ON THE ASYMPTOTIC PROPORTION OF CONNECTED MATROIDS

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Abstract. Very little is known about the asymptotic behaviour of classes of matroids. We make a number of conjectures about such behaviours. For example, we conjecture that asymptotically almost every matroid: has a trivial automorphism group; is arbitrarily highly connected; and, is not representable over any field. We prove one result: The proportion of labeled n-element matroids that are connected is asymptotically at least 1/2.

1. Introduction

In their monograph on combinatorial geometries, Brylawski and Kelly make the following intriguing remark [3, p. 89]:

> "It is an exercise in random matroids to show that most matroids are not coordinatizable over any field (or even any division ring)."

To the best of our knowledge, this exercise has yet to be successfully completed. Indeed, there are almost no results on the asymptotic behaviour of classes of matroids. This seems to be due to the lack of a successful model of a random matroid (although random subsets of projective spaces have been studied by Oxley and Kelly [5, 6] and by Kordecki [7, 8]). Even the most elementary questions about the properties of "almost all" matroids are currently unanswered.

In this introduction we collate some of those questions. The remainder of the article is dedicated to an investigation of the proportion of labeled n-element matroids that are connected. In particular, for a positive integer n, let $l(n)$ be the number of matroids on the ground set $\{1, \ldots, n\}$, so that $l(n)$ is the number of labeled n-element matroids. We prove the following:

Theorem 1.1. *For a positive integer n, let $l_c(n)$ be the number of connected matroids on the ground set $\{1, \ldots, n\}$. For every $\epsilon > 0$, there exists an integer N such that*

$$\frac{l_c(n)}{l(n)} \geq \frac{1}{2} - \epsilon$$

whenever $n \geq N$.

The first author was supported by a FRST Science & Technology post-doctoral fellowship. The second and fourth authors were supported by the Marsden Fund of New Zealand.

Thus we have shown that $\lim_{n\to\infty} l_c(n)/l(n) \geq 1/2$, provided that the limit exists. We strongly believe that $\lim_{n\to\infty} l_c(n)/l(n) = 1$. However, for the moment this remains unproved.

In the following conjectures, we make no distinction between labeled and unlabeled matroids. This is because we believe that asymptotically almost every matroid is *asymmetric* (that is, has no non-trivial automorphism). To make this more precise, we let $u(n)$ be the number of unlabeled n-element matroids, and $A_u(n)$ be the set of such matroids with trivial automorphism groups. We believe that the following statement holds.

Conjecture 1.2. The limit $\lim_{n\to\infty} |A_u(n)|/u(n)$ exists and is equal to one.

A matroid *property* is a class of matroids that is closed under isomorphism. Let $\mathcal{P}$ be a matroid property. Then $\mathcal{P}_u(n)$ and $\mathcal{P}_l(n)$ denote, respectively, the set of unlabeled and labeled n-element matroids with property $\mathcal{P}$.

Lemma 1.3. *Assume that Conjecture* 1.2 *holds. Let* $\mathcal{P}$ *be a matroid property. Then*

$$\lim_{n\to\infty} \frac{|\mathcal{P}_u(n)|}{u(n)}$$

exists and is equal to L *if and only if*

$$\lim_{n\to\infty} \frac{|\mathcal{P}_l(n)|}{l(n)}$$

exists and is equal to L.

Thus, if we assume that Conjecture 1.2 holds, then our conjectures about asymptotic behaviour coincide for labeled and unlabeled matroids. This phenomenon has also been noted in the context of graph theory (see [1, p. 1462]). Because the proof of Lemma 1.3 is not central to our arguments, we relegate it to an appendix.

In what follows we assume Conjecture 1.2. The statement that *asymptotically almost every* matroid has property $\mathcal{P}$ means that both $|\mathcal{P}_u(n)|/u(n)$ and $|\mathcal{P}_l(n)|/l(n)$ tend to one as n tends to infinity.

Conjecture 1.4. Asymptotically almost every matroid is connected.

Recall that a matroid M on the ground set E is k-*connected* if and only if there is no partition (X, Y) of E such that $|X|, |Y| \geq k'$, and $r(X) + r(Y) - r(M) < k'$, for some $k' < k$.

Conjecture 1.5. For any fixed integer $k > 1$, asymptotically almost every matroid is k-connected.

Recall that a rank-r matroid is *paving* if every circuit contains at least r elements. Welsh [12] was prompted by the catalogue of matroids produced by Blackburn, Crapo, and Higgs [2] to ask whether "most" matroids are paving. We conjecture that this is true in a strong sense:

Conjecture 1.6. Asymptotically almost every matroid is paving.

It is an easy exercise to prove that, if asymptotically almost every matroid belongs to the class $\mathcal{P}$, then asymptotically almost every matroid belongs to $\mathcal{P} \cap \{M^* \mid M \in \mathcal{P}\}$. A matroid M is *sparse* paving if both M and M^* are paving. Therefore Conjecture 1.6 implies that asymptotically almost every matroid is sparse paving. We conjecture that any fixed sparse paving matroid is present as a minor in asymptotically almost every matroid.

Conjecture 1.7. Let N be a fixed sparse paving matroid. Asymptotically almost every matroid has an N-minor.

Recall that the *Vámos matroid*, V_8, is a self-dual rank-4 paving matroid [11] (see [9, Example 2.1.22]). The next conjecture would be implied by a positive answer to Conjecture 1.7, and is perhaps more approachable.

Conjecture 1.8. Asymptotically almost every matroid has a V_8-minor.

A simple counting argument shows that if $\mathbb{F}$ is a finite field, then asymptotically almost every matroid is not representable over $\mathbb{F}$. It is a relatively straightforward exercise to show that a result due to Ronyai, Babai, and Ganapathy [10] implies that this phenomenon holds for any fixed field $\mathbb{F}$. We conjecture something stronger:

Conjecture 1.9. Asymptotically almost every matroid is not representable over any field.

Since V_8 is not representable over any field, a positive answer to Conjecture 1.8 would imply Conjecture 1.9

Welsh asked whether the number of non-isomorphic n-element matroids with rank r is maximum when $r = \lfloor n/2 \rfloor$ (see [12, P20]). We make a stronger conjecture.

Conjecture 1.10. Asymptotically almost every matroid M satisfies

$$\frac{|E(M)| - 1}{2} \leq r(M) \leq \frac{|E(M)| + 1}{2}.$$

2. Loopless and coloopless matroids predominate

We now turn to the proof of Theorem 1.1. Our strategy is to prove that connected matroids make up at least half of the set of loopless and coloopless labeled matroids on n elements. To show that this implies Theorem 1.1, we must establish that loopless and coloopless matroids asymptotically predominate in the set of labeled matroids. This section is devoted to that task.

Let M be a matroid. Recall that a *modular cut* of M is a collection, $\mathcal{F}$, of flats of M such that:

(i) if $F_0 \in \mathcal{F}$ and F_1 is a flat containing F_0, then $F_1 \in \mathcal{F}$; and,
(ii) if $F_0, F_1 \in \mathcal{F}$, and $r(F_0) + r(F_1) = r(F_0 \cap F_1) + r(F_0 \cup F_1)$, then $F_0 \cap F_1 \in \mathcal{F}$.

A *single-element extension* of M is a matroid M_0 on the ground set $E(M) \cup e$, where $e \notin E(M)$, such that $M_0 \backslash e = M$. It is well-known that the single-element extensions of M are in bijective correspondence with the modular cuts of M [9, Section 7.2]. If F is a flat of M, then the set of flats that contain F is a modular cut of M. The single-element extension that corresponds to this modular cut is said to be a *principal extension*.

Proposition 2.1. *Let M be a rank-r matroid. Then M has at least 2^r flats.*

Proof. Simply take any basis of M and form the closures of every subset of that basis. □

Proposition 2.2. *Let $n \geq 2$ be an integer. Then $l(n) \geq 2^{(n-3)/2} l(n-1)$.*

Proof. Consider the $l(n-1)$ matroids on the ground set $\{1, \ldots, n-1\}$. By duality, at least $l(n-1)/2$ of these have rank no less than $(n-1)/2$. Each of these matroids has at least $2^{(n-1)/2}$ flats, by Proposition 2.1. We construct the principal extensions on $\{1, \ldots, n\}$ corresponding to these flats. These extensions are all distinct, as two distinct matroids on the set $\{1, \ldots, n-1\}$ cannot have identical single-element extensions on the set $\{1, \ldots, n\}$. Thus there are at least $2^{(n-1)/2} l(n-1)/2$ distinct matroids on the ground set $\{1, \ldots, n\}$. The result follows. □

Theorem 2.3. *For a positive integer n, let $l^{\circ}(n)$ be the number of matroids on the ground set $\{1, \ldots, n\}$ that have at least one loop or coloop. If $n \geq 2$, then*

$$\frac{l^{\circ}(n)}{l(n)} \leq \frac{n}{2^{(n-5)/2}},$$

and hence $l^{\circ}(n)/l(n) \to 0$ as $n \to \infty$.

Proof. Suppose that i is an integer in $\{1, \ldots, n\}$, and that M is a matroid on $\{1, \ldots, n\}$ in which i is a loop. Let $l_i^{\circ}(n)$ be the number of such matroids. We claim that $l_i^{\circ}(n) = l(n-1)$. Consider the function which takes each such M to the matroid obtained from M by deleting i and relabeling every element $j \in \{i+1, \ldots, n\}$ with $j-1$. This is clearly a bijection between the matroids on $\{1, \ldots, n\}$ in which i is a loop, and the matroids on the set $\{1, \ldots, n-1\}$. Hence we have established the claim.

Exactly the same argument shows that $l(n-1)$ is the number of matroids on $\{1, \ldots, n\}$ in which i is a coloop. By taking the sum as i ranges over $\{1, \ldots, n\}$, we see that $l^{\circ}(n)$ is at most $2nl(n-1)$. Now Proposition 2.2 shows that

$$\frac{l^{\circ}(n)}{l(n)} \leq \frac{2nl(n-1)}{2^{(n-3)/2} l(n-1)} = \frac{n}{2^{(n-5)/2}}.$$ □

3. The loopless and coloopless case

In this section we show that connected matroids make up at least half of the loopless and coloopless matroids on $\{1, \ldots, n\}$. As a first step, we partition the disconnected but loopless and coloopless matroids on $\{1,\ldots,n\}$ into two classes.

Let $\mathcal{M}_1$ be the set of loopless and coloopless disconnected matroids on $\{1,\ldots,n\}$ that either have at least three connected components, or that have no connected component of rank one. Let $\mathcal{M}_2$ be the set of loopless and coloopless disconnected matroids on $\{1,\ldots,n\}$ that have precisely two connected components, at least one of which has rank one.

Suppose that M is a matroid with rank at least one. Let $\mathcal{I}(M)$ be the family of independent sets of M. The *truncation* of M, denoted by $T(M)$, is the matroid on the ground set $E(M)$ with $\{I \in \mathcal{I}(M) : |I| \leq r(M) - 1\}$ as its independent sets.

Proposition 3.1. *Let M be a loopless matroid with rank at least two. Then $T(M)$ is connected. In particular, if $M \in \mathcal{M}_1$ then $T(M)$ is connected.*

Proof. Let e and f be elements of $E(M)$. Since M has no loops, either $\{e, f\}$ is a circuit, or it is independent. If $\{e, f\}$ is a circuit of M, then it is a circuit of $T(M)$, as $r(M) \geq 2$. If $\{e, f\}$ is independent in M, then it is contained in a basis B of M. The set B is a circuit of $T(M)$. Thus e and f are contained in a common circuit of $T(M)$ in either case. Therefore $T(M)$ is connected. As each member of $\mathcal{M}_1$ is loopless and has rank at least two, the result follows. □

Lemma 3.2. *Let M_1 and M_2 be members of $\mathcal{M}_1$. If $T(M_1) = T(M_2)$, then $M_1 = M_2$.*

Proof. Suppose that $T(M_1) = T(M_2)$. We will show that M_1 and M_2 have exactly the same set of circuits. Note that M_1 and M_2 must have the same rank. Let r be this common rank. As M_1 and M_2 both belong to $\mathcal{M}_1$, it follows that each connected component of M_i has rank at most $r-2$, for $i = 1, 2$. Thus every circuit of M_i has rank at most $r-2$, and is therefore a non-spanning circuit of $T(M_i)$. On the other hand, a non-spanning circuit of $T(M_i)$ is also a circuit of M_i.

The previous paragraph establishes that the circuits of M_i are precisely the non-spanning circuits of $T(M_i)$. Since $T(M_1) = T(M_2)$, it follows that $T(M_1)$ and $T(M_2)$ have the same set of non-spanning circuits, and hence M_1 and M_2 have the same set of circuits. □

Suppose that M_1 and M_2 are matroids on the ground set E and that $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$ are their families of independent sets respectively. Recall that the *union* of M_1 and M_2, denoted $M_1 \vee M_2$, is the matroid on the ground set E, with $\{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1),\, I_2 \in \mathcal{I}(M_2)\}$ as its family of independent sets.

Next we assume that M is a member of $\mathcal{M}_2$. Thus M has no loops or coloops, and M has precisely two connected components, N_1 and N_2. Let E, E_1, and E_2 be the ground sets of M, N_1 and N_2 respectively. By relabeling as necessary, we assume that N_1 is the connected component of M with rank one. (Recall that $E = \{1, \ldots, n\}$. If both N_1 and N_2 have rank one, we assume that N_1 is the component containing the element 1.)

Let N_1' be the rank-one uniform matroid on the ground set E. Let N_2' be obtained from N_2 by adding the elements of E_1 as loops. Thus N_1' and N_2' are both matroids on the ground set E. We define $\Psi(M)$ to be $N_1' \vee N_2'$. We remark here that $\Psi(M)$ can also be defined using the *free product* of Crapo and Schmidt [4]. Alternatively, $\Psi(M)$ can be obtained by freely coextending N_2 by an element $e \in E_1$, and then adding $E_1 - e$ in parallel to e.

Proposition 3.3. *Suppose that C is a circuit of N_2 and that e is an element in E_1. Then $C \cup e$ is a circuit of $\Psi(M)$. Conversely, if C' is a circuit of $\Psi(M)$ that contains e, then $C' - e$ is a circuit of N_2.*

Proof. Suppose that $C \cup e$ is independent in $\Psi(M)$. Then $C \cup e$ is the disjoint union of I_1 and I_2, where I_i is independent in N_i' for $i = 1, 2$. Note that I_2 cannot contain any element of E_1, for any such element is a loop of N_2'. Thus I_2 must be a proper subset of C. It follows that I_1 contains e and at least one element of C. But such a pair is a circuit of N_1'. Thus we have a contradiction and we conclude that $C \cup e$ is dependent in $\Psi(M)$.

However, if $x \in C$, then $C - x$ is independent in N_2' and both $\{e\}$ and $\{x\}$ are independent in N_1'. It follows easily that $C \cup e$ is indeed a circuit of $\Psi(M)$.

To prove the converse, we note that if $C' - e$ were independent in N_2, then C' would be independent in $\Psi(M)$. Thus $C' - e$ contains a circuit of N_2. The arguments in the previous paragraphs show that $C' - e$ must in fact be a circuit of N_2, for otherwise C' properly contains a circuit of $\Psi(M)$. $\square$

Proposition 3.4. *There are no loops in $\Psi(M)$, and E_1 is the unique non-trivial parallel class of $\Psi(M)$.*

Proof. Let e be a member of E. Then $\{e\}$ is independent in N_1', and hence in $\Psi(M)$. Thus $\Psi(M)$ has no loops.

Suppose that e and f are contained in E_1 and that $\{e, f\}$ is independent in $\Psi(M)$. Then $\{e, f\}$ is the disjoint union of I_1 and I_2, independent sets of N_1' and N_2' respectively. Note that I_2 is non-empty, as $\{e, f\}$ is dependent in N_1', so I_2 contains either e or f. But both these elements are loops of N_2', so we have a contradiction. Thus $\{e, f\}$ is a circuit of $\Psi(M)$. Suppose that $x \in E_2$. If $e \in E_1$, then $\{e\}$ is independent in N_1', and $\{x\}$ is independent in N_2'. Therefore $\{e, x\}$ is independent in $\Psi(M)$, so E_1 is indeed a parallel class of $\Psi(M)$.

Suppose that P is a non-trivial parallel class of $\Psi(M)$ other than E_1, and let $\{x, y\}$ be a parallel pair in P. Then $\{x, y\}$ must be dependent in N_2, and as N_2 is loopless, $\{x, y\}$ is a parallel pair of N_2. Now Proposition 3.3 implies that $\{x, y, e\}$ is a circuit of $\Psi(M)$, where e is any member of E_1. Since this circuit properly contains $\{x, y\}$ we have a contradiction and deduce that E_1 is the only non-trivial parallel class in $\Psi(M)$. $\square$

Proposition 3.5. *Let M be a member of $\mathcal{M}_2$. Then $\Psi(M)$ is connected.*

Proof. It suffices to show that if $e \in E_1$ and $f \in E_2$, then there is a circuit of $\Psi(M)$ that contains $\{e, f\}$. As M has no coloops, there is a circuit C of N_2 that contains f. Proposition 3.3 says that $C \cup e$ is a circuit of $\Psi(M)$, so we are done. □

Lemma 3.6. *The function that takes each member M of $\mathcal{M}_2$ to $\Psi(M)$ is injective.*

Proof. We will prove this fact by demonstrating that M can be canonically recovered from $\Psi(M)$. Proposition 3.4 says that E_1 is the unique non-trivial parallel class of $\Psi(M)$. Suppose that $e \in E_1$. If C is a circuit of N_2, then $C \cup e$ is a circuit of $\Psi(M)$, so C is a circuit of $\Psi(M)/e$. Suppose that $C \subseteq E_2$ is a circuit of $\Psi(M)/e$. If C is a circuit of $\Psi(M)$, then C must be a dependent set of N_2. Let C' be a circuit of N_2 that is contained in C. Then $C' \cup e$ is a circuit of $\Psi(M)$, so C' must be properly contained in C. But then C' is a circuit of $\Psi(M)/e$ that is properly contained in C. This contradiction means that $C \cup e$ is a circuit of $\Psi(M)$. Now Proposition 3.3 asserts that C is a circuit of N_2.

We have shown that the matroid obtained from $\Psi(M)/e$ by deleting $E_1 - e$ is equal to N_2. Thus M can be recovered from $\Psi(M)$ by contracting any element from its unique parallel class, deleting the resulting loops, and adding the unique parallel class of $\Psi(M)$ to the resulting matroid as a connected component. This completes the proof. □

Proposition 3.7. *Suppose that M is a member of $\mathcal{M}_2$ on the ground set E. If $f \in E$ is contained in a non-spanning circuit of $\Psi(M)$, then there is a non-spanning circuit of $\Psi(M)$ that contains f and an element from the unique non-trivial parallel class of $\Psi(M)$.*

Proof. Suppose that f is contained in a non-spanning circuit C of $\Psi(M)$. If $f \in E_1$ then the result is obvious, as E_1 is a non-trivial parallel class of $\Psi(M)$. Therefore we suppose that $f \in E_2$. Assume that there is no non-spanning circuit of $\Psi(M)$ that both contains f and meets E_1. In particular, C contains no element of E_1.

Let e be an element of E_1. The set C is not a circuit of N_2, for otherwise Proposition 3.3 implies that $C \cup e$ is a circuit of $\Psi(M)$ that properly contains C. However C must be dependent in N_2, so C properly contains at least one circuit of N_2. Suppose that C contains two distinct circuits C_1 and C_2 of N_2. Then $C_1 \cup e$ and $C_2 \cup e$ are circuits of $\Psi(M)$ by Proposition 3.3. As $|C_1 \cup e| \leq |C|$ and $|C_2 \cup e| \leq |C|$, it follows that $C_1 \cup e$ and $C_2 \cup e$ are non-spanning circuits of $\Psi(M)$. Thus f is contained in neither C_1 nor C_2. Now $((C_1 \cup e) \cup (C_2 \cup e)) - e$ contains a circuit of $\Psi(M)$, by circuit-exchange, and this circuit must be C. Thus $C_1 \cup C_2 = C$. But this is a contradiction, as $f \notin C_1 \cup C_2$. Therefore C contains precisely one circuit C_1 of N_2.

Let x be an element of C_1. Then $C - x$ is independent in N_2'. However $\{x\}$ is independent in N_1', so C is independent in $\Psi(M)$. This contradiction completes the proof. □

Lemma 3.8. *Suppose that $M_1 \in \mathcal{M}_1$ and that $M_2 \in \mathcal{M}_2$. Then $T(M_1) \neq \Psi(M_2)$.*

Proof. Suppose that $T(M_1) = \Psi(M_2)$. Proposition 3.4 implies that $T(M_1)$ contains a unique non-trivial parallel class E_1. As the members of $\mathcal{M}_1$ have rank at least three, E_1 is a parallel class of M_1. Let f be an element from a connected component of M_1 that does not contain E_1. Now f is contained in a circuit of M_1. We remarked in the proof of Lemma 3.2 that the circuits of M_1 are the non-spanning circuits of $T(M_1)$. Therefore f is in a non-spanning circuit of $T(M_1)$, and by Proposition 3.7 must be contained in a non-spanning circuit of $T(M_1)$ that also contains a member of E_1. This means that f is contained in a circuit of M_1 that meets E_1, which is a contradiction. □

Recall that $l_c(n)$ is the number of connected matroids on the ground set $\{1, \ldots, n\}$.

Lemma 3.9. *Let n be a positive integer, and let $l^{\o}(n)$ be the number of loopless and coloopless matroids on the ground set $\{1, \ldots, n\}$. Then $l_c(n) \geq l^{\o}(n)/2$.*

Proof. We note that $l^{\o}(1) = 0$, so the result is true if $n = 1$. Henceforth we assume that $n > 1$. Let M be a loopless and coloopless matroid that is not connected, and consider the function that takes each such M to $T(M)$ if $M \in \mathcal{M}_1$ and to $\Psi(M)$ if $M \in \mathcal{M}_2$. The image of M is connected by Propositions 3.1 and 3.5. Moreover this function is injective by Lemmas 3.2, 3.6, and 3.8. Therefore the number of connected matroids on $\{1, \ldots, n\}$ is at least as large as the number of disconnected but loopless and coloopless matroids on $\{1, \ldots, n\}$. Since the sum of these two numbers is $l^{\o}(n)$ the result follows. □

Now we can prove our main result. We recall that $l(n)$, $l_c(n)$, $l^{\circ}(n)$, and $l^{\o}(n)$ are, respectively, the number of: matroids; connected matroids; matroids with at least one loop or coloop; and loopless and coloopless matroids, on the ground set $\{1, \ldots, n\}$.

Theorem 1.1. *For every $\epsilon > 0$, there exists an integer N such that $l_c(n)/l(n) \geq 1/2 - \epsilon$ whenever $n \geq N$.*

Proof. We start by observing that

$$\begin{aligned} \frac{l(n)}{l^{\o}(n)} &= \frac{l(n)}{l(n) - l^{\circ}(n)} \\ &= \frac{1}{1 - l^{\circ}(n)/l(n)} \end{aligned}$$

so Theorem 2.3 implies that $l(n)/l^{\varnothing}(n)$ tends to 1 from above as $n \to \infty$. Now, by applying Lemma 3.9, we see that

$$\begin{aligned}\frac{l_c(n)}{l(n)} &= \frac{l_c(n)/l^{\varnothing}(n)}{l(n)/l^{\varnothing}(n)} \\ &\geq \frac{1/2}{l(n)/l^{\varnothing}(n)}\end{aligned}$$

so the result follows. □

4. Appendix

In this section we prove Lemma 1.3. Recall that $l(n)$ and $u(n)$ stand respectively for the number of labeled and unlabeled n-element matroids. Moreover $A_l(n)$ and $A_u(n)$ are respectively the number of labeled and unlabeled n-element matroids with no automorphism other than the trivial one. We shall use $\overline{A}_l(n)$ and $\overline{A}_u(n)$ to denote the sets of labeled and unlabeled n-element matroids with at least one non-trivial automorphism.

Throughout this section we assume that Conjecture 1.2 holds. In other words, we use the following hypothesis.

Hypothesis 1. The limit $\lim_{n\to\infty} |A_u(n)|/u(n)$ exists and is equal to one.

We will make frequent use of the following fact:

Fact 1. *Let $\{X_n\}_{n\geq 1}$ be a sequence of sets. For every positive integer n, let Y_n and Z_n be subsets of X_n. Suppose that $\lim_{n\to\infty} |Y_n|/|X_n|$ and $\lim_{n\to\infty} |Z_n|/|X_n|$ exist, and are equal to L and 1 respectively. Then*

$$\lim_{n\to\infty} \frac{|Y_n \cap Z_n|}{|X_n|} = L.$$

Proof. If $\lim_{n\to\infty} |Z_n|/|X_n| = 1$, then $\lim_{n\to\infty} |Y_n \cup Z_n|/|X_n| = 1$, so writing $|Y_n \cap Z_n|/|X_n|$ as

$$\frac{|Y_n|}{|X_n|} + \frac{|Z_n|}{|X_n|} - \frac{|Y_n \cup Z_n|}{|X_n|}$$

and taking the limit as n tends to infinity gives the result. □

Fact 2.

$$\lim_{n\to\infty} \frac{l(n)}{n!u(n)} = 1.$$

Proof. The number of labeled matroids associated with the unlabeled n-element matroid M is $n!/|\mathrm{Aut}(M)|$, where $\mathrm{Aut}(M)$ is the automorphism group of M. Let $\mathcal{M}_n$ be the set of unlabeled n-element matroids. Then

$$l(n) = \sum_{M\in\mathcal{M}_n} \frac{n!}{|\mathrm{Aut}(M)|}.$$

On the other hand,

$$n!u(n) = \sum_{M\in\mathcal{M}_n} n!,$$

so $l(n) \leq n!u(n)$, and thus $l(n)/n!u(n) \leq 1$. Now

$$l(n) \geq \sum_{M \in A_u(n)} \frac{n!}{|\mathrm{Aut}(M)|} = n!|A_u(n)|.$$

Therefore

$$1 \geq \frac{l(n)}{n!u(n)} \geq \frac{n!|A_u(n)|}{n!u(n)} = \frac{|A_u(n)|}{u(n)}.$$

The last expression tends to one as $n \to \infty$, by our hypothesis, so the result follows. □

The last fact implies that $\lim_{n\to\infty} n!u(n)/l(n) = 1$.

Fact 3. *The limit* $\lim_{n\to\infty} |A_l(n)|/l(n)$ *exists and is equal to one.*

Proof. Note that both $\lim_{n\to\infty} |A_u(n)|/u(n)$ and $\lim_{n\to\infty} n!u(n)/l(n)$ exist and are equal to one. Therefore

$$\begin{aligned}
1 &= \lim_{n\to\infty} \frac{n!|A_u(n)|}{n!u(n)} \cdot \lim_{n\to\infty} \frac{n!u(n)}{l(n)} \\
&= \lim_{n\to\infty} \frac{n!|A_u(n)|}{n!u(n)} \cdot \frac{n!u(n)}{l(n)} \\
&= \lim_{n\to\infty} \frac{n!|A_u(n)|}{l(n)} \\
&= \lim_{n\to\infty} \frac{|A_l(n)|}{l(n)}
\end{aligned}$$

□

Proof of Lemma 1.3. Suppose that $\lim_{n\to\infty} |\mathcal{P}_u(n)|/u(n) = L$. Then

$$\lim_{n\to\infty} \frac{|\mathcal{P}_u(n) \cap A_u(n)|}{u(n)} = L,$$

by Hypothesis 1 and Fact 1. Now

$$\frac{|\mathcal{P}_l(n) \cap A_l(n)|}{l(n)} = \left(\frac{n!|\mathcal{P}_u(n) \cap A_u(n)|}{n!u(n)}\right)\left(\frac{n!u(n)}{l(n)}\right).$$

The limits, as $n \to \infty$, of the bracketed expressions exist, and are equal to L and 1 respectively. Therefore

$$\lim_{n\to\infty} \frac{|\mathcal{P}_l(n) \cap A_l(n)|}{l(n)} = L.$$

But

$$\frac{|\mathcal{P}_l(n)|}{l(n)} = \frac{|\mathcal{P}_l(n) \cap A_l(n)|}{l(n)} + \frac{|\mathcal{P}_l(n) \cap \overline{A}_l(n)|}{l(n)}$$

and it follows easily from Fact 3 that $|\overline{A}_l(n)|/l(n)$, and hence $|\mathcal{P}_l(n) \cap \overline{A}_l(n)|/l(n)$, tends to zero as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{|\mathcal{P}_l(n)|}{l(n)} = \lim_{n\to\infty} \frac{|\mathcal{P}_l(n) \cap A_l(n)|}{l(n)} = L,$$

as desired. The proof of the converse is similar. □

5. Acknowledgements

We thank Noga Alon, Douglas Kelly, Steve Noble, and James Oxley for helpful discussions.

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