M_2}(Z) = r - 2$. Moreover $r - 2 = r_{M_2}(Z \cup Y') = r_{M_2}(X' \cup Y')$, from our earlier conclusion that $Y' \subseteq \text{cl}_{M_2}(Z)$. However, it follows from Lemma 5.2 (vii) that $r_{M_2}(X' \cup Y') \geq r - 1$. Therefore we have a contradiction, and we conclude that $r^* = r + 1$, so $|Z| = r - 1$. Thus $r_{M_2}(Z) = r - 1$.

Let $W = E(M) - (X' \cup Y') = \{3, 4, 5, 6\}$. We already know that $\text{cl}_{M_1}(Z)$ does not meet Y'. Suppose that there is some element $w \in W$ such that $w \in \text{cl}_{M_1}(Z)$. Then there is a circuit C of M_1 such that $C \subseteq Z \cup w$ and $w \in C$. As Z is a circuit of M_1, it follows that there is an element $z \in Z$ such that $z \notin C$. Therefore C does not contain $X' = Z$, so C is a circuit of M_2. Thus $\text{cl}_{M_2}(Z)$ contains w and Y', and therefore Z is spanning in M_2. But this is a contradiction as $r_{M_2}(Z) = r - 1$. We conclude that Z is a flat of M_1.

Recall that $M_1 = M_B$ and that Z is a circuit and a flat of M_1 with cardinality $r - 1$. Consider M_1/Z. This is a loopless rank-2 binary matroid on the ground set $W \cup Y'$. Obviously M_1/Z contains no more than three parallel classes. As $|W| = 4$, we deduce that some parallel class of M_1/Z contains two distinct elements of W, say w and w'. Therefore there is a circuit C of M_1 such that $C \subseteq Z \cup \{w, w'\}$ and $w, w' \in C$. Note that C

must meet Z, for w and w' are not parallel in $M_B\backslash X/Y = N$, so they are not parallel in M_1.

Let $C' = (Z-C)\cup\{w, w'\}$. Since M_1 is binary, C', which is the symmetric difference of C and Z, is a disjoint union of circuits of M_1. Any circuit in C' that contains w must also contain w', for $w \notin \mathrm{cl}_{M_1}(Z)$. Note that $C' \cap Z$ is a proper subset of Z, as $C \cap Z$ is non-empty. These observations imply that C' must in fact be a circuit of M_1. Moreover, $C' \cap Z$ is non-empty, as C cannot contain the circuit Z.

Both C and C' are circuits of M_2 since neither contains Z. Thus M_2 has a circuit contained in $(C \cup C') - w'$. This circuit must contain w, so $w \in \mathrm{cl}_{M_2}(Z)$. Hence, by Lemma 5.5 (vii), Z is spanning in M_2; a contradiction. □

Corollary 5.8. *Both M and M_B are simple.*

Proof. Certainly M is simple as it is 3-connected and $|E(M)| \geq 10$. If M_B contains a circuit of at most two elements, then that set contains a distinguishing set. But Lemma 5.7 implies that any distinguishing set has cardinality at least four. □

Corollary 5.9. *Suppose that Z is a distinguishing set of M and M_B. Then Z is a circuit in one of M and M_B and a basis in the other. Moreover, $r^*(M) \in \{r(M), r(M)+1, r(M)+2\}$.*

Proof. It follows from Lemma 5.7 that any distinguishing set of M and M_B is in fact a minimal distinguishing set. The fact that Z is a circuit in one of M and M_B and a basis in the other now follows easily.

Lemma 5.5 implies that $X' \subseteq Z$ and that $|X'| = r^*(M) - 2$. Thus $r^*(M) - 2 \leq |Z| = r(M)$ by Lemma 5.7. The corollary follows from our assumption that $r(M) \leq r^*(M)$. □

We now set to the task of showing that M and M_B have a unique distinguishing set.

Lemma 5.10. *Let $\{M_1, M_2\} = \{M, M_B\}$ and suppose that the distinguishing set Z is a circuit in M_1 and a basis in M_2. Then*

(i) if M_1 is binary, then Z is a hyperplane of M_1; and
(ii) if Z is not a hyperplane of M_1, then $r(M) = r^*(M)$.

Moreover $|\mathrm{cl}_{M_1}(Z)| \leq |Z| + 1$.

Proof. Let $r = r(M)$ and let $r^* = r^*(M)$. We note that $X' \subseteq Z$ and that $|X'| = r^* - 2$ by Lemma 5.5. Corollary 5.9 states that $r^* \in \{r, r+1, r+2\}$. Note that $|Z - X'| = r - r^* + 2$, so $r^* = r$ if and only if $Z - X'$ contains exactly two elements. We prove the following claim:

5.10.1. *Suppose that y is in $\mathrm{cl}_{M_1}(Z) - Z$. Then $(Z - X') \cup y$ is a circuit of both M_1 and M_2.*

Proof. There is a circuit C of M_1 such that $y \in C$ and $C \subseteq Z \cup y$. Assume that C is not a circuit of M_2. If C is a distinguishing set, then $X' \subseteq C$. On the other hand, if C is not a distinguishing set, then C is dependent in M_2 and C must properly contain a circuit C' of M_2. Since C' is a proper subset of the circuit C of M_1, it follows that C' is a distinguishing set of M_1 and M_2, and therefore $X' \subseteq C'$. Hence $X' \subseteq C$ in either case.

Choose e in X'. Then $e \in Z \cap C$. By circuit elimination in M_1, there is a circuit $C' \subseteq (Z - e) \cup y$ such that $y \in C'$. Note that C' does not contain X', so C' is a circuit of M_2 by Proposition 5.6. Therefore we can relabel C' with C, and assume that C is a circuit of both M_1 and M_2 such that $C \subseteq Z \cup y$ and $y \in C$.

If C does not avoid X', then $C \cap X'$ contains an element e and, by circuit elimination in M_1, there is a circuit C' of M_1 such that $y \in C'$ and $C' \subseteq (Z - e) \cup y$. Since C' does not contain X', Proposition 5.6 implies that C' is a circuit of M_2. Thus Z is independent in M_2, but $Z \cup y$ contains two distinct circuits of M_2, namely C and C'. This contradiction means that C avoids X', so $C \subseteq (Z - X') \cup y$. But $|Z - X'| \leq 2$ and M_2 is simple by Corollary 5.8. Thus $C = (Z - X') \cup y$ is a circuit of both M_1 and M_2. □

Suppose that Z is not a hyperplane of M_1. As $r_{M_1}(Z) = |Z| - 1 = r - 1$, there must be some element y in $\text{cl}_{M_1}(Z) - Z$. Then (5.10.1) implies that $(Z - X') \cup y$ is a circuit of M_1. As $|Z - X'| \leq 2$ and M_1 is simple, we conclude that $|Z - X'| = 2$, and that therefore $r = r^*$ by our earlier observation. We have shown that statement (ii) of the lemma holds.

Suppose that $M_1 = M_B$. Then M_1 is binary and $((Z - X') \cup y) \triangle Z$ is a disjoint union of circuits of M_1. Thus $X' \cup y$ contains a circuit C' of M_1 that contains y. Clearly $|C'| \leq |X'| + 1 = r^* - 1 = r - 1$. Therefore C' cannot be a distinguishing set by Lemma 5.7. Hence C' is dependent in M_2. If C' is not a circuit of M_2, then it properly contains a circuit of M_2 and this circuit is a distinguishing set with cardinality less than r, a contradiction. Therefore C' is a circuit of M_2. Note that $C' \neq (Z - X') \cup y$, so $Z \cup y$ contains two distinct circuits of M_2. This is a contradiction. Therefore $|\text{cl}_{M_1}(Z)| > |Z|$ implies that M_1 is not binary. It follows that if M_1 is binary, then Z is a hyperplane of M_1. This completes the proof of statement (i).

We may now assume that M_1 is non-binary, so that M_2 is binary. Suppose that y_1 and y_2 are distinct elements of $\text{cl}_{M_1}(Z) - Z$. Then $(Z - X') \cup y_1$ and $(Z - X') \cup y_2$ are both circuits of M_2 by (5.10.1). By taking the symmetric difference of these circuits, we deduce that $\{y_1, y_2\}$ is a disjoint union of circuits of M_2, and this is a contradiction. It follows that $\text{cl}_{M_1}(Z)$ can contain at most one element not in Z. This completes the proof. □

Lemma 5.11. *Suppose that Z is a distinguishing set for M and M_B. Then Z is a circuit-hyperplane in M_B and a basis in M.*

Proof. Let $\{M_1, M_2\} = \{M, M_B\}$, and assume that Z is a circuit in M_1 and a basis in M_2. If M_1 is binary, then Z is a hyperplane of M_1 by Lemma 5.10 and there is nothing left to prove, so we assume that $M_1 = M$ and $M_2 = M_B$.

Note that Z does not meet Y' by Lemma 5.5 (viii). Suppose that Z is a hyperplane of M_1. Then

$$|Y' - \mathrm{cl}_{M_1}(Z)| = |Y'| = r(M) - 2 \geq 2.$$

On the other hand, if Z is not a hyperplane of M_1, then $r(M) = r^*(M)$ and $|\mathrm{cl}_{M_1}(Z)| \leq |Z| + 1$ by Lemma 5.10. In this case, $r(M) = |E(M)|/2 \geq 5$. Hence

$$|Y' - \mathrm{cl}_{M_1}(Z)| \geq |Y'| - 1 = r(M) - 3 \geq 2.$$

In either case, $Y' - \mathrm{cl}_{M_1}(Z)$ contains distinct elements y_1 and y_2.

Clearly Z is a circuit of M_1/y_i for $i = 1, 2$. Lemma 5.5 (iv) implies that $M_1/y_i = M_2/y_i$. Therefore Z is a circuit of M_2/y_i and, as Z is independent in M_2, this means that $Z \cup y_i$ is a circuit of M_2. Therefore $Z \cup y_1$ and $Z \cup y_2$ are distinct circuits of the binary matroid M_2, so $\{y_1, y_2\}$ is a union of circuits in M_2. This contradiction completes the proof. □

Lemma 5.12. *Suppose that Z_1 and Z_2 are distinct distinguishing sets for M and M_B. Then*

(i) $|Z_1| = |Z_2| = r(M) = r^*(M)$*;*
(ii) $Z_1 - X'$ *and* $Z_2 - X'$ *are disjoint sets;*
(iii) $|Z_1 - X'| = |Z_2 - X'| = 2$*;*
(iv) $Z_1 \triangle Z_2 = \{3, 4, 5, 6\}$*; and*
(v) $Z_1 \triangle Z_2$ *is a circuit of* M*.*

Proof. Let $r = r(M)$ and $r^* = r^*(M)$. From Lemma 5.11, we see that Z_1 and Z_2 are circuit-hyperplanes of M_B, so $|Z_1| = |Z_2| = r$. Moreover, $X' \subseteq Z_i \subseteq E(M) - Y'$ by Lemma 5.5. Note that $r^* - 2 = |X'| \leq |Z_i| = r$ for $i = 1, 2$. As $r \leq r^*$, this means that $|Z_i - X'| \leq 2$. Since

$$Z_1 \triangle Z_2 \subseteq (Z_1 - X') \cup (Z_2 - X'), \tag{5.1}$$

it follows that $|Z_1 \triangle Z_2| \leq 4$. Moreover $|Z_1 \triangle Z_2|$ is even, as $|Z_1| = |Z_2|$.

Now $Z_1 \triangle Z_2$ is a disjoint union of circuits in the simple matroid M_B. It follows that $Z_1 \triangle Z_2$ is a circuit, and that $|Z_1 \triangle Z_2| = 4$. Equation (5.1) implies that $Z_1 - X'$ and $Z_2 - X'$ must be disjoint sets, each of cardinality two. Since

$$Z_i - X' \subseteq E(M) - (X' \cup Y') = \{3, 4, 5, 6\}$$

for $i = 1, 2$, we have that $Z_1 \triangle Z_2 = \{3, 4, 5, 6\}$ is a circuit. As $|Z_1 - X'| = |Z_2 - X'| = 2$, it follows that $|Z_1| = |Z_2| = |X'| + 2 = r^*$, so we are done. □

Proposition 5.13. *Suppose that Z is a distinguishing set for M and M_B. If $e \in E(M) - Z$, then $Z \cup e$ is a circuit of M.*

Proof. As Z is a basis of M, there is a circuit C of M such that $e \in C$ and $C \subseteq Z \cup e$. Lemma 5.11 implies that C cannot be a distinguishing set, so C is dependent in M_B. But there is only one circuit of M_B that is contained in $Z \cup e$, namely Z itself. Therefore C contains Z, so $C = Z \cup e$ is a circuit of M, as desired. □

For the next step we will need a result due to Kahn and Seymour [11] (see [19, Lemma 10.3.7]).

Lemma 5.14. *Let N_1 and N_2 be matroids having distinct elements a and b such that the following conditions hold:*

(i) N_1 and N_2 are distinct connected matroids having a common ground set;
(ii) $N_1\backslash a = N_2\backslash a$ and $N_1\backslash b = N_2\backslash b$;
(iii) $N_1\backslash a\backslash b = N_2\backslash a\backslash b$ and this matroid is connected; and
(iv) $\{a, b\}$ is not a cocircuit of N_1 or of N_2.

Then at most one of N_1 and N_2 is ternary.

Lemma 5.15. *There is a unique distinguishing set for M and M_B.*

Proof. Let Z_1 and Z_2 be distinct distinguishing sets. Lemma 5.12 says that $Z_1 - X'$ and $Z_2 - X'$ are disjoint sets of cardinality two, and both $Z_1 - X'$ and $Z_2 - X'$ are contained in $\{3, 4, 5, 6\}$. If Z were any other distinguishing set, then $Z - X'$ would be disjoint from $Z_1 - X'$ and $Z_2 - X'$, but $Z - X'$ would also be contained in $\{3, 4, 5, 6\}$. Since this is impossible it follows that Z_1 and Z_2 are the only distinguishing sets for M and M_B. Suppose that $Z_1 - X' = \{a, b\}$ and $Z_2 - X' = \{c, d\}$, where $\{a, b, c, d\} = \{3, 4, 5, 6\}$. We deduce from Lemma 5.12 that $r(M) = r^*(M)$.

5.15.1. *If $e \in E(M) - X'$ then $M\backslash e$ is non-binary.*

Proof. Since $X' = Z_1 \cap Z_2$, we can assume without loss of generality that $e \notin Z_1$. Let x and y be distinct elements in $E(M)-(Z_1\cup e)$. Proposition 5.13 implies that $Z_1 \cup x$ and $Z_1 \cup y$ are circuits of $M\backslash e$. If $M\backslash e$ is binary, then this would imply that $\{x, y\}$ is a union of circuits in M; a contradiction. Therefore $M\backslash e$ is non-binary. □

Suppose that $e \in E(M) - Z_1$. Then $Z_1 \cup e$ is a circuit of M by Proposition 5.13. This observation means that if A is a proper subset of $E(M)-Z_1$, then $M\backslash A$ is connected. Similarly, if A is a proper subset of $E(M) - Z_2$, then $M\backslash A$ is connected.

We have shown in (5.15.1) that $M\backslash a$ and $M\backslash b$ are non-binary, and therefore ternary. Obviously $M\backslash a\backslash b$ is ternary. Suppose that $M\backslash a\backslash b$ is represented over GF(3) by the matrix $[I_r|A]$. It is known [6] that ternary matroids are uniquely representable over GF(3). One consequence of this is that there are column vectors $\mathbf{a}$ and $\mathbf{b}$ over GF(3) such that $[I_r|A|\mathbf{a}]$ and $[I_r|A|\mathbf{b}]$ represent $M\backslash b$ and $M\backslash a$ respectively over GF(3). Let M_T be the ternary matroid that is represented over GF(3) by the matrix $[I_r|A|\mathbf{a}|\mathbf{b}]$. Thus $M_T\backslash a = M\backslash a$ and $M_T\backslash b = M\backslash b$.

Let e be an arbitrary element in Y'. We wish to show that $M\backslash e = M_T\backslash e$. We know that $M\backslash e$ is non-binary and hence ternary, by (5.15.1). Certainly $M_T\backslash e\backslash a = M\backslash e\backslash a$ and $M_T\backslash e\backslash b = M\backslash e\backslash b$. Moreover, our earlier observation implies that $M\backslash e$ and $M\backslash e\backslash a$ are connected. If $M_T\backslash e$ is not connected, then a must be a loop or a coloop in $M_T\backslash e$. This means that a is a loop of M_T, or $\{a,e\}$ is a series pair in M_T. But M_T contains no loops as M contains no loops. Furthermore $\{a,e\}$ is not a series pair of M_T, as $M_T\backslash a = M\backslash a$ is connected. Therefore $M_T\backslash e$ is connected. The set $\{a,b\}$ is not a cocircuit of either $M_T\backslash e$, or $M\backslash e$, for $M\backslash e\backslash a = M_T\backslash e\backslash a$ and $M\backslash e\backslash b = M_T\backslash e\backslash b$, and both these matroids are connected. Finally, $M\backslash e\backslash a\backslash b = M_T\backslash e\backslash a\backslash b$ and this matroid is connected since Z_2 avoids all of e, a, and b. We have shown that the hypotheses of Lemma 5.14 apply to $M\backslash e$ and $M_T\backslash e$. Since $M\backslash e$ and $M_T\backslash e$ are both ternary, Lemma 5.14 implies that $M\backslash e$ and $M_T\backslash e$ are not distinct. Therefore $M\backslash e = M_T\backslash e$.

The matroids M and M_T are distinct as M is not ternary. Let Z be a distinguishing set for M and M_T. We have deduced that $M\backslash x = M_T\backslash x$ for every $x \in Y' \cup \{a,b\} = E(M) - Z_2$. Thus $Y' \cup \{a,b\} \subseteq Z$. But $|Y'| = r^*(M) - 2 = r(M) - 2$, so $|Z| \geq r(M)$. However, $|Z| \leq r(M)$, so $|Z| = r(M)$, and $Z = Y' \cup \{a,b\}$. Therefore there is a unique distinguishing set for M and M_T, and Proposition 2.8 implies that Z is a circuit-hyperplane in one of these matroids and a basis in the other.

Suppose that Z is a basis of M. Then Proposition 2.8 states that $Z \cup e$ is a circuit of M for all e in $E(M) - Z$. Since $Z \cup e$ contains neither Z_1 nor Z_2, we deduce that $Z \cup e$ is a circuit of M_B for all $e \in E(M) - Z$. If e and e' are distinct elements in $E(M) - Z$, then $\{e,e'\}$ is a union of circuits in M_B, a contradiction. Therefore, from Proposition 2.8, we conclude that Z is a circuit-hyperplane of M, and M_T is obtained from M by relaxing Z.

We know from (5.15.1) that $M\backslash c$ is non-binary, and hence ternary. We have already noted that $M\backslash c$ is connected. Moreover, Proposition 2.8 implies that $Z \cup e$ is a circuit of M_T for every $e \in E(M) - Z$. Thus $M_T\backslash c$ is connected. We have shown that if $y_1, y_2 \in Y'$, then $M\backslash y_i = M_T\backslash y_i$ for $i \in \{1,2\}$. Therefore $M\backslash c\backslash y_i = M_T\backslash c\backslash y_i$. Also $M\backslash c\backslash y_1\backslash y_2 = M_T\backslash c\backslash y_1\backslash y_2$ and this last matroid is connected, since Z_1 avoids c, y_1, and y_2. Finally, $\{y_1,y_2\}$ is not a cocircuit of $M\backslash c$ or of $M_T\backslash c$, for $M\backslash c\backslash y_i = M_T\backslash c\backslash y_i$ for $i = 1,2$, and these matroids are connected. We apply Lemma 5.14. Since both $M\backslash c$ and $M_T\backslash c$ are ternary, we conclude that $M\backslash c = M_T\backslash c$. But Z is a circuit of $M\backslash c$, and a basis of $M_T\backslash c$. This contradiction completes the proof. □

Lemma 5.16. *Suppose that Z is a distinguishing set for M and M_B. Then $E(M) - Z$ is a circuit-hyperplane of M. Moreover the matroid obtained from M by relaxing $E(M) - Z$ is ternary.*

Proof. Much of the argument in this lemma is similar to that in Lemma 5.15. Note that Z is a circuit-hyperplane of M_B by Lemma 5.11. Since Z is the unique distinguishing set by Lemma 5.15, we see from Proposition 2.8 that M is obtained from M_B by relaxing Z.

Suppose that $e \in E(M) - Z$. Let a and b be distinct elements in $E(M) - (Z \cup e)$. Then $Z \cup a$ and $Z \cup b$ are circuits of $M\backslash e$ by Proposition 5.13. If $M\backslash e$ were binary, then $\{a,b\}$ would be a union of circuits in M. This contradiction implies that $M\backslash e$ is ternary for every element $e \in E(M) - Z$.

The fact that $Z \cup e$ is a circuit of M for every $e \in E(M) - Z$ means that $M \backslash A$ is connected for every proper subset of $E(M) - Z$.

Choose elements $a, b \in E(M) - Z$. Then $M \backslash a$, $M \backslash b$, and $M \backslash a \backslash b$ are all ternary. Suppose that these three matroids are represented over GF(3) by the matrices $[I_r|A|\mathbf{b}]$, $[I_r|A|\mathbf{a}]$, and $[I_r|A]$ respectively. Let M_T be the ternary matroid represented over GF(3) by $[I_r|A|\mathbf{a}|\mathbf{b}]$, so that $M_T \backslash a = M \backslash a$ and $M_T \backslash b = M \backslash b$.

Suppose that $e \in E(M) - (Z \cup \{a, b\})$. Then $M \backslash e$ is ternary. Furthermore, $M \backslash e \backslash a = M_T \backslash e \backslash a$ and $M \backslash e \backslash b = M_T \backslash e \backslash a$, and these matroids are both connected. We have already observed that $M \backslash e$ is connected. If $M_T \backslash e$ is not connected, then a is a loop or a coloop in $M_T \backslash e$. But M_T has no loops, and $\{a, e\}$ is not a series pair of M_T as $M_T \backslash a = M \backslash a$, and $M \backslash a$ is connected. We also note that $M \backslash e \backslash a \backslash b = M_T \backslash e \backslash a \backslash b$, and this last matroid is connected. Finally, $\{a, b\}$ is not a series pair of $M \backslash e$ or $M_T \backslash e$ as $M \backslash e \backslash a = M_T \backslash e \backslash a$ and $M \backslash e \backslash b = M_T \backslash e \backslash b$ are connected.

We have shown that Lemma 5.14 applies to $M \backslash e$ and $M_T \backslash e$. Since both these matroids are ternary, we deduce that $M \backslash e = M_T \backslash e$.

Let Z' be a distinguishing set for M and M_T. Then $E(M) - Z \subseteq Z'$, so $r^*(M) = |E(M) - Z| \leq |Z'|$. But $|Z'| \leq r(M) \leq r^*(M)$, so $Z' = E(M) - Z$. Thus $E(M) - Z$ is the unique distinguishing set for M and M_T, and one of these matroids is obtained from the other by relaxing $E(M) - Z$.

Suppose that M is obtained from M_T by relaxing the circuit-hyperplane $E(M) - Z$. Then $(E(M) - Z) \cup e$ is a circuit of M for all e in Z. Thus $(E(M) - Z) \cup e$ is a circuit of M_B for all $e \in Z$. It follows that M_B has a circuit of size at most two. This contradiction shows that M_T is obtained from M by relaxing the circuit-hyperplane $E(M) - Z$, and this completes the proof. □

To complete the proof of Theorem 5.1, we suppose that Z is a distinguishing set for M and M_B. Then Z is a circuit-hyperplane of M_B by Lemma 5.11. Lemma 5.15 says that Z is unique, so M is obtained from M_B by relaxing the circuit-hyperplane Z (Proposition 2.8). Also, $E(M) - Z$ is a circuit-hyperplane of M, and the matroid M_T produced by relaxing $E(M) - Z$ in M is ternary by Lemma 5.16. It is an easy exercise to see that $E(M) - Z$ is a circuit-hyperplane of M_B, and that relaxing both Z and $E(M) - Z$ in M_B produces M_T. Thus, if we can show that M_B is 3-connected, the result follows by renaming Z with J and $E(M) - Z$ with K.

Suppose that (X_1, X_2) is a k-separation of M_B for some $k < 3$. As M is 3-connected, (X_1, X_2) is not a k-separation of M. Thus $r_M(X_i) > r_{M_B}(X_i)$, where $\{i, j\} = \{1, 2\}$. It is easy to see that this means $X_i = Z$. Therefore both X_1 and X_2 are circuit-hyperplanes of M_B, meaning that

$$1 \geq r_{M_B}(X_1) + r_{M_B}(X_2) - r(M_B) = r(M_B) - 2.$$

Therefore $r(M) = r(M_B) \leq 3$, which contradicts the hypotheses of the theorem. □

We close this section with some simple consequences of Theorem 5.1.

Corollary 5.17. *Let M be a 3-connected excluded minor for $\mathcal{M}$ such that $|E(M)| \geq 10$ and both the rank and corank of M exceed three. Let M_B be the binary matroid supplied by Theorem 5.1, and let J and K be the circuit-hyperplanes that partition $E(M_B)$. Then*

(i) $r(M) = r^*(M)$;
(ii) $|E(M)|$ *is divisible by* 4;
(iii) every non-spanning circuit of M has even cardinality;
(iv) every non-cospanning cocircuit of M has even cardinality;
(v) M_B contains no triangles and no triads; and
(vi) the matroid obtained from M_B by relaxing K is an excluded minor for $\mathcal{M}$.

Proof. Statement (i) is clear. Observe that J and K are both circuits and cocircuits of M_B. As M_B is binary, this means that $|J| = |K|$ is even. Therefore $|E(M)| = |J| + |K|$ is a multiple of 4.

Any non-spanning circuit of M is also a circuit in M_B, and must therefore meet both J and K in an even number of elements. This proves statement (iii). Statement (iv) follows by duality.

If T is a triangle of M_B, then it is also a triangle of M, which contradicts statement (iii). If T^* is a triad of M_B, then it cannot be a triad of M, by statement (iv). Since M is obtained from M_B by relaxing the circuit-hyperplane J, it follows that T^* must be $E(M) - J = K$. As $|E(M)| = |J| + |K| = 2|K|$, this means that $|E(M)| = 6$, a contradiction.

Finally, let M' be the matroid obtained from M_B by relaxing K. Suppose that M' is binary. Then, for any two elements $j, j' \in J$, both $K \cup j$ and $K \cup j'$ are circuits of M', and hence $\{j, j'\}$ is a circuit of M'. This implies that M_B contains a parallel pair, and this contradicts the fact that M_B is 3-connected. Suppose that M' is ternary. Then the matroid, M_T, obtained from M' is relaxing J is also ternary, and Lemma 2.9 implies that M' is binary, a contradiction. Therefore M' does not belong to $\mathcal{M}$. By applying Proposition 2.6 we see that a single-element deletion or contraction of M' is equal to a single-element deletion or contraction of either the binary matroid M_B or the ternary matroid M_T. The result follows. □

6. Almost-regular matroids

In this section we establish a connection between the excluded minors for $\mathcal{M}$ and Truemper's class of almost-regular matroids, defined in Section 2.6.

Theorem 6.1. *Let M be an excluded minor for $\mathcal{M}$ with $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$. Let M_B be the binary matroid supplied by Theorem 5.1, so that $E(M)$ is partitioned into two circuit-hyperplanes, J and K, of M_B. Then $M_B\backslash e$ and M_B/e are almost-regular, for every element $e \in E(M)$. In particular, if $e \in J$, then $M_B\backslash e$ is almost-regular, with* con $= J - e$ *and* del $= K$, *and M_B/e is almost-regular with* con $= K$ *and* del $= J - e$. *If*

$e \in K$, then $M_B\backslash e$ is almost-regular, with con $= K - e$ *and* del $= J$, *and M_B/e is almost-regular with* con $= J$ *and* del $= K - e$.

Proof. Theorem 5.1 states that relaxing both J and K in M_B produces a ternary matroid M_T. Let e be an element in J. Let $con = J - e$ and let $del = K$. It follows from Proposition 2.6 that if f is in K, then $M_B\backslash e\backslash f = M_T\backslash e\backslash f$. Hence $M_B\backslash e\backslash f$ is both binary and ternary, and is therefore regular. On the other hand, if $f \in J-e$, then $M_B\backslash e/f = M_T\backslash e/f$, so $M_B\backslash e/f$ is regular.

Next we show that $M_B\backslash e$ itself is not regular. Suppose that it is. Then, in particular, $M_B\backslash e$ is ternary. Note that K is a circuit-hyperplane of $M_B\backslash e$, and that relaxing this circuit-hyperplane in $M_B\backslash e$ produces $M_T\backslash e$, by Proposition 2.6. Therefore $M_B\backslash e$ and $M_T\backslash e$ are both ternary matroids, and the second is produced from the first by relaxing K. Lemma 2.9 asserts that there is an enlarged wheel G such that K is the rim of G and $M_B\backslash e = M(G)$. Now M_B is simple, so G contains no parallel edges. Since $J - e$ makes up the spoke edges of G, and $|K| = |J - e| + 1$, it follows that the rim of G contains precisely one series pair. But M_B contains no series pair, as it is 3-connected. Therefore M_B contains at least one triad, contradicting Corollary 5.17. Hence $M_B\backslash e$ is not regular.

We note that $J - e$ is a cocircuit of $M_B\backslash e$, so any circuit of this matroid meets $J - e$ in an even number of elements. Similarly, K is a circuit of $M_B\backslash e$, so any cocircuit of $M_B\backslash e$ meets K in a set of even cardinality. We conclude that $M_B\backslash e$ is almost-regular.

Next we consider M_B/e. Let $con = K$ and let $del = J - e$. If $f \in K$, then $M_B/e/f = M_T/e/f$, and if $f \in J - e$, then $M_B/e\backslash f = M_T/e\backslash f$, so both these matroids are regular. Suppose that M_B/e is regular. Now $J - e$ is a circuit-hyperplane of M_B/e, and the matroid produced from M_B/e by relaxing $J - e$ is M_T/e. Therefore M_B/e is the cycle matroid of an enlarged wheel G, and $J - e$ is the rim of G. Since M_B is 3-connected, it follows that M_B/e has no series pairs. As the rim of G has cardinality $r(M) - 1$ and the complement of the rim contains $r(M)$ elements, this means that G must contain a parallel pair. Therefore M_B contains a triangle, so we have a contradiction to Corollary 5.17. Finally, we observe that K is a cocircuit of M_B/e, so any circuit of this matroid meets K in an even number of elements, and $J - e$ is a circuit of M_B/e, so any cocircuit meets $J - e$ in a set with even cardinality. It follows that M_B/e is almost-regular.

An identical argument works in the case that $e \in K$. □

Proposition 6.2. *Let M be an excluded minor for $\mathcal{M}$ with $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$. Let M_B be the binary matroid supplied by Theorem* 5.1. *Then $M_B\backslash e$ and M_B/e are internally* 4*-connected, for every e in $E(M)$.*

Proof. By duality it suffices to prove that $M_B\backslash e$ is internally 4-connected. Since M_B is 3-connected, $M_B\backslash e$ is certainly 2-connected. Suppose that $M_B\backslash e$ is not 3-connected. Since $M_B\backslash e$ is almost-regular, Theorem 22.1 of [27] implies that $M_B\backslash e$ must contain a series pair. But this implies that

M_B contains a triad, a contradiction to Corollary 5.17. If $M_B\backslash e$ is not internally 4-connected, then [27, Theorem 22.1] implies that $M_B\backslash e$ contains both a triangle and a triad. Thus M_B contains a triangle, and again we have a contradiction to Corollary 5.17. □

7. Reduction to a finite list of excluded minors

We are now ready to proceed with the proof of Theorem 1.1. In what follows, M will be an excluded minor for $\mathcal{M}$ such that $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$. Theorem 5.1 supplies us with the matroid M_B. We consider three cases. In the first, M_B has an R_{10}-minor; in the second, M_B has an R_{12}-minor; and, in the last case, M_B has no R_{10}-minor and no R_{12}-minor. In each case, we bound the size of $|E(M)|$, and thereby reduce the remainder of the proof to a finite case check.

7.1. **The R_{10} case.** In this section we consider the easiest case, namely when M_B has an R_{10}-minor. The arguments of this section closely follow those of Truemper in Section 26 of [27].

The matroid N_{11} plays an important role in Truemper's characterization of the almost-regular matroids. It is the rank-5 binary matroid with eleven elements obtained from R_{10} by adding an element z so that z is in a triangle. Since the automorphism group of R_{10} is transitive on pairs of elements ([25, p. 328]), N_{11} is well-defined up to isomorphism. As R_{10} contains no triangles, it follows that z is in no parallel pair of N_{11}. Therefore N_{11} is 3-connected. Since R_{10} is a splitter for the class of regular matroids (Proposition 2.12), it follows that N_{11} is not regular. However, it is not difficult to see that N_{11} is almost-regular. The following matrix is a reduced representation of N_{11} over GF(2).

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Deleting the last column of this matrix produces a reduced representation of R_{10}.

Proposition 7.1. *The matroid N_{11}/z is not regular.*

Proof. Let A be the matrix displayed above, so that $[I_5|A]$ represents N_{11}. Suppose that the columns of $[I_5|A]$ are labeled with the integers $1, \ldots, 11$, so that z corresponds to the column labeled by 11. By pivoting on the first entry in column 11 and then deleting the first row, and columns 1, 6, and 7, we see that N_{11}/z has an F_7^*-minor, and is therefore not regular. □

Proposition 7.2. *Suppose that N is a 3-connected almost-regular matroid such that $|E(N)| = 11$ and N has an R_{10}-minor. Then N is isomorphic to either N_{11} or N_{11}^*.*

Proof. Since R_{10} is self-dual, we can assume that N is a extension of R_{10} by the element z. We will be done if we can show that z is contained in a triangle of N.

Consider the partition (del, con) of $E(N)$. The set *con* is non-empty, by definition. Suppose that *con* contains only a single element. This element is contained in a circuit, as N is connected. But this circuit meets *con* in precisely one element, which contradicts the definition of almost-regular matroids. Thus we can choose an element $e \in con$ such that $e \neq z$.

Suppose that N/e is 3-connected. It is regular as $e \in con$. Since N/e has rank four it has neither an R_{10}- nor an R_{12}-minor, and is therefore either graphic or cographic by Lemma 2.13. Every single-element contraction of R_{10} is isomorphic to $M^*(K_{3,3})$, so N/e is a 3-connected cographic extension of $M^*(K_{3,3})$ by the element z. But it is easy to see that no such cographic matroid exists, so we have a contradiction.

We now know that N/e is not 3-connected. As N/e is a single-element extension of $M^*(K_{3,3})$, a 3-connected matroid, it follows that z is in a parallel pair in N/e. Therefore z is in a triangle in N. Thus we are done. □

Proposition 7.3. *Let e be an element of $E(N_{11})$ such that no triangle of N_{11} contains $\{e, z\}$. Let M be the binary matroid obtained by adding the element f to N_{11} so that $\{e, f, z\}$ is a triangle. Then $M\backslash e\backslash z$ is not regular.*

Proof. Note that z is contained in at least one triangle in $M\backslash f$. Let $\{a, b, z\}$ be such a triangle, and let M' be $M/a\backslash z$. We start by showing that M' is simple. Since M is simple by construction, if M' is not simple, there is a triangle T of M such that $a \in T$, but T avoids z. Note that $M'\backslash f$ is isomorphic to a single-element contraction of R_{10}, and is therefore simple. Thus $f \in T$. Let x be the single element in $T - \{a, f\}$.

Note that x is not equal to b, for that would imply that f and z are parallel in M. Also x is not equal to e, as that would imply that a and z are parallel in M. It follows that a, b, e, f, x, z are distinct elements of M. But $\{a, b, z\}$, $\{e, f, z\}$, and $\{a, f, x\}$ are triangles of M. The symmetric difference of these sets is the triangle $\{b, e, x\}$. Therefore $M\backslash z\backslash f \cong R_{10}$ contains a triangle, and this is a contradiction. Hence M' is simple.

Since $M\backslash z\backslash f \cong R_{10}$, it follows that $M'\backslash f$ is a single-element contraction of R_{10}, and is therefore isomorphic to $M^*(K_{3,3})$. Moreover, $\{a, b, z\}$ and $\{e, f, z\}$ are triangles of M, meaning that $\{a, b, e, f\}$ is a circuit of M, so $\{b, e, f\}$ is a triangle of M'. Thus M' is isomorphic to the matroid obtained from $M^*(K_{3,3})$ by adding the element f so that it forms a triangle with b and e. Since M' is simple, there is no triangle of $M^*(K_{3,3})$ that contains b and e. Thus b and e correpond to edges with no vertex in common in the graph $K_{3,3}$.

Let g be one of the two edges of $K_{3,3}$ that has a common vertex with both b and e. Therefore g is in triangles of $M^*(K_{3,3})$ with both b and e. Assume that g is in a triangle of M' with f. The symmetric difference of this triangle with $\{b, e, f\}$ is a four-element circuit of $M^*(K_{3,3})$ that contains $\{b, e, g\}$. But it is easy to check that no four-element bond of $K_{3,3}$ contains $\{b, e, g\}$, so this is impossible. Therefore the triangles of M' that contain g are also triangles of $M^*(K_{3,3})$. This means that g is in exactly two triangles of $M^*(K_{3,3})$. Since M' has ten elements, it follows that $\mathrm{si}(M'/g)$ has seven elements. As $\mathrm{si}(M'/g)$ has rank three, this implies that $\mathrm{si}(M'/g)$ is isomorphic to F_7, and is therefore non-regular. As e is in a parallel pair in M'/g, it follows that $\mathrm{si}(M'/g)$ is isomorphic to a minor of $M'/g\backslash e$, so $M'\backslash e$ is non-regular. Moreover, M' is a minor of $M\backslash z$, so $M\backslash z\backslash e$ is non-regular, as desired. □

The following result is the key step in this part of the case analysis (see also [27, Theorem 26.1]).

Lemma 7.4. *Let N be an internally 4-connected almost-regular matroid having an R_{10}-minor. Then N is isomorphic to either N_{11} or N_{11}^*.*

Proof. Since N is not regular, it cannot be isomorphic to R_{10}. By the Splitter Theorem (Theorem 2.10), there is a 3-connected minor N_0 of N such that N_0 is a single-element extension or coextension of R_{10}. Proposition 7.2 implies that N_0 is isomorphic to either N_{11} or N_{11}^*. By exploiting duality, we can assume the former. Let z be the distinguished element of $E(N_0)$ such that $N_0\backslash z \cong R_{10}$ and z is contained in a triangle of N_0.

If N is equal to N_0, we are done, so assume that N_0 is a proper minor of N. Since N_0/z is non-regular by Proposition 7.1, it follows that N/z is non-regular. Thus $N\backslash z$ is regular and has a proper R_{10}-minor. But R_{10} is a splitter for the class of regular matroids, so $N\backslash z$ is not 3-connected. As N is 3-connected, we see that $N\backslash z$ is certainly 2-connected.

Suppose that (X_1, X_2) is a 2-separation of $N\backslash z$, and that $|X_1|, |X_2| \geq 3$. Then both $(X_1 \cup z, X_2)$ and $(X_1, X_2 \cup z)$ are 3-separations of N, and we have a contradiction to the fact that N is internally 4-connected. We deduce from this that if (X_1, X_2) is a 2-separation of $N\backslash z$, then either X_1 or X_2 is a series pair of $N\backslash z$. This implies that $\mathrm{co}(N\backslash z)$ is 3-connected. As $\mathrm{co}(N\backslash z)$ is regular with an R_{10}-minor, $\mathrm{co}(N\backslash z)$ must in fact be isomorphic to R_{10}.

Consider a series pair P of $N\backslash z$, and suppose that $P \subseteq E(N_0)$. Then $N_0\backslash z$ must contain a cocircuit of size at most two, and this is a contradiction, as $N_0\backslash z \cong R_{10}$. Since N is 3-connected, the series pairs of $N\backslash z$ are pairwise disjoint. Therefore we can find a set S containing exactly one element from each series pair of $N\backslash z$ such that S does not meet $E(N_0)$. Note that $N\backslash z/S \cong \mathrm{co}(N\backslash z)$. Thus $|E(N\backslash z/S)| = 10$. But $E(N\backslash z/S)$ contains $E(N_0\backslash z)$, and this set also has cardinality ten. Thus every element of $E(N)$ not in S is an element of N_0.

Let P be a series pair of $N\backslash z$. Then $P \cup z$ is a triad of N. Let s be the unique element in $P \cap S$. Suppose that N_0 is a minor of $N\backslash s$. Then N_0 contains $(P - s) \cup z$, and this set is a series pair of $N\backslash s$. Thus N_0 contains a cocircuit of size at most two, a contradiction. Therefore N_0 is not a minor of $N\backslash s$, for any element $s \in S$. It follows that $N_0 = N/S$.

Next we suppose that P is a series pair of $N\backslash z$, that $P = \{e, s\}$ where $s \in S$, and that there is no triangle of N_0 that contains both $\{e, z\}$. Consider the matroid $N/(S - s)$. This matroid cannot be regular, since it has N_0 as a minor. Hence it is almost-regular, by Proposition 2.16. Note that $N/(S - s)/s$ is not regular, so $N/(S - s)\backslash s$ must be regular. However, $P \cup z$ is a triad of $N/(S - s)$. Let M be the binary matroid obtained from N_0 by adding an element so that it forms a triangle with z and e. Then $N/(S-s)\backslash s/e \cong M\backslash e\backslash z$. The last matroid is not regular by Proposition 7.3. Thus we have a contradiction, and conclude that if P is a series pair of $N\backslash z$, then the single element in $P - S$ is contained in a triangle of N_0 with z.

Suppose that there are distinct triangles T_1 and T_2 of N_0 such that $z \in T_1 \cap T_2$, and there are elements $e_1 \in T_1 - z$ and $e_2 \in T_2 - z$ such that e_i is contained in the series pair $\{e_i, s_i\}$ of $N\backslash z$ for $i = 1, 2$. Let N' be $N/(S - \{s_1, s_2\})$. Then N' is not regular, since it has N_0 as a minor. Thus N' is almost-regular.

Note that $N'/s_1/s_2/z = N_0/z$ is non-regular by Proposition 7.1. But e_1 is in a parallel pair of N_0/z, so $N'/s_1/s_2/z\backslash e_1$, and hence $N'\backslash e_1$, is non-regular. It follows that N'/e_1 is regular.

We observe that $N_0\backslash z$, and hence $N'\backslash z$, has an R_{10}-minor. But $\{e_1, s_1\}$ is a series pair of $N'\backslash z$, so $N'\backslash z/e_1$, and hence N'/e_1 has an R_{10}-minor. Thus N'/e_1 is regular with a proper R_{10}-minor. We will obtain a contradiction by showing that N'/e_1 is 3-connected.

First we show that N' is 3-connected. The matroid $N'/s_1/s_2$ is 3-connected, as it is isomorphic to N_{11}. Neither s_1 nor s_2 is a loop of N', so if N' is not 3-connected, it contains a cocircuit of size at most two. Hence so does N, a contradiction. Thus N' is 3-connected.

Suppose that N'/e_1 is not simple. Then there is a triangle T of N' that contains e_1. The triad $\{e_1, s_1, z\}$ must meet T in two elements. If s_1 were in T, then $N'/s_1/s_2 = N_0$ would contain a circuit of size at most two, a contradiction. Therefore $z \in T$. The triad $\{e_2, s_2, z\}$ must meet T in two elements, and s_2 is not in T, by the previous argument, so $T = \{e_1, e_2, z\}$. Now T_1, T_2, and $\{e_1, e_2, z\}$ are triangles of N_0, and as T_1 and T_2 are distinct, this implies the existence of a parallel pair in N_0. This contradiction means that N'/e_1 is simple.

Suppose that (X_1, X_2) is a 2-separation of N'/e_1. As N' is 3-connected, it contains no series pairs, so neither does N'/e_1. We have already shown that N'/e_1 has no parallel pairs. Now it follows easily that $|X_1|, |X_2| \geq 4$. Note that $N'/e_1/s_1/s_2 \cong N_0/e_1$ and the last matroid is obtained from $M^*(K_{3,3})$ by adding a single parallel element. Thus if (Y_1, Y_2) is a 2-separation of $N'/e_1/s_1/s_2$, then either Y_1 or Y_2 is a parallel pair. Now Proposition 2.3 implies that $\{s_1, s_2\}$ must be contained in either X_1 or X_2. Without loss of generality, we assume the former. It follows that $X_1 - \{s_1, s_2\}$ is the unique parallel pair of $N'/e_1/s_1/s_2$. Thus $|X_1| = 4$.

As N'/e_1 is simple, $r_{N'/e_1}(X_1) \geq 3$. Thus $r_{N'/e_1}(X_2) \leq r(N'/e_1) - 2$, so X_1 contains at least two cocircuits of N'/e_1. This implies the existence of a cocircuit of size at most two in N', and we have a contradiction.

This argument shows that there is a triangle T of N_0, such that if P is a series pair of $N\backslash z$, then the unique element in $P - S$ is contained in T. There is a circuit $C \subseteq T \cup S$ of N such that C contains T. But C must be equal to T, for otherwise C meets a triad of N in three elements. Thus T is a triangle of N which meets at least one triad. This is impossible in an internally 4-connected matroid, so we have arrived at a contradiction that completes the proof of the lemma. □

Now we can state the conclusion of this analysis.

Lemma 7.5. *Suppose that M is an excluded minor for the class $\mathcal{M}$ such that $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$. Let M_B be the binary matroid supplied by Theorem* 5.1. *If M_B has an R_{10}-minor, then M is a single-element extension of N_{11} or N_{11}^*, and hence $|E(M)| = 12$.*

Proof. By duality we can assume that there is an element $e \in E(M)$ such that $M_B\backslash e$ has an R_{10}-minor. Then $M_B\backslash e$ is almost-regular and internally 4-connected by Theorem 6.1 and Proposition 6.2. The result follows from Lemma 7.4. □

7.2. **The R_{12} case.** In this section we assume that M is an excluded minor for $\mathcal{M}$ with $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$, and that M_B, the matroid supplied by Theorem 5.1, has an R_{12}-minor.

Recall that Truemper graphs were defined in Section 2.8. We use these graphs repeatedly in this section and the next.

Proposition 7.6. *Let $G = (R, S)$ be a simple Truemper graph. Assume that both R and S contain at least two edges, and that every vertex is incident with at least one cross edge. Then either:*

(i) G contains a triangle;
(ii) an internal vertex of G has degree three; or
(iii) G has an XX*-minor.*

Proof. Let $r_1, \ldots, r_m$ and $s_1, \ldots, s_n$ be the vertices of R and S respectively. Thus $m, n \geq 3$. Assume that the result fails, and that G is a counterexample, but that the result holds for graphs with fewer edges than G.

We first suppose that $m = 3$. Consider s_2 and s_{n-1}. Because G is a counterexample, both these vertices meet at least two cross edges. Neither can be adjacent to r_2, for that implies that G contains a triangle. Thus s_2 and s_{n-1} are adjacent to r_1 and r_3. If s_1 were adjacent to r_1 or r_3, then G would contain a triangle. Thus s_1 is adjacent to precisely one vertex in R, namely r_2. Similarly, s_n is adjacent to r_2, and no other vertex in R. But now the edges r_1s_2, r_2s_1, r_2s_n, r_3s_2 give rise to an XX-minor. This contradiction means that $m > 3$ and, by symmetry, $n > 3$.

Proposition 2.20 implies that there is an edge joining two terminal vertices. By relabeling if necessary, we assume that there is an edge e joining r_1 and s_1. Suppose that both r_1 and s_1 meet at least two cross edges in G. Then the hypotheses of the proposition apply to $G\backslash e$, so our minimality assumption implies that $G\backslash e$ contains either a triangle, an XX-minor, or an internal vertex with degree three. However, in any of these cases, the result also holds for G, and we have a contradiction. Hence either r_1 or s_1 has degree exactly two. By symmetry, we assume that r_1 has degree two.

Let f be the edge r_1r_2. Assume that s_1 has degree greater than two. Since $m, n > 3$, the hypotheses of the proposition apply to $G\backslash e/f$. Therefore $G\backslash e/f$ contains (a) an XX-minor, (b) an internal vertex with degree three, or (c) a triangle. If $G\backslash e/f$ has an XX-minor, then so does G, and we have a contradiction. The internal vertices of $G\backslash e/f$ are internal vertices of G, and the degree of such a vertex in $G\backslash e/f$ is the same as its degree in G. Therefore (b) cannot occur. Finally, we suppose that (c) occurs. Then $G\backslash e/f$ has a triangle, but G does not. Thus f is contained in a cycle of length four in $G\backslash e$. But f is a pendant edge in this graph, and we have a contradiction.

We may now assume that the degree of s_1 is two. Let g be the edge s_1s_2. The result holds for $G\backslash e/f/g$, so $G\backslash e/f/g$ has an XX-minor, an internal vertex with degree three, or a triangle. The first two cases quickly lead to contradictions. Thus $G\backslash e/f/g$ has a triangle, but G does not. Therefore there is a cycle of $G\backslash e$ that contains either f or g. As these are pendant edges in $G\backslash e$, we have a contradiction. □

Truemper introduced a particular almost-regular matroid, V_{13}. There is a distinguished element z in V_{13} such that $V_{13}\backslash z$ is isomorphic to R_{12}. The dual matroid, V_{13}^*, has the reduced representation shown in Figure 8. Let A_0 be the matrix in Figure 8. We assume that the columns of $[I_7|A_0]$ are labeled $a_1, \ldots, a_6, z, b_1, \ldots, b_6$. Thus the rows of A_0 correspond in a natural way with the columns of the identity matrix, as reflected by the labels in Figure 8.

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	1	0	1	1	0	0
a_2	0	1	1	1	0	0
a_3	1	0	1	0	1	1
a_4	0	1	0	1	1	1
a_5	1	0	1	0	1	0
a_6	0	1	0	1	0	1
z	0	0	0	1	0	1

FIGURE 8. A reduced representation of V_{13}^*.

The next result follows from Theorem 25.9 of [27].

Lemma 7.7. *Let N be a 3-connected almost-regular matroid having an R_{12}-minor. Then N has a minor isomorphic to either V_{13} or V_{13}^*.*

Proposition 7.8. *The matroid $V_{13}^*\backslash z$ is non-regular.*

Proof. By considering the matrix in Figure 8 it is relatively straightforward to verify that

$$V_{13}^*/\{a_3, a_6, b_2, b_6\}\backslash\{a_2, z\}$$

is isomorpic to F_7. □

Suppose that A is a matrix, and that X (respectively Y) is a set of rows (columns) of A. We use $A[X, Y]$ to denote the submatrix of A induced by X and Y.

Lemma 7.9. *Suppose that N is an almost-regular matroid with a minor N_0 such that $N_0 \cong V_{13}^*$. Let $E(N_0) = \{a_1, \ldots, a_6, b_1, \ldots, b_6, z\}$, and assume that A_0 is a reduced representation of N_0 over* GF(2)*, where A_0 is the matrix in Figure* 8*. Let A be a reduced representation of N over* GF(2)*, and assume that $\{a_1, \ldots, a_6, z\}$ label rows of A, while $\{b_1, \ldots, b_6\}$ label columns. Then, up to row and column permutations, A has the form shown in Figure* 9*, and the following conditions hold:*

(i) $A[A_1, B_2]$ is the zero matrix; and
(ii) $A[A_2, B_1]$ has rank three, while $A[A_2 - z, B_1]$ has rank two.

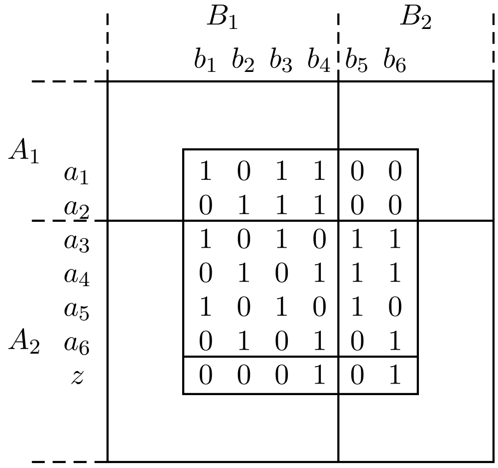


FIGURE 9

Proof. Proposition 7.8 implies that $N\backslash z$ is non-regular, so N/z is regular. Recall that $V_{13}^*/z \cong R_{12}$. Thus V_{13}^*/z has a 3-separation (X_1, X_2) such that $|X_1| = |X_2| = 6$. In particular, $X_1 = \{a_1, a_2, b_1, b_2, b_3, b_4\}$ and $X_2 = \{a_3, a_4, a_5, a_6, b_5, b_6\}$, so (X_1, X_2) is the 3-separation of V_{13}^*/z indicated by the division of the matrix in Figure 8.

Now N/z is a regular matroid with an R_{12}-minor, and therefore N/z has a 3-separation (Y_1, Y_2) such that $X_i \subseteq Y_i$ for $i = 1, 2$ (see [25, (9.2)]). From this fact, Truemper deduces that any reduced representation of N must be as is illustrated in Figure 9. (Note that the figure (25.12) of [27] contains an error. The upper right submatrix should consist of zeroes.) He concludes, moreover, that $A[A_1, B_2]$ is the zero matrix, $A[A_2, B_1]$ has rank three, and $A[A_2 - z, B_1]$ has rank two (see [27, p. 294]). □

Suppose that A is any matrix of the form in Figure 9, and that A is a reduced representation of an almost-regular matroid N. We let $A_{11} = A_1 - \{a_1, a_2\}$ and $B_{11} = B_1 - \{b_1, b_2, b_3, b_4\}$. Similarly, we let $A_{22} = A_2 - \{a_3, a_4, a_5, a_6, z\}$ and $B_{22} = B_2 - \{b_5, b_6\}$. If the column $b \notin \{b_1, \ldots, b_6\}$ has zero entries for all rows in A_{11}, then we shall say that b is a *righthand* column. Otherwise, we shall say that b is a *lefthand* column. Similarly, if a is a row of $A - \{a_1, \ldots, a_6, z\}$, and the row vector $A[\{a\}, B_1]$ is in the row space of $A[\{a_3, a_4\}, B_1]$, then we shall say that a is a *lower* row. Otherwise we say that a is an *upper* row. Note that the rank conditions upon the matrix $A[A_2, B_1]$ mean that if b is a lefthand column, then the entry in column b and row a, where $a \in A_2 - z$, is completely determined by the entries of b in rows a_3 and a_4.

Truemper studies the matroid $N/A_{11}\backslash B_{11}$, that is, the matroid with the reduced representation $A[A_2 \cup \{a_1, a_2\}, B_2 \cup \{b_1, b_2, b_3, b_4\}]$. He starts by considering the rows of the matrix $A[A_{22}, \{b_1, \ldots, b_6\}]$. Any such row must be one of the following vectors (see [27, (25.15)]).

	I	[1 0 1 0 0 0]	II	[0 0 0 0 1 0]
(7.1)	III	[0 1 0 1 0 1]	IV	[1 1 1 1 0 0]
	V	[1 0 1 0 1 1]	VI	[0 0 0 0 0 1]

If a is an element of A_{22} that corresponds to a row of type I, then we shall say that a is *type I element*, and so on.

Consider the family of graphs illustrated in Figure 10. In this diagram all solid edges are present, while all dashed edges represent (possibly empty) paths. Thus, for example, the vertices 2 and 3 may be equal. We will use G_0 to stand for a graph of this type. We let R (respectively S) be the path consisting of the horizontal edges joining vertices 1 and 7 (respectively 8 and 14).

Lemma 7.10. *Suppose that N is an almost-regular matroid with a reduced representation A, where A is as shown in Figure 9. Then $N/A_{11}\backslash B_{11}$ is equal to a graft of the form $M(G, D)$, where G is obtained from G_0 by adding edges between R and S, and $D = \{1, 7, 8, 14\}$. Here the graft element is b_2. In the graph G:*

(i) *the subpath of R between 2 and 3 consists of type I elements.*
(ii) *the subpath of R between 4 and 5 consists of type II elements.*
(iii) *the subpath of R between 6 and 7 consists of type III elements.*
(iv) *the subpath of S between 8 and 9 consists of type IV elements.*

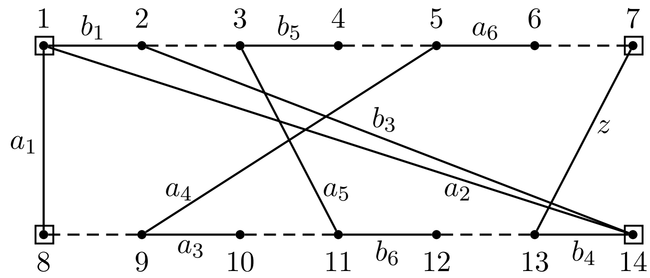


FIGURE 10. The graph G_0.

(v) the subpath of S between 10 *and* 11 *consists of type V elements.*
(vi) the subpath of S between 12 *and* 13 *consists of type VI elements.*

Proof. This follows immediately from Lemma 25.20 of [27]. □

Note that the graph G in Lemma 7.10 is a Truemper graph, as defined in Section 2.8. We remark that the cross edges added to G_0 to obtain G are precisely the members of B_{22}. Similarly, every element in A_{22} is an edge that appears in one of the paths represented by dashed edges.

Proposition 7.11. *Suppose that N is an almost-regular matroid and that A is a reduced representation of M, where A is a matrix of the type in Figure* 9*. Let (G, D) be the graft supplied by Lemma* 7.10*, so that $M(G, D) = N/A_{11}\backslash B_{11}$. Let v be an internal vertex of G other than* 2 *or* 13*, and let C^* be the set of edges incident to v in G. Then C^* is a cocircuit of N.*

Proof. This follows by examining the matrix in Figure 9. (See [27, p. 298].) □

Now we are ready to prove the concluding result in this case.

Lemma 7.12. *Suppose that M is an excluded minor for the class $\mathcal{M}$ such that $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$. Let M_B be the binary matroid supplied by Theorem* 5.1*. Then M_B has no R_{12}-minor.*

Proof. Let us assume that lemma fails, and that M_B does have a minor isomorphic to R_{12}. Corollary 5.17 implies the following:

7.12.1. *M_B has no triangles and no triads.*

7.12.2. *By exploiting duality, we can assume that there is an element e of $E(M_B)$ such that $M_B\backslash e$ is internally* 4*-connected, almost-regular, and has a V_{13}^*-minor.*

Proof. Since M_B is not regular, it follows that M_B has a proper R_{12}-minor. Theorem 2.10 implies that there is an element $e \in E(M_B)$ such that either $M_B\backslash e$ or M_B/e is 3-connected with an R_{12}-minor.

Suppose that $M_B\backslash e$ is 3-connected with an R_{12}-minor. Theorem 6.1 says that $M_B\backslash e$ is almost-regular, so $M_B\backslash e$ has either a V_{13}- or a V_{13}^*-minor, by

Lemma 7.7. If $M_B\backslash e$ has a V_{13}^*-minor, then we are done, since $M_B\backslash e$ is internally 4-connected by Proposition 6.2, We return to the case that $M_B\backslash e$ has an V_{13}-minor later.

Assume that M_B/e is 3-connected with an R_{12}-minor. Then M_B/e has either a V_{13}- or a V_{13}^*-minor. Assume that it has a V_{13}-minor. Then $M_B^*\backslash e$ is internally 4-connected with a V_{13}^*-minor. Now M^* is also an excluded minor for the class $\mathcal{M}$, and by swapping the labels on J and K, we see that M_B^* is a binary matroid with an R_{12}-minor that satisfies Theorem 5.1. That is, M_B^* is 3-connected, and J and K are disjoint circuit-hyperlanes of M_B^* that partition its ground set. Moreover, since M is obtained from M_B^* by relaxing J, it follows that M^* is obtained from M_B^* by relaxing K. Clearly the matroid obtained from M_B^* by relaxing J and K is ternary. Therefore we are free to relabel M^* as M and M_B^* as M_B. Hence we can assume that $M_B\backslash e$ is internally 4-connected with a V_{13}^*-minor, so in this case we are done.

We have shown that the claim is true (up to duality) if $M_B\backslash e$ is 3-connected with a V_{13}^*-minor, or if M_B/e is 3-connected with a V_{13}-minor. Therefore we assume that either $M_B\backslash e$ is 3-connected with a V_{13}-minor, or M_B/e is 3-connected with a V_{13}^*-minor. If the former case holds, then M_B^*/e is 3-connected with a V_{13}^*-minor. By switching to the dual if required, we can assume in either case that M_B/e is 3-connected with a V_{13}^*-minor.

It follows from Lemma 7.9 that we can assume M_B/e has a reduced representation A over GF(2), where A is as shown in Figure 9. If $B_{11} \cup B_{22}$ is non-empty, then there is an element $f \in B_{11} \cup B_{22}$ such that $M_B/e\backslash f$, and hence $M_B\backslash f$, has a V_{13}^*-minor. As $M_B\backslash f$ is internally 4-connected and almost-regular we can complete the proof by relabeling f as e. Therefore we assume that $B_{11} \cup B_{22} = \varnothing$.

Lemma 7.10 implies that $M_B/e/A_{11}$ is equal to a graft $M(G, D)$. As B_{22} is empty, no cross edges are added to G_0 to obtain G. But this means that the set of edges incident with vertex 5 in G is a triad of $M(G, D) = M_B/e/A_{11}$. Thus M_B contains a triad. This contradicts 7.12.1. □

In what follows, we will utilize 7.12.2, and assume that e is an element of M_B such that $M_B\backslash e$ is an internally 4-connected almost-regular matroid with a V_{13}^*-minor. Thus we can assume, by Lemma 7.9, that $M_B\backslash e$ has a reduced representation, A, over GF(2), of the type shown in Figure 9. There is a column which we can add to A so that the resulting matrix is a reduced representation of M_B over GF(2). We will abuse notation, and refer to this column as e.

7.12.3. *The set A_{11} is non-empty.*

Proof. By considering the six possibilities for rows of $A[A_{22}, \{b_1, \ldots, b_6\}]$ shown in Equation (7.1) on page 38, we see that the columns of A labeled by b_1 and b_3 are identical in all rows except a_2, and possibly rows in A_{11}. Thus, if A_{11} is empty, then $\{a_2, b_1, b_3\}$ is a triangle of $M_B\backslash e$, and hence of M_B. This contradiction completes the proof of the claim. □

7.12.4. *The set $B_{11} \cup B_{22}$ is non-empty.*

Proof. Suppose that $B_{11} \cup B_{22}$ is empty. By 7.12.3, there is an element a in A_{11}. Let A_a be the matrix obtained from A by adding the column e, and then deleting the row a. Now M_B/a is almost-regular by Theorem 6.1, and A_a is a reduced representation of M_B/a. Lemma 7.9 implies that A_a must have the form illustrated in Figure 9. This means that either:

(i) the column e has zero entries in all rows labeled by $A_{11} - a$; or
(ii) the entries of e in $A_2 - z$ are completely determined by the entries of e in a_3 and a_4.

In the first case, we call e a righthand column of A_a, and in the second we call it a lefthand column. If e is a righthand column of A_a, then we let B'_{11} be $B_{11} = \varnothing$, and if e is a lefthand column, we let B'_{11} be $\{e\}$.

In either case, $M_B/A_{11}\backslash B'_{11}$ is equal to a graft $M(G, D)$, as described in Lemma 7.10. But G is obtained from a graph G_0, either by adding a single edge (if e is a righthand column), or by adding no edges at all (if e is a lefthand column). If the second case applies, then the set of edges incident with the vertex 5 is a triad of $M(G, D)$, and of M_B/a, by Proposition 7.11. Thus M_B contains a triad, a contradiction.

We may now assume that e is a righthand column, and that we obtain G by adding the edge e to the graph G_0. It is easy to check that all the dashed edges in Figure 10 must represent empty paths, for otherwise G_0 has at least three internal vertices (other than 2 and 13) of degree two or three. This means that G contains an internal vertex of degree at most three, so M_B/a has a cocircuit of size at most three. This argument shows that A_a has no lower rows. A lower row of A is also a lower row of A_a, so this argument shows that A has no lower row, and that therefore $A_{22} = \varnothing$.

We now know that G_0 has exactly three internal vertices with degree three: those in Figure 10 labeled by 5, 11, and 13. Proposition 7.11 implies that the edge e must join 5 and 11 in G. Now $\{a_5, b_5, e\}$ is a triangle of $M(G, D) = M_B/A_{11}$. By considering the matrix in Figure 9, we see that the column e has non-zero entries in rows a_3 and a_4, and that if s is any other row in A_2, then e has a zero in row s.

Suppose now that e has a zero entry in row a. Then e contains precisely two non-zero entries: in rows a_3 and a_4. This means that $\{a_5, b_5, e\}$ is a triangle of M_B, and we have a contradiction. Therefore e contains precisely three non-zero entries: in rows a, a_3, and a_4.

Now we suppose that $A_{11} - a$ is non-empty, and that a' is an element in this set. We let $A_{a'}$ be the matrix obtained from A by adding the column e, and deleting the row a'. As before, $A_{a'}$ is a reduced representation of the almost-regular matroid M_B/a', and $A_{a'}$ must have the form described in Lemma 7.9. But e has a non-zero entry in $A_{11} - a'$, so e cannot be a righthand column of $A_{a'}$. Thus e is a lefthand column of $A_{a'}$, and M_B/A_{11} is equal to a graft $M(G', D')$. In this case, G' is obtained from a graph G_0 by adding no edges. Thus $M(G', D')$ contains a triad at the internal vertex 5, and hence M_B/a' contains a triad by Proposition 7.11. This contradiction means that $A_1 - a$ is empty.

We have shown that $A_{22} = \varnothing$, and that $|A_{11}| = 1$. Since $B_{11} \cup B_{22} = \varnothing$, we conclude that $|E(M_B \backslash e)| = 14$. Thus $|E(M_B)| < 16$, and Corollary 5.17 implies that $|E(M_B)| \leq 12$. This is a contradiction as M_B has a proper R_{12}-minor. □

By virtue of 7.12.4, there is a column $b \in B_{11} \cup B_{22}$. Consider the matrix A_b produced by adding the column e to A and then deleting b. Then A_b is a reduced representation $M_B \backslash b$, an almost-regular matroid with a V_{13}^*-minor. Thus A_b is of the form described in Lemma 7.9. Thus e is either a righthand or a lefthand column of A_b. We say that e is a right or lefthand element, according to which of these cases is true. Clearly this definition is independent of our choice of b.

By 7.12.3, there is a row a in A_{11}. Let A_a be the matrix obtained from A by adding the column e, and deleting a. Thus A_a is a reduced representation of the almost-regular matroid M_B/a. If e is a lefthand element, then let $B'_{11} = B_{11} \cup e$, and otherwise let $B'_{11} = B_{11}$. Now consider $M_B/A_{11} \backslash B'_{11}$. Lemma 7.10 says that this matroid is equal to a graft $M(G, D)$, where G is obtained from a member of the family illustrated in Figure 10 by adding cross edges.

7.12.5. *The vertices* 1 *and* 14 *have degree three in* G.

Proof. Let X be the set of edges that are incident with 1 in G. Assume that $X - \{a_1, a_2, b_1\}$ is non-empty, and let b be an element of this set, so that either $b \in B_{22}$ or $b = e$ (in which case e is a righthand element). Then $X \cup b_2$ is a cocircuit of $M(G, D) = M_B/A_{11} \backslash B'_{11}$. By examining the matrix in Figure 9, we see that this means that the column of A labeled by b has a non-zero entry in the one of the rows labeled by a_1 or a_2. This means that b cannot be equal to e, for if it were e would not be a righthand element. Thus $b \in B_{22}$, and this contradicts the fact that $A[A_1, B_2]$ is the zero matrix.

Now let X be the set of edges incident with 14. If $X - \{a_2, b_3, b_4\}$ is non-empty, we can deduce, using the same type of argument, that either e is a righthand column, and has a non-zero entry in row a_2, or that some member of B_{22} has a non-zero entry in row a_2. In either case, we have a contradiction that completes the proof. □

Let G' be the graph obtained from G by deleting a_1, a_2, and b_3. We obtain G'' from G' by contracting b_1 and b_4, and possibly two other edges: if vertex 2 has degree two in G', then we contract both of its incident edges, and if 8 has degree one in G', then we contract its incident edge. Every vertex in G'' must be incident with at least one cross edge, for otherwise G contains an internal vertex with degree two and, in this case, Proposition 7.11 would imply that M_B/a, and hence M_B, contains a series pair. Certainly the two vertex-disjoint paths in G'' contain at least two edges each, so we can apply Proposition 7.6 to G''.

If G'' has an XX-minor, then $M/A_{11}\backslash B'_{11}$ has a minor isomorphic to AG$(3,2)$, and is therefore neither regular nor almost-regular. This contradicts Proposition 2.16. The internal vertices of G'' are internal vertices of G, and 2 and 13 are not internal vertices of G''. The degree of an internal vertex in G'' equals its degree in G. Therefore no internal vertex of G'' has degree three, by Proposition 7.11 and 7.12.1. We conclude from Proposition 7.6 that G'' contains a triangle T. Now G'' can be obtained from $G' = G\backslash a_1\backslash a_2\backslash b_3$ by contracting pendant edges, so T is also a triangle of G', and hence of G.

Clearly T must contain at least one element corresponding to a column of A_a. Since T is a triangle of G'', it does not contain b_1, b_2, b_3, or b_4. Thus any column contained in T is either a member of B_2, or is equal to e (in which case e is a righthand element). This implies that any column in T has zero entries in any row in A_{11}. It follows that T is a triangle of $M[A] = M_B$. This contradiction completes the proof of the lemma. □

7.3. **The no R_{10} and no R_{12} case.** The two previous sections mean that we now need only consider the case that the binary matroid M_B has no minor isomorphic to R_{10} or R_{12}. Recall that switching in a graft is defined in Section 2.7.

Lemma 7.13. *Suppose that N is an internally 4-connected almost-regular matroid and assume that N has no R_{10}- or R_{12}-minor. Suppose also that $N = M(G, D)$ for some graft (G, D). If D is minimal under switching, then $|D| = 4$ and $G = (R, S)$ is a Truemper graph. Moreover*

(i) the set del *consists of all path edges, along with the graft element;*
(ii) the set con *consists of all cross edges; and*
(iii) the vertices in D are precisely the terminal vertices of G.

Proof. Theorem 23.41 of [27] proves this result in the case that N is an irreducible almost-regular matroid. A close examination of [27, Section 23] up to the proof of Theorem 23.41 reveals that the hypothesis of N being irreducible is not needed. Truemper shows that an irreducible almost-regular matroid is necessarily internally 4-connected [27, Theorem 22.1], and the proof of Theorem 23.41 holds under the weaker hypothesis that N is internally 4-connected. □

Lemma 7.14. *Let $G = (R, S)$ be a Truemper graph with no* XX*-minor. Assume that the cross edges of G form a spanning path P and that the end-vertices of P are terminal vertices of G. If both R and S contain at least four vertices, then G contains distinct triangles T_1, T_2, and T_3, two of which are edge-disjoint.*

Proof. Assume that G is a minimal counterexample to the proposition. Thus $|V(R)| \geq 4$ and $|V(S)| \geq 4$. Suppose the terminal vertices of G are $\{v_1, v_2, v_3, v_4\}$ and that the end-vertices of P are v_1 and v_4. Let e_1 and e_4 respectively be the cross edges incident with v_1 and v_4. Now v_2 and v_3

are incident with exactly two cross edges each. It follows that we can find distinct cross edges e_2 and e_3 such that e_2 is incident with v_2 and e_3 is incident with v_3, and neither e_2 nor e_3 joins v_2 to v_3. Since no cross edge joins v_1 to v_4, we conclude, by applying Proposition 2.20 to $\{e_1, e_2, e_3, e_4\}$, that one of v_1 or v_4 is adjacent to one of v_2 or v_3. We will assume without loss of generality that v_1 is adjacent to v_2.

Suppose that $\max\{|V(R)|, |V(S)|\} > 4$. If $|V(R)| = |V(S)|$, then R and S each contain one of the vertices v_1 and v_4. In this case, we will assume by relabelling if necessary that v_1 is in R. If $|V(R)| \neq |V(S)|$, then let us assume, by relabelling if necessary, that $|V(R)| > |V(S)|$. In this case, both v_1 and v_4 are contained in R. Thus v_1 is in R and $|V(R)| > 4$ in either case, so $R - v_1$ contains at least four vertices. Moreover, $P - v_1$ is a spanning path of $G - v_1$ and the end-vertices of $P - v_1$ are v_2 and v_4, which are terminal vertices of the Truemper graph $G - v_1 = (R - v_1, S)$. By our assumption on the minimality of G, it follows that $G - v_1$ contains distinct triangles T_1, T_2, and T_3, two of which are edge-disjoint. This implies that G is not a counterexample to the proposition, so we must assume that $|V(R)| = |V(S)| = 4$.

It remains only to show that the result holds when both R and S have exactly four vertices each. This is easily done: we simply construct all relevant Truemper graphs $G = (R, S)$ where R and S have vertices r_1, r_2, r_3, r_4 and s_1, s_2, s_3, s_4 respectively. We identify (v_1, v_2, v_4) with (r_1, s_1, s_4). Thus r_1 is adjacent to s_1 and the cross edges form a spanning path with end-vertices r_1 and s_4. Ignoring automorphisms, there are exactly twelve such graphs. These are obtained from the graphs in Figure 12 on page 57 by deleting the extra edge joining r_1 and s_4. Four of the twelve graphs have XX-minors, marked by heavy edges. The remaining eight graphs each contain three triangles, two of which are edge-disjoint. Therefore the proposition holds in the case that $|V(R)| = |V(S)| = 4$, and hence holds in general. □

Lemma 7.15. *Let $G = (R, S)$ be a Truemper graph and assume that the cross edges of G form a spanning cycle. Let the vertices of R and S be $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ respectively where $n \geq 3$. Assume that r_1 is adjacent to both s_1 and s_n and that s_n is not adjacent to r_2. Suppose that f is the edge r_1r_2 and that g is the edge s_1s_2. If s_1 is not adjacent to r_2, then let $G' = G/f$. Otherwise let $G' = G/f/g$. In either case, G' is a 3-connected graph. Moreover, if T is the edge set of a triangle of G and T is also a triangle in G', then G'/T is 2-connected.*

Proof. We start by proving the following claim.

7.15.1. *Suppose that u and v are distinct vertices of G and that $\{r_1, s_1\} \cap \{u, v\} = \varnothing$. There are three paths P_1, P_2, and P_3, such that u and v are the end-vertices of P_1, P_2, and P_3, and:*

(i) P_1, P_2, and P_3 are internally disjoint;
(ii) at most one of $P_1 - \{u, v\}$, $P_2 - \{u, v\}$, and $P_3 - \{u, v\}$ meets $\{r_1, r_2\}$;

(iii) if s_1 is adjacent to r_2, then at most one of $P_1 - \{u,v\}$, $P_2 - \{u,v\}$, and $P_3 - \{u,v\}$ meets $\{s_1, s_2\}$; and

(iv) if T is a triangle of G, then at most two of $P_1 - \{u,v\}$, $P_2 - \{u,v\}$, and $P_3 - \{u,v\}$ meet the vertices of T.

Proof. The proof of the claim is divided into several cases and subcases.

Case 1. $u = s_i$ and $v = s_j$ where $1 < i < j \leq n$.

We let P_1 be the path $s_i, \ldots, s_j$ and let P_2 be the path with vertex sequence $s_i, \ldots, s_1, r_1, s_n, \ldots, s_j$. Since every vertex of G is incident with two cross edges, there are vertices r_{i_1} and r_{j_1} such that $s_i r_{i_1}$ and $s_j r_{j_1}$ are edges. Since $1 < i < n$, it follows that r_1 is not adjacent to s_i. Thus we can choose i_1 so that $2 < i_1$. Similarly, by using the assumption that s_n is not adjacent to r_2, we can assume that $2 < j_1$. We let P_3 be the path formed by $s_i r_{i_1}$ and $s_j r_{j_1}$ and the segment of R between r_{i_1} and r_{j_1}.

It is easy to see that condition (i) is satisfied. Since $2 < i_1, j_1$, it also follows that (ii) is satisfied, and it is clear that (iii) holds. To see that condition (iv) is satisfied, we note that the vertex set of any triangle in G contains either two adjacent vertices in R or two adjacent vertices in S. Since $2 < i_1, j_1$, it follows that no triangle of G can meet all three of the sets $P_1 - \{u,v\}$, $P_2 - \{u,v\}$, and $P_3 - \{u,v\}$.

Case 2. $u = r_i$ and $v = r_j$, where $1 < i < j \leq n$.

We let P_1 be the path $r_i, \ldots, r_j$. Assume that u is adjacent to s_{i_1} and s_{i_2} and that v is adjacent to s_{j_1} and s_{j_2} where $1 \leq i_1 < i_2 \leq n$ and $1 \leq j_1 < j_2 \leq n$.

Case 2.1. $j_1 \leq i_1$.

In this case $j_1 < i_2$. We let P_2 be the path $r_i, \ldots, r_1, s_1, \ldots, s_{j_1}, r_j$ and we let P_3 be the path formed from $r_i s_{i_2}$, $r_j s_{j_2}$, and the segment of S between s_{i_2} and s_{j_2}. It is clear that conditions (i) and (ii) are satisfied. If more than one of these three sets has a non-empty intersection with $\{s_1, s_2\}$, then $j_1 = 1$ and either $i_2 = 2$ or $j_2 = 2$. As $j_1 = 1$, we have s_1 adjacent to both r_1 and r_j, and therefore s_1 is not adjacent to r_2. Thus (iii) is satisfied.

If condition (iv) is violated, then $j_1 + 1 \in \{i_2, \ldots, j_2\}$, and some triangle contains s_{j_1}, s_{j_1+1}, and a vertex w in $\{r_{i+1}, \ldots, r_{j-1}\}$. Thus either $i_2 = j_1 + 1$ or $j_2 = j_1 + 1$. In the first case, $i_1 = j_1$, so the only vertices in R that s_{j_1} is adjacent to are r_i and r_j. Thus the triangle cannot exist. In the second case, the cross edges contain the cycle $\{w s_{j_1}, s_{j_1} r_j, r_j s_{j_2}, s_{j_2} w\}$. This is a contradiction as $n \geq 3$ and the cross edges form a spanning cycle.

Case 2.2. $i_2 \leq j_2$ and $i_1 < j_1$.

In this case, $i_1 < j_2$. We let P_2 be the path $r_i, \ldots, r_1, s_n, \ldots, s_{j_2}, r_j$ and we let P_3 be the path formed from $r_i s_{i_1}$ and $r_j s_{j_1}$ and the segment of S between s_{i_1} and s_{j_1}. As in the previous case, it is easy to check that conditions (i) and (ii) hold. Moreover, (iii) holds as $i_1 < j_1 < j_2$.

If (iv) is violated, then $j_2 - 1 \in \{s_{i_1}, \ldots, s_{j_1}\}$ and there is a triangle with vertices s_{j_2}, s_{j_2-1} and $w \in \{r_{i+1}, \ldots, r_{j-1}\}$. Thus $j_1 = j_2 - 1$ and the cross edges contain the cycle $\{ws_{j_1}, s_{j_1}r_j, r_js_{j_2}, s_{j_2}w\}$; a contradiction.

Case 2.3. $i_1 < j_1$ and $j_2 < i_2$.

We let P_2 be $r_i, s_{i_1}, \ldots, s_{j_1}, r_j$ and we let P_3 be $r_i, s_{i_2}, \ldots, s_{j_2}, r_j$. Because $j_1 < j_2$, it follows that condition (i) holds, and it is obvious that (ii) and (iii) hold. The only way in which (iv) can fail is if $j_2 = j_1 + 1$ and there is a triangle with vertices s_{j_1}, s_{j_2} and $w \in \{r_{i+1}, \ldots, r_{j-1}\}$. In this case, the cross edges contain the cycle $\{ws_{j_1}, s_{j_1}r_j, r_js_{j_2}, s_{j_2}w\}$.

Case 3. $u = r_i$ and $v = s_j$ where $1 < i, j \leq n$.

Suppose that u is adjacent to s_{i_1} and s_{i_2} where $1 \leq i_1 < i_2 \leq n$ and that v is adjacent to r_{j_1} and r_{j_2}, where $1 \leq j_1 < j_2 \leq n$.

Case 3.1. $i \leq j_2$.

Case 3.1.1. $j \leq i_2$

We let P_1 be the path $r_i, s_{i_2}, \ldots, s_j$ and let P_2 be the path $r_i, \ldots, r_{j_2}, s_j$. We also let P_3 be the path $r_i, \ldots, r_1, s_1, \ldots, s_j$. It is easy to see that conditions (i), (ii), (iii), and (iv) are satisfied.

Case 3.1.2. $i_2 < j$.

We let P_1 be the path $r_i, s_{i_2}, \ldots, s_j$, we let P_2 be the path $r_i, \ldots, r_{j_2}, s_j$, and we let P_3 be the path $r_i, \ldots, r_1, s_n, \ldots, s_j$. In this case, the result holds.

Case 3.2. $j_2 < i$.

Case 3.2.1. $i_1 \leq j$ and $j \leq i_2$.

We let P_1 be the path $r_i, s_{i_1}, \ldots, s_j$, we let P_2 be the path $r_i, s_{i_2}, \ldots, s_j$, and we let P_3 be the path $r_i, \ldots, r_{j_2}, s_j$. It is easy to see that conditions (i), (ii), (iii), and (iv) hold.

Case 3.2.2. $j < i_1$.

Let P_1 be the path $r_i, s_{i_1}, \ldots, s_j$, let P_2 be the path $r_i, \ldots, r_{j_2}, s_j$, and let P_3 be the path $r_i, s_{i_2}, \ldots, s_n, r_1, \ldots, r_{j_1}, s_j$. Statement (i) holds. If (ii) fails, then $j_2 = 2$, so $j_1 = 1$. Since the only vertices in S adjacent to r_1 are s_1 and s_n, it follows that $j = n$. But then $j_2 < i_1 \leq n$, so we have a contradiction. Clearly (iii) is satisfied.

If condition (iv) fails, then either: $j_2 = j_1 + 1$ and some triangle contains r_{j_1}, r_{j_2} and some vertex in $\{s_{j+1}, \ldots, s_{i_1}\}$; or $i_2 = i_1 + 1$, and some triangle contains s_{i_1}, s_{i_2}, and some vertex in $\{r_{j_2}, \ldots, r_{i-1}\}$. In either of these cases, the set of cross edges contains a cycle of length four, which is a contradiction as we have assumed $n \geq 3$.

Case 3.2.3. $i_2 < j$.

Let P_1 be the path $r_i, s_{i_2}, \ldots, s_j$, let P_2 be the path $r_i, \ldots, r_{j_2}, s_j$, and let P_3 be the path $r_i, s_{i_1}, \ldots, s_1, r_1, \ldots, r_{j_1}, s_j$. Clearly (i) is true. If (ii) is not true, then $j_1 = 1$ and $j_2 = 2$. This implies that $j = n$ and that s_n is adjacent to r_2, a contradiction. For (iii) to be false, we must have $i_1 = 1$ and $i_2 = 2$, and s_1 is adjacent to r_2. Thus r_i is adjacent to s_1. However, $j_2 < i$, so $2 < i$. Thus s_1 is adjacent to three vertices in R: r_1, r_2, and r_i. This is a contradiction.

We again see that if (iv) fails then the cross edges of G contain a cycle of length four, a contradiction.

We have now exhausted all possible cases, so the claim must hold. □

We continue with the proof of the lemma. First suppose that s_1 is adjacent to r_2. Then $G' = G/f/g$. Let T be an arbitrary triangle of G that is also a triangle in G'. Suppose that u' and v' are distinct vertices of G'. Let u and v be vertices of G that correspond to u' and v' respectively. Since r_1 is identified with r_2 and s_1 is identified with s_2 in G', we may assume that $\{u, v\} \cap \{r_1, s_1\} = \varnothing$. Claim 7.15.1 says that there are three internally disjoint paths in G joining u to v, and conditions (ii) and (iii) imply that these paths lead to three internally disjoint paths in G' joining u' to v'. Since u' and v' were arbitrary distinct vertices in G', this means that G' is 3-connected. Moreover, condition (iv) implies the existence of two internally disjoint paths in G'/T joining u' to v'. Thus G'/T is 2-connected.

Next we suppose that s_1 is not adjacent to r_2. In this case, $G' = G/f$. Suppose that G' is not 3-connected. Then there are subsets $X, Y \subseteq V(G')$ such that (i) $X \cup Y = V(G')$; (ii) $|X \cap Y| \leq 2$; (iii) neither $X - Y$ nor $Y - X$ is empty; and (iv) no edge of G' joins a vertex in $X - Y$ to a vertex in $Y - X$.

Let u' and v' be vertices in $X - Y$ and $Y - X$ respectively, and let u and v be vertices of G which correspond to u' and v'. Since r_1 is identified with r_2 in G', we may assume that neither u nor v is equal to r_1. If neither u nor v is equal to s_1, then Claim 7.15.1 implies there are three internally disjoint paths joining u to v in G, and that furthermore these paths lead to three internally disjoint paths from u' to v' in G'. This is a contradiction as any path from u' to v' contains a vertex in $X \cap Y$. Thus we assume that $u = s_1$. Since s_1 is not incident with f, this means $u' = s_1$. As u' was an arbitrary vertex in $X - Y$, it follows that $X - Y = \{u\}$. Now any vertex that is adjacent with u in G' must be in $X \cap Y$. However, u is adjacent to distinct vertices s_2, r_1, and r_i in G, where $2 < i \leq n$, and these three vertices are distinct in G'. Thus $|X \cap Y| > 2$, a contradiction.

Next we suppose that T is an arbitrary triangle of G and that T is a triangle in G'. Suppose that G'/T is not 2-connected. Then there are subsets $X, Y \subseteq V(G'/T)$ such that: (i) $X \cup Y = V(G/T')$; (ii) $|X \cap Y| \leq 1$; (iii) neither $X - Y$ nor $Y - X$ is empty; and (iv) no edge of G'/T joins a vertex in $X - Y$ to a vertex in $Y - X$.

Assume that u' and v' are vertices in $X - Y$ and $Y - X$ respectively, and let u and v be corresponding vertices of G. We may assume that neither u

nor v is r_1. If neither u nor v is s_1, then there are three internally disjoint paths between u and v, and these paths lead to two internally disjoint paths in G'/T, a contradiction. Thus $u = s_1$, without loss of generality, and if we assume that u is also a vertex of G'/T, then $X - Y = \{u\}$. Now every vertex adjacent to u must be in $X \cap Y$. Since $|X \cap Y| \leq 1$, this means that all vertices of G that are adjacent to u must be identified in G'/T. Thus the vertices of T are s_2, r_1, and r_i. But $n \geq 3$, so r_1 is not adjacent to s_2, and we have a contradiction. This completes the proof of the lemma. □

Definition 7.16. Suppose that M is a connected matroid. A triangle T of M is a *separating triangle* if M/T is not connected.

Lemma 7.17. *Let G' be a graph such that $M(G')$ is connected, and let T_1, T_2, and T_3 be distinct non-separating triangles of $M(G')$. If M' is a single-element coextension of $M(G')$, and none of T_1, T_2, or T_3 is a triangle in M', then M' is not cographic.*

Proof. Assume that M' is a coextension of $M(G')$ by the element e. Suppose that M' is cographic, so that $M' = M^*(H)$ for some connected graph H. Now T_1, T_2, and T_3 are triads in

$$M^*(G') = (M'/e)^* = M(H\backslash e).$$

Thus T_1, T_2, and T_3 are minimal edge cut-sets in $H\backslash e$.

Let the two components of $H\backslash e\backslash T_1$ be H_1 and H_2. If both H_1 and H_2 contain at least one edge, then $M(H\backslash e\backslash T_1)$, and hence $M^*(H\backslash e\backslash T_1)$, is not connected. But $M^*(H\backslash e\backslash T_1) = M(G')/T_1$, so this contradicts the fact that T_1 is not a separating triangle of $M(G')$. Thus we assume that H_1 contains no edges. As H_1 is connected, it follows that H_1 must contain a single vertex, so T_1 is the set of edges incident with a vertex v_1 in $H\backslash e$. The same argument implies that T_2 and T_3 are the sets of edges incident with vertices v_2 and v_3 in $H\backslash e$.

None of T_1, T_2, or T_3 is a minimal edge cut-set in H, so e must be incident in H with the distinct vertices v_1, v_2, and v_3, an impossibility. □

Proposition 7.18. *Suppose that M is an excluded minor for the class $\mathcal{M}$ such that $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$, and that M_B is the binary matroid supplied by Theorem* 5.1. *Assume that M_B has no R_{10}-minor. Then there are distinct elements e and d in $E(M_B)$ such that either $M_B/e\backslash d$ or $M_B^*/e\backslash d$ is graphic.*

Proof. Let e be an arbitrary element of $E(M_B)$. Then M_B/e is almost-regular by Theorem 6.1, so $E(M_B/e)$ can be partitioned into non-empty *del* and *con* sets. Let d be an element in *del*. Then $M_B/e\backslash d$ is regular.

Proposition 6.2 implies that M_B/e is internally 4-connected. If $M_B/e\backslash d$ is not 3-connected, then M_B/e must contain a triad, which contradicts Corollary 5.17. As M_B has no R_{10}- or R_{12}-minor (by Lemma 7.12), Lemma 2.13 implies that $M_B/e\backslash d$ is either graphic or cographic. If $M_B/e\backslash d$ is graphic, then we are done. Therefore we assume that $M_B/e\backslash d$ is cographic. In this case,

$$(M_B/e\backslash d)^* = M_B^*/d\backslash e$$

is graphic and the result follows by swapping the labels on e and d. □

We can now prove the main result in this part of the case analysis.

Lemma 7.19. *Let M be an excluded minor for the class $\mathcal{M}$ with $|E(M)| \geq 10$ and $r(M), r^*(M) \geq 4$. Let M_B be the binary matroid supplied by Theorem* 5.1. *If M_B has no R_{10}-minor then $|E(M)| \leq 16$.*

Proof. Note that M_B has no R_{12}-minor, by Lemma 7.12. Let us assume that $|E(M)| > 16$.

Corollary 5.17 implies the following fact:

7.19.1. *M_B has no triangles and no triads.*

By virtue of Proposition 7.18, and by switching to the dual if necessary, we will henceforth assume that e and d are distinct elements of $E(M_B)$ such that $M_B/e\backslash d$ is graphic. Thus M_B/e is almost-regular and a graft, where d is the graft element. By Proposition 6.2 and Lemma 7.13, we can assume that $M_B/e = M(G, D)$, where $G = (R, S)$ is a Truemper graph and D is exactly the set of terminal vertices of G. Proposition 2.19 implies that:

7.19.2. *G has no* XX*-minor.*

By virtue of Corollary 5.17 (vi), we can relabel J and K if necessary, so we assume that $e \in J$. It follows from Theorem 6.1 that M_B/e is almost-regular with *del* $= J - e$ and *con* $= K$. By Lemma 7.13, *del* $= J - e$ consists of the path edges of G along with d; *con* $= K$ consists of the cross edges. Thus $d \in J$. But K is a spanning circuit of M_B/e. From this, we deduce the following.

7.19.3. *The paths R and S have the same length, and the cross edges form a spanning cycle of G.*

Suppose that the vertices of R and S are $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ respectively. Since we are assuming $|E(M)| > 16$, it follows that $n \geq 5$. By Corollary 2.21, we may assume the following without loss of generality.

7.19.4. *r_1 is adjacent to both s_1 and s_n.*

Both s_1 and s_n cannot be adjacent to r_2, otherwise the cross edges contain the cycle $\{r_1s_1, s_1r_2, r_2s_n, s_nr_1\}$. Thus we will assume the following.

7.19.5. *s_n is not adjacent to r_2.*

Let f and g be the edges r_1r_2 and s_1s_2 respectively. First let us suppose that s_1 is adjacent to r_2. Let G' be $G/f/g$ and let M' be $M_B\backslash d/f/g$. Thus M' is a coextension of $M(G')$ by the element e.

Note that $G - \{r_1, s_1\}$ is a subgraph of G'. Furthermore, $G - \{r_1, s_1\} = (R - r_1, S - s_1)$ is a Truemper graph and both $R - r_1$ and $S - s_1$ contain at least four vertices. The cross edges of $G - \{r_1, s_1\}$ form a spanning path joining the terminal vertex r_2 to the terminal vertex s_n. From Lemma 7.14, we conclude that $G - \{r_1, s_1\}$, and hence G', contains distinct triangles, T_1, T_2, and T_3, such that at least two of these triangles are edge-disjoint.

Since T_1, T_2, and T_3 are triangles of $M_B/e\backslash d$, but M_B has no triangles, $T_1 \cup e$, $T_2 \cup e$, and $T_3 \cup e$ are circuits of M_B. Suppose that $T_1 \cup e$ is not a circuit in $M' = M_B\backslash d/f/g$. Then there is a circuit C of M_B such that $C - \{f, g\}$ is properly contained in $T_1 \cup e$, and $C \cap \{f, g\}$ is non-empty. Now $C - \{e, f, g\}$ is a dependent subset of T_1 in $M'/e = M(G')$. Since T_1 is a triangle of $M(G')$, this means that $C - \{e, f, g\} = T_1$. As $C - \{e, f, g\}$ is a proper subset of $T_1 \cup e$ it follows that $e \notin C$. Because every circuit of M_B has even cardinality, this means that either $C = T_1 \cup f$ or $C = T_1 \cup g$. By taking the symmetric difference of $T_1 \cup e$ with $T_1 \cup f$ or $T_1 \cup g$, we deduce that either $\{e, f\}$ or $\{e, g\}$ is a union of circuits in M_B. Since this is a contradiction, it follows that $T_1 \cup e$ is a circuit of M'. The same argument shows that $T_i \cup e$ is a circuit of M' for each i in $\{1, 2, 3\}$.

Lemma 7.15 asserts that G' is 3-connected. Clearly G' is loopless. Therefore $M(G')$ is connected. As T_1, T_2, and T_3 are triangles of $G - \{r_1, s_1\}$, and hence of G, Lemma 7.15 also implies that G'/T_i is 2-connected for all $i \in \{1, 2, 3\}$. There are no parallel edges in $G - \{r_1, s_1\}$, and therefore no loops in G'/T_i. Therefore $M(G'/T_i)$ is connected, so T_1, T_2, and T_3 are non-separating triangles of $M(G')$.

As d is a member of J, Theorem 6.1 states that $M_B\backslash d$ is almost-regular with $del = K$ and $con = J - d$. Both f and g are path edges of G, and are therefore in $J - e$, so $f, g \in con$. Thus $M' = M_B\backslash d/f/g$ is regular. Furthermore, G' is a 3-connected graph by Lemma 7.15, and M' is a single-element coextension of $M(G')$. It is not difficult to check that M' can be obtained from a 3-connected matroid M'' by a sequence of parallel or series extensions. Since M' has no R_{10}- or R_{12}-minor, Lemma 2.13 tells us that M'' is either graphic or cographic. Therefore M' is either graphic or cographic.

As T_1, T_2, and T_3 are non-separating triangles of $M(G') = M'/e$, and none of T_1, T_2, or T_3 is a triangle in M', Lemma 7.17 tells us that M' is not cographic. Therefore M' is graphic. Thus $M' = M(H)$ for some connected graph H, where e is an edge of H and $M(H/e) = M(G')$. Neither G' nor H/e has any isolated vertices, and G' is 3-connected by Lemma 7.15. It follows from Whitney's 2-isomorphism theorem (see [19, Theorem 5.3.1]) that $H/e = G'$.

Suppose that e is incident with vertices v_0 and v_1 in H, and let v be the vertex of $H/e = G'$ that results from identifying v_0 and v_1.

We will suppose that v has degree at most four. Since M_B is 3-connected having no triads and $M(G') = M_B\backslash d/e/f/g$, both v_0 and v_1 have degree three in H. Thus if T is the set of edges incident with v_0 in H, then $T \cup d$ is a cocircuit of M_B that contains d and e. As $\{d, e\}$ is contained in J, and both J and K are circuits of M_B, it follows that either $T - e \subseteq J$ or $T - e \subseteq K$. If $T - e \subseteq J$, then J contains the cocircuit $T \cup d$ and, as J is a cocircuit of

M_B, this means that $J = T \cup d$. This implies that $|E(M_B)| = 2|J| = 8$, a contradiction. Therefore $T - e \subseteq K$, so the two edges other than e that are incident with v_0 in H are members of K, implying that they are cross edges of G. The same argument shows that the two edges other than e that are incident with v_1 in H are cross edges of G. Thus v is incident with precisely four edges in G', and they are all cross edges of G. But no such vertex of G' exists, so we conclude that v has degree at least five in G'.

We may assume that r_2 and s_2 are vertices of G'. Then they are the only two vertices of degree at least five. Thus $v = r_2$ or $v = s_2$. Since $T_1 \cup e$, $T_2 \cup e$, and $T_3 \cup e$ are circuits in $M' = M(H)$, it follows that all of T_1, T_2, and T_3 are incident with v in $H/e = G'$, and hence in $G - \{r_1, s_1\}$. But r_2 and s_2 have degree at most three in $G - \{r_1, s_1\}$, so no pair of triangles in $\{T_1, T_2, T_3\}$ can be edge-disjoint, a contradiction.

This completes the argument in the case that s_1 is adjacent to r_2. The argument when s_1 is not adjacent to r_2 is very similar. Let G' be G/f and let M' be $M_B\backslash d/f$. Both $R - r_1$ and S contain at least four vertices, and $G - r_1 = (R - r_1, S)$ is a Truemper graph in which the cross edges form a spanning path joining two terminal vertices. Thus $G - r_1$, and hence G', contains distinct triangles T_1, T_2, and T_3, two of which are edge-disjoint. The sets $T_1 \cup e$, $T_2 \cup e$, and $T_3 \cup e$ are all circuits of M_B and of M'.

We observe that $M_B\backslash d$ is almost-regular with $\mathit{del} = K$ and $\mathit{con} = J - d$. Since $f \in J - d$, it follows that M' is regular. Hence M' is graphic or cographic. Since T_1, T_2, and T_3 are non-separating triangles of $M(G')$, it follows that M' is not cographic.

We now know that $M' = M(H)$ for some graph H, where $H/e = G'$. If v is the vertex of H/e formed by identifying the two end-vertices of e, then v must have degree at least five, so $v = r_2$. Thus T_1, T_2, and T_3 are incident with r_2 in $G - r_1$. However, r_2 has degree three in $G - r_1$, so no two of T_1, T_2, and T_3 are edge-disjoint, a contradiction.

This completes the proof of the lemma. □

8. Case-checking

The results in Section 7 mean that the proof of our main theorem is reduced to a finite case check. In this section we develop the tools required for such a check, and we prove our principal result. We start by deducing some information about representations of the binary matroid M_B.

Lemma 8.1. *Suppose that M is an excluded minor for $\mathcal{M}$ such that $|E(M)| \geq 10$, while $r(M), r^*(M) \geq 4$. Let $r = r(M)$, and let M_B be the rank-r binary matroid supplied by Theorem* 5.1*, so that M_B contains two disjoint circuit-hyperplanes, J and K. For all j in J and all k in K, there is a matrix $A(j,k)$ such that M_B is represented over* GF(2) *by the following matrix.*

$$\begin{array}{c|c|c|c} j & K-k & k & J-j \\ \left[\begin{array}{c} 1 \\ \hline \mathbf{0} \end{array}\right. & \begin{array}{c} \mathbf{0}^T \\ \hline I_{r-1} \end{array} & \begin{array}{c} 0 \\ \hline \mathbf{1} \end{array} & \left.\begin{array}{c} \mathbf{1}^T \\ \hline A(j,k) \end{array}\right] \end{array}$$

Proof. It is clear that $(K-k)\cup j$ is a basis of M_B. Moreover $(K-k)\cup k = K$ is a circuit, and no element of $J-j$ is spanned by $K-k$. The result follows. □

Before proving the next result, we give an alternative reduced representation of T_{12}. Suppose that the columns in the original representation in Figure 1 are labeled $1, 2, \ldots, 12$. It is easily checked that $\{5, 2, 10, 4, 6, 8\}$ is a basis. By considering fundamental circuits with respect to this basis, we see that if the columns of the representation $[I_6|A]$ are labeled $5, 2, 10, 4, 6, 8, 12, 11, 3, 9, 1, 7$, then A must be as follows.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Lemma 8.2. *T'_{12} is the unique 12-element excluded minor for $\mathcal{M}$.*

Proof. Let M be a 12-element excluded minor for $\mathcal{M}$. Then M is 3-connected, and $r(M), r^*(M) \geq 4$, by Lemmas 3.2 and 3.3. Theorem 5.1 implies that there is a binary matroid M_B having two complementary circuit-hyperplanes, J and K, such that M is obtained from M_B by relaxing J. Corollary 5.17 implies that $r(M_B) = r^*(M_B) = 6$.

We start by proving that M_B has no R_{10}-minor. Assume otherwise. By duality we can assume that there is an element $e \in E(M_B)$ such that $M_B\backslash e$ has an R_{10}-minor. Theorem 6.1 and Proposition 6.2 imply that $M_B\backslash e$ is an internally 4-connected almost-regular matroid. As $|E(M_B\backslash e)| = 11$ and $r(M_B\backslash e) = 6$, Lemma 7.4 implies that $M_B\backslash e \cong N_{11}^*$. Therefore $M_B\backslash e$ is represented by $[I_6|A]$, where A is the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Assume the columns of $[I_6|A]$ are labeled $1, \ldots, 11$. It is routine to check that $M_B\backslash e$ has a unique circuit-hyperplane, namely $\{1, 2, 7, 8, 9, 11\}$. Therefore the complement of this set in M_B, namely $\{3, 4, 5, 6, 10, e\}$ is a circuit-hyperplane. But this set properly contains $\{3, 4, 5, 10\}$, which is a circuit of M_B. Therefore M_B has no R_{10}-minor, as desired.

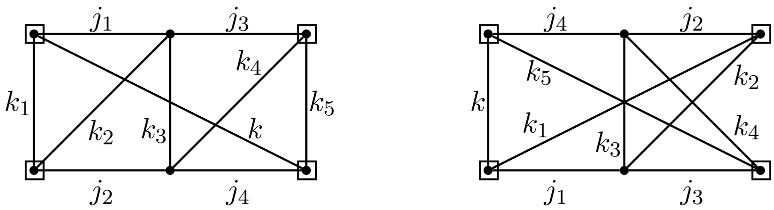


FIGURE 11. Two Truemper graphs.

Lemma 7.12 says that M_B has no R_{12}-minor. By using Proposition 7.18 and duality, we can assume there are distinct elements $e, d \in E(M_B)$ such that $M_B/e\backslash d$ is graphic. By Corollary 5.17 (vi), we assume that $e \in J$. As M_B/e is almost-regular and internally 4-connected, Lemma 7.13 says that it is isomorphic to a graft $M(G, D)$, where $G = (R, S)$ is a Truemper graph. As $(del, con) = (J - e, K)$ by Theorem 6.1, the cross edges of G comprise K, and therefore form a spanning cycle of G. Thus R and S both contain exactly three vertices. Since G has no XX-minor, we can assume by Corollary 2.21 that r_1 is adjacent to both s_1 and s_3. We enumerate the Truemper graphs having these properties, and we see that G must be one of the two (isomorphic) graphs in Figure 11. In either case, we let $j = e$, and we let k be the edge labeled as such in Figure 11. If the elements of $K - k$ and $J - j$ are ordered $k_1, \ldots, k_5$ and $j_1, \ldots, j_5$ respectively (where j_5 is the graft element d), then $A(j, k)$ is the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus M_B is isomorphic to T_{12}, so M is isomorphic to T'_{12}, as desired. □

Lemma 8.3. *There is no* 16*-element excluded minor for* $\mathcal{M}$.

Proof. Suppose that M is a 16-element excluded minor for $\mathcal{M}$, and that M_B is the binary matroid appearing in Theorem 5.1. Recall that $\mathrm{AG}(3, 2)$ has the following reduced representation.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

We will deduce that M_B has a minor isomorphic to $\mathrm{AG}(3, 2)$. Since every proper minor of M_B is either regular or almost-regular (Proposition 2.16 and Theorem 6.1), and $\mathrm{AG}(3, 2)$ is neither, this will yield a contradiction.

Let J and K be the complementary circuit-hyperplanes of M_B. Now M_B has no R_{10}- or R_{12}-minor, by Lemmas 7.5 and 7.12. As in the proof of

Lemma 8.2, we deduce that, up to duality, there are elements e and d in M_B such that $M_B/e\backslash d$ is graphic, and $M_B/e \cong M(G, D)$. Here $G = (R, S)$ is a Truemper graph, the cross edges of G form a spanning path, and both R and S have exactly four vertices. We assume that $e \in J$. We also assume that r_1 is adjacent to both s_1 and s_4. The twelve Truemper graphs satisfying these constraints are enumerated in Figure 12 (we ignore symmetries).

Four of these Truemper graphs have XX-minors, and so can be disregarded. In the remaining cases, we assume that $j = e$. One of the edges in G is labeled by k. We also assume that the elements of $J - j$ and $K - k$ are ordered $j_1, \ldots j_7$ and $k_1, \ldots, k_7$ respectively (where j_7 is the graft element d). Now it easy to see that $A(j, k)$ is one of the following three matrices.

$$\begin{bmatrix} 1&0&0&0&0&0&1\\ 1&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0&0&1\\ 1&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&1&1&1&0&1\\ 0&0&1&0&1&1&1\\ 0&0&0&0&0&1&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0&0&1\\ 1&1&0&0&0&0&0\\ 1&1&1&0&0&0&1\\ 1&0&1&1&0&0&1\\ 0&0&1&1&1&0&1\\ 0&0&1&0&1&1&1\\ 0&0&0&0&0&1&1 \end{bmatrix}$$

In each of these three cases, we demonstrate that M_B has an AG$(3, 2)$-minor. Recall that M_B has the following reduced representation

$$A = \left[\begin{array}{c|c} 0 & \mathbf{1}^T \\ \hline \mathbf{1} & A(j,k) \end{array}\right],$$

where the columns of $[I_8|A]$ are labeled $j, k_1, \ldots, k_7, k, j_1, \ldots, j_7$. Suppose that $A(j, k)$ is equal to the first of the three matrices above. Then it is straighforward to confirm that

$$M_B/\{k, k_2, k_5, k_7\}\backslash\{j, j_3, j_6, j_7\} \cong \text{AG}(3, 2).$$

Similarly, if $A(j, k)$ is the second displayed matrix, then

$$M_B/\{k_2, k_3, k_4, k_6\}\backslash\{j, j_2, j_4, j_5\} \cong \text{AG}(3, 2)$$

and if $A(j, k)$ is the third displayed matrix, then

$$M_B/\{k_1, k_2, k_4, k_6\}\backslash\{j, j_2, j_5, j_7\} \cong \text{AG}(3, 2).$$

This completes the proof. □

We are now ready to prove our main theorem, which we restate here.

Theorem 1.1. *The excluded minors for the class of matroids that are binary or ternary are* $U_{2,5}$, $U_{3,5}$, $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, $U_{2,4} \oplus_2 F_7^*$, AG$(3, 2)'$, *and* T_{12}'.

Proof. Let M be an excluded minor for $\mathcal{M}$. If M is not 3-connected, or if the rank or corank of M is less than four, then M is isomorphic to one of $U_{2,5}$, $U_{3,5}$, $U_{2,4} \oplus F_7$, $U_{2,4} \oplus F_7^*$, $U_{2,4} \oplus_2 F_7$, or $U_{2,4} \oplus_2 F_7^*$, by Lemmas 3.2 and 3.3. Thus we assume that M is 3-connected, and that $r(M), r^*(M) \geq 4$. Hence

$|E(M)| \geq 8$. If $|E(M)| = 8$ then $M \cong \mathrm{AG}(3,2)'$, by Lemma 4.1. Thus we assume that $|E(M)| \geq 9$. This implies that $|E(M)| \geq 10$, by Lemma 4.6.

Now we apply Theorem 5.1 to deduce the existence of a binary matroid M_B such that M is obtained from M_B by relaxing a circuit-hyperplane. Lemma 7.12 says that M_B has no R_{12}-minor. If M_B has an R_{10}-minor, then $|E(M)| = 12$, by Lemma 7.5. On the other hand, if M_B has no R_{10}-minor, then $|E(M)| \leq 16$, by Lemma 7.19. Therefore we have established that $|E(M)| \leq 16$. Corollary 5.17 implies that we need only consider the case that $|E(M)| = 12$ or 16. If $|E(M)| = 12$ then $M \cong T_{12}'$, by Lemma 8.2, and Lemma 8.3 implies that $|E(M)| \neq 16$. Therefore the proof is complete. □

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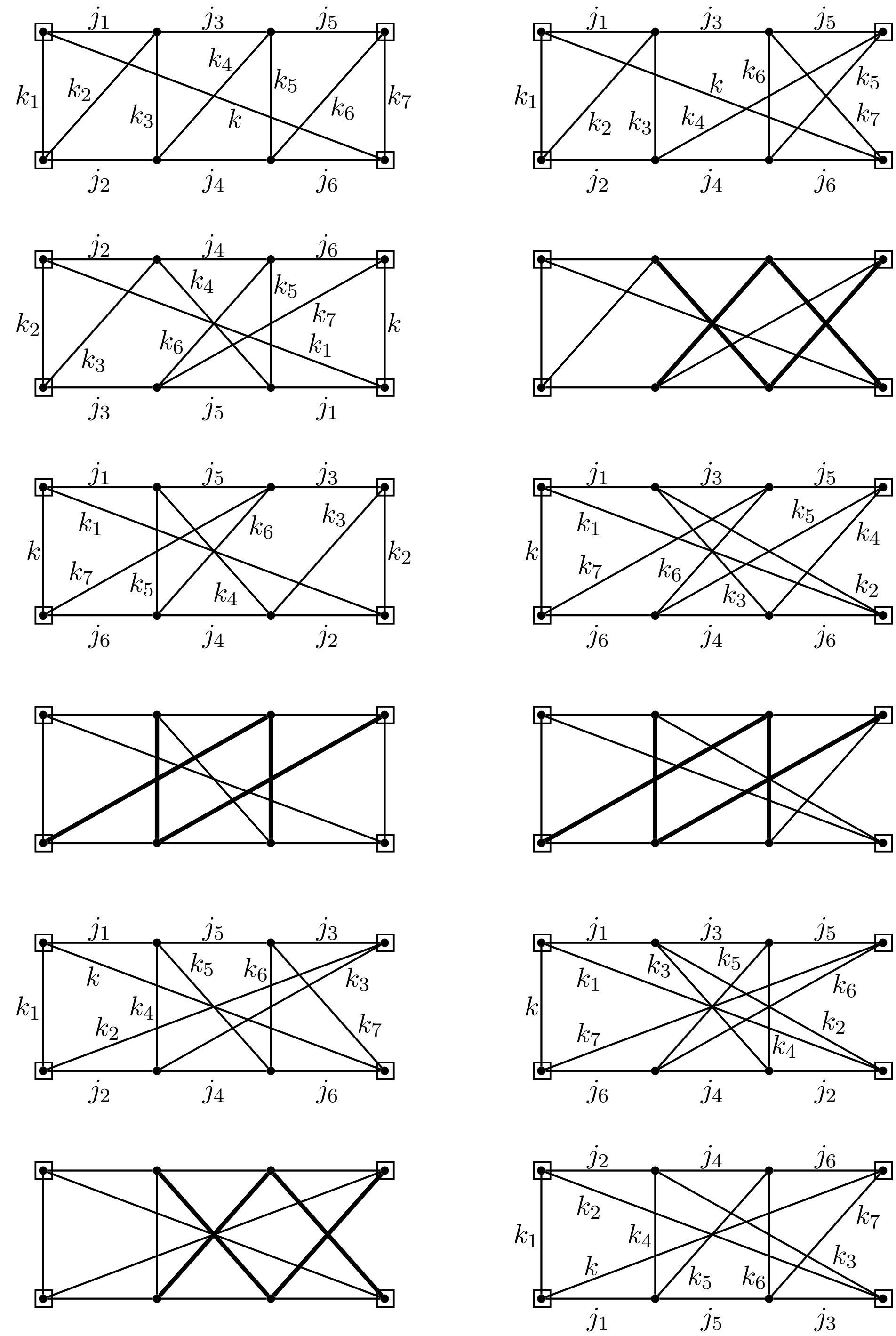

FIGURE 12. Twelve Truemper graphs.