ON MAINTAINING MATROID 3-CONNECTIVITY RELATIVE TO A FIXED BASIS

GEOFF WHITTLE AND ALAN WILLIAMS

Abstract. We show that for any 3-connected matroid M on a ground set of at least four elements such that M does not contain any 4-element fans, and any basis B of M, there exists a set $K \subseteq E(M)$ of four distinct elements such that for all $k \in K$, $\mathrm{si}(M/k)$ is 3-connected whenever $k \in B$, and $\mathrm{co}(M\backslash k)$ is 3-connected whenever $k \in E(M) - B$. Moreover, we show that if no other elements of $E(M) - K$ satisfy this property, then M necessarily has path-width 3.

1. Introduction

If we are given a standard representation of a matroid M over a field, then we are effectively dealing with a matroid relative to a fixed basis B. Ideally we would like tools—analogous to Tutte's Wheels and Whirls Theorem [4]—that would enable inductive arguments to be made for such matroids. Valuable information, displayed by the representation, can be lost by pivoting, so the goal is to either contract elements from B or delete elements from $E(M) - B$ without losing connectivity.

Recall that a matroid M has *path-width* 3 if there is an ordering $(e_1, ..., e_n)$ of $E(M)$ such that $\{e_1, ..., e_i\}$ is 3-separating for all $i \in \{1, ..., n\}$. In this paper we prove the following theorem.

Theorem 1.1. *Let M be a 3-connected matroid with no 4-element fans where $|E(M)| \geq 4$. Let B be a basis of M. Then there exists a set K with $|K| \geq 4$ such that for all $k \in K$, either*

(i) *$k \in B$ and $\mathrm{si}(M/k)$ is 3-connected, or*
(ii) *$k \in E(M) - B$ and $\mathrm{co}(M\backslash k)$ is 3-connected.*

Moreover, if $|K| = 4$, then M has path-width 3.

1991 *Mathematics Subject Classification.* 05B35.

The first author was supported by a grant from the Marsden Fund of New Zealand.

The second author was partially supported by a Victoria University Doctoral Scholarship.

This theorem is related to the following theorem of Oxley, Semple and Whittle [3].

Theorem 1.2. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M. Then either*

(i) *B contains an element B such that M/b is 3-connected, or*
(ii) *$E(M) - B$ contains an element b^* such that $M\backslash b^*$ is 3-connected.*

In one sense our theorem is stronger that Theorem 1.2 in that we find at least four elements that can be removed. On the other hand, to obtain four elements, we do need to weaken the connectivity slightly. The techniques of this paper are closely related to those in [3].

The paper is structured as follows. The next section deals with the prerequisite definitions and some necessary results on connectivity. In Section 3, we begin by showing the existence of key elements which can removed from a suitable matroid in the appropriate manner whilst maintaining 3-connectivity. Section 4 then deals with arranging these elements in such a way as to facilitate the arguments contained in Section 5, where we work with the notion of path-width; concluding with our proof of Theorem 1.1. Finally in Section 6, we discuss some related open problems.

2. Preliminaries

Let M be a matroid with ground set E. The *connectivity function* of M; denoted by λ_M (or λ when there is no ambiguity), is defined on subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k-1$. A k-separating partition $(X, E-X)$ is a *k-separation* if $|X|, |E-X| \geq k$. A k-separating set X, or a k-separating partition $(X, E - X)$, or a k-separation $(X, E - X)$ is *exact* if $\lambda_M(X) = k - 1$. M is *n-connected* if M has no k-separation for any $k < n$. A k-separation $(X, E - X)$ is *vertical* if $r(X), r(E - X) \geq k$, while a k-separation $(X, E-X)$ is *cyclic* if both X and $E-X$ contain circuits. If $(X, \{e\}, Y)$ is a partition of E where both $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are vertical k-separations of M and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$, we also say that $(X, \{e\}, Y)$ is a vertical k-separation of M. Similarly, a partition $(X, \{e\}, Y)$ of M is cyclic if both $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are cyclic k-separations and $e \in \mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y)$.

Lemma 2.1. *Let M be a k-connected matroid. A k-separation $(X, E-X)$ of M is a vertical k-separation of M if and only if it is a cyclic*

k-separation of M^*. Thus, a partition $(P, \{e\}, Q)$ of M is a vertical k-separation of M if and only if it is a cyclic k-separation of M^*.

It is easily verified that the connectivity function λ of M is submodular, that is, for all $X, Y \subseteq E$,

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y).$$

From this, the next lemma is readily deduced.

Lemma 2.2. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$. Then the following hold.*

(i) *If $|X \cap Y| \geq 2$, then $X \cup Y$ is 3-separating, and*
(ii) *If $|E(M) - (X \cup Y)| \geq 2$, then $X \cap Y$ is 3-separating.*

We shall use the phrase "by uncrossing" to refer to an application of Lemma 2.2.

A *path of* 3*-separations* in a matroid M is an ordered partition $\mathbb{P} = (P_0, P_1, ..., P_r)$ of $E(M)$ with the property that $\lambda(P_0 \cup ... \cup P_i) = 2$ for all $i \in \{0, ..., r-1\}$. The members of $\mathbb{P}$ are called *steps* of $\mathbb{P}$. Empty steps are permitted. Note that both vertical and cyclic 3-separations are examples of paths of 3-separations. The following lemma on paths of 3-separations is elementary.

Lemma 2.3. *A 3-connected matroid M has path-width 3 if and only if there is a path $\mathbb{P} = (P_0, ..., P_r)$ of 3-separations such that $|P_0| = |P_r| = 2$, and $|P_i| = 1$ for all $i \in \{1, ..., r-1\}$.*

A *segment* in a matroid M is a subset L of $E(M)$ such that $M|L \cong U_{2,k}$ for some $k \geq 3$, while a *cosegment* of M is a segment of M^*. We shall use the notation Γ_4 to denote the class of matroids whose ground set can be partitioned (L_4, L_4^*) where L_4 is a 4-element segment and L_4^* is a 4-element cosegment. A 4*-element fan* of a matroid M is a subset F of $E(M)$ with $|F| = 4$ such that there exists an ordering (f_1, f_2, f_3, f_4) of the elements of F where $\{f_1, f_2, f_3\}$ is a triangle and $\{f_2, f_3, f_4\}$ is a triad. If M is a 3-connected matroid, with some basis B, and $e \in E(M)$, we say that e is *removable with respect to* B if either:

(i) $e \in B$ and $\mathrm{si}(M/e)$ is 3-connected, or
(ii) $e \notin B$ and $\mathrm{co}(M \backslash e)$ is 3-connected.

The property of being removable is well behaved under duality. We omit the obvious proof of the next lemma.

Lemma 2.4. *Let M be a 3-connected matroid, and let B be a basis of M. Then $e \in E(M)$ is removable with respect to B if and only if e is removable with respect to the basis $E(M) - B$ in M^*.*

The following eight lemmas will be used throughout the paper. The first five are elementary, as is the eighth. The sixth appears in [5], while a proof of the seventh can be found in [3].

Lemma 2.5. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M)-\{e\}$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 2.6. *Let X be an exactly 3-separating set in a 3-connected matroid M, and suppose that $e \in E(M)-X$. Then $X \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}^{(*)}(X)$.*

Lemma 2.7. *Let M be a 3-connected matroid, B a basis of M, and let K denote the set of elements which are removable with respect to B. If S is a segment of M with $|S| \geq 4$, then $|S \cap (E(M)-B) \cap K| \geq |S|-2$.*

Lemma 2.8. *Let M be a 3-connected matroid with $r(M) \geq 4$. Suppose that C^* is a rank-3 cocircuit of M. If there exists some $c \in C^*$ such that $r(C^*-\{c\})=3$, then $\mathrm{co}(M \backslash c)$ is 3-connected.*

Lemma 2.9. *Let M be a 3-connected matroid with a rank-3 cocircuit C^* such that $|C^*| \geq 4$ and $r(M) \geq 4$. Let B be a basis of M. Then there exists some $c \in C^* \cap (E(M)-B)$ such that $\mathrm{co}(M \backslash c)$ is 3-connected.*

Lemma 2.10. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $e \in C^*$ has the property that $\mathrm{cl}_M(C^*)-\{e\}$ contains a triangle of M/e, then $\mathrm{si}(M/e)$ is 3-connected.*

Lemma 2.11. *Let M be a 3-connected matroid with $e \in E(M)$. If $\mathrm{si}(M/e)$ is not 3-connected, then there exists a vertical 3-separation $(X,\{e\},Y)$ of M.*

Lemma 2.12. *Let M be a 3-connected matroid with $e \in E(M)$. If $(X,\{e\},Y)$ is a vertical 3-separation of M, then $(\mathrm{cl}(X)-\{e\},\{e\},Y-\mathrm{cl}(X))$ is also a vertical 3-separation of M.*

Lemmas 2.11 and 2.12 will be used repeatedly. Use of Lemma 2.11 will be made freely and without reference, while we shall use the phrase "by closing X" to refer to an application of Lemma 2.12 followed by a relabelling whereby $(\mathrm{cl}(X)-\{e\},\{e\},Y-\mathrm{cl}(X))$ becomes $(X,\{e\},Y)$.

We require the next technical result.

Lemma 2.13. *Let M be a 3-connected matroid with a triad $\{a,b,c\}$ and a circuit $\{a,b,c,d\}$. Then at least one of the following holds.*

(i) *Either $\mathrm{co}(M \backslash a)$ or $\mathrm{co}(M \backslash c)$ is 3-connected.*
(ii) *There exists $a', c' \in E(M)$ such that both $\{a,a',b\}$ and $\{c,c',b\}$ are triangles.*

(iii) *There exists* $r \in E(M)$ *such that* $\{a, b, c, r\}$ *is a cosegment.*

Proof. We shall assume that neither (i) nor (ii) holds and show that this forces (iii). As neither $\mathrm{co}(M\backslash a)$ nor $\mathrm{co}(M\backslash c)$ are 3-connected, it follows from the dual of Lemma 2.11 that there are cyclic 2-separations (P, Q) and (V, W) of $M\backslash a$ and $M\backslash c$ respectively.

Consider (P, Q). We can assume without loss of generality that $d \in Q$. If $\{b, c\} \subseteq Q$, then $(P, Q \cup \{a\})$ is a 2-separation of M as $a \in \mathrm{cl}(\{b, c, d\})$. So assume $b \in P$. Then $c \in \mathrm{cl}^*_{M\backslash a}(P)$ by Lemma 2.5, and it follows that $(P \cup \{c\}, Q - \{c\})$ is a cyclic 2-separation of $M\backslash a$. Thus, relabelling for clarity, we have a cyclic 2-separation $(X \cup \{b, c\}, Y)$ of $M\backslash a$ where $d \in Y$. Similarly, we obtain a cyclic 2-separation $(S \cup \{b, a\}, T)$ of $M\backslash c$ where $d \in T$.

2.13.1. *Neither* $\{b, c\}$ *nor* $\{b, a\}$ *is contained in a triangle.*

Proof. Suppose $\{b, c, c'\}$ is a triangle. Then by our original assumption, $\{b, a\}$ is not contained in a triangle. Consider the 2-separation $(S \cup \{b, a\}, T)$ of $M\backslash c$. As $S \cup \{a, b\}$ contains a circuit and $\{b, a\}$ is not in a triangle, we have $|S| \geq 2$. If $c' \in S$, then $c \in \mathrm{cl}_M(S \cup \{b, a\})$, so that $(S \cup \{a, b, c\}, T)$ is a 2-separation of M. Thus $c' \in T$. Now we have

$$\begin{aligned}\lambda_{M\backslash c/b}(T) &= r_{M\backslash c/b}(T) + r_{M\backslash c/b}(S \cup \{a\}) - r(M\backslash c/b)\\ &= r_{M\backslash c}(T) + r_{M\backslash c}(S \cup \{a, b\}) - r(M)\\ &= r(M\backslash c) + 1 - r(M) = 1\end{aligned}$$

Note that $r_{M\backslash c}(\{a, b, c', d\}) = 3$, hence $r_{M\backslash c/b}(\{a, c', d\}) = 2$. Also, $r_{M\backslash c/b}(\{c', d\}) = 2$, so $a \in \mathrm{cl}_{M\backslash c/b}(\{c', d\}) \subseteq \mathrm{cl}_{M\backslash c/b}(T)$. Therefore it must be that $\lambda_{M\backslash c/b}(T \cup \{a\}) = 1$. From this, we can deduce that $\lambda_{M\backslash c}(T \cup \{a, b\}) = 1$. But then $c \in \mathrm{cl}(\{a, b, d\})$ implying that $\lambda_M(T \cup \{a, b, c\}) = 1$; contradicting the fact that M is 3-connected. Therefore $\{b, c\}$, and similarly $\{b, a\}$ cannot be contained in a triangle. □

From the fact that neither $\{b, a\}$ nor $\{b, c\}$ is contained in a triangle and the fact that both $X \cup \{b, c\}$ and $S \cup \{b, a\}$ contain circuits, we deduce that $|X| \geq 2$ and $|S| \geq 2$. Let $M' = M\backslash\{a, b, c\}$.

2.13.2. $c \in \mathrm{cl}(X \cup \{b\})$ *and* $a \in \mathrm{cl}(S \cup \{b\})$.

Proof. Suppose that $c \notin \mathrm{cl}(X \cup \{b\})$. Then $r(X) = r(X \cup \{b, c\}) - 2$. But $r(M') = r(M) - 1$. We have $r(X \cup \{b, c\}) + r(Y) \leq r(M) + 1$, meaning that $r(X) + r(Y) \leq r(M')$ so that (X, Y) is a separation of M'. But $r(Y \cup \{a, b, c\}) \leq r(Y) + 2$ so that $r(X) + r(Y \cup \{a, b, c\}) \leq r(M) + 1$, giving $(X, Y \cup \{a, b, c\})$ a 2-separating partition of M; which is contradictory as $|X| \geq 2$. Similarly $a \in \mathrm{cl}(S \cup \{b\})$. □

2.13.3. $\lambda_{M'}(X) = \lambda_{M'}(Y) = \lambda_{M'}(S) = \lambda_{M'}(T) = 1.$

Proof. Using 2.13.2,

$$\begin{aligned}\lambda_{M'}(X) &= r_{M'}(X) + r_{M'}(Y) - r(M) + 1\\ &= r_{M\backslash a}(X \cup \{b\}) + r_{M\backslash a}(Y) - r(M)\\ &= r_{M\backslash a}(X \cup \{b, c\}) + r_{M\backslash a}(Y) - r(M) = 1 = \lambda_{M'}(Y).\end{aligned}$$

The result that $\lambda_{M'}(S) = \lambda_{M'}(T) = 1$ follows similarly. □

2.13.4. *If $Z \subseteq X$ or $Z \subseteq S$, then $\lambda_M(Z) \leq \lambda_{M'}(Z) + 1$.*

Proof. Suppose $Z \subseteq X$. Let $Z' = E(M') - Z$. Note that $d \in Z'$ so $r(Z' \cup \{a, b, c\}) \leq r(Z') + 2$. Then

$$\begin{aligned}\lambda_M(Z) &= r_{M'}(Z) + r(Z' \cup \{a, b, c\}) - r(M') - 1\\ &\leq r_{M\prime}(Z) + r_{M'}(Z') - r(M') + 1 = \lambda_{M'}(Z) + 1,\end{aligned}$$

and similarly, if $Z \subseteq S$, then $\lambda_M(Z) \leq \lambda_{M'}(Z) + 1$. □

2.13.5. *If $Z \subseteq Y - \{d\}$ or $Z \subseteq T - \{d\}$, then $\lambda_M(Z) = \lambda_{M'}(Z)$.*

Proof. Suppose $Z \subseteq Y - \{d\}$. Let $Z' = E(M') - Z$. Note that $d \in Z'$ and $X \subset Z'$. By 2.13.2, $c \in \mathrm{cl}(Z' \cup \{b\})$, but $a \in \mathrm{cl}(Z' \cup \{b, c\})$ so $r(Z' \cup \{a, b, c\}) = r(Z') + 1$. Therefore $\lambda_M(Z) = r(Z) + r(Z' \cup \{a, b, c\}) - r(M) = r(Z) + r(Z') - r(M') = \lambda_{M'}(Z)$. as required. Similarly, if $Z \subseteq T - \{d\}$, then $\lambda_M(Z) = \lambda_{M'}(Z)$.. □

2.13.6. *$T \cap X \neq \emptyset$, and $S \cap Y \neq \emptyset$.*

Proof. If $T \cap X = \emptyset$, then $X \subseteq S$, and by 2.13.2, $\{a, c\} \subseteq \mathrm{cl}(S \cup \{b\})$, so that $(S \cup \{a, b, c\}, T)$ is a 2-separation of M. Similarly $S \cap Y \neq \emptyset$. □

Applying submodularity of the connectivity function together with 2.13.3, we have $\lambda_{M'}(T \cup X) + \lambda_{M'}(T \cap X) \leq \lambda_{M'}(T) + \lambda_{M'}(X)$, so that $\lambda_{M'}(S \cap Y) + \lambda_{M'}(T \cap X) \leq 2$.

But by 2.13.5, $\lambda_{M'}(S \cap Y) = \lambda_M(S \cap Y)$ and $\lambda_{M'}(T \cap X) = \lambda_M(T \cap X)$. This result combined with 2.13.6 implies that we must have $|S \cap Y| = |T \cap X| = 1$. As $(X \cup \{b, c\}, Y)$ is cyclic, it follows that $|Y \cap T| \geq 2$.

2.13.7. $\lambda_{M'}(T \cap Y) \geq 2.$

Proof. By 2.13.2, $\{a, c\} \subseteq \mathrm{cl}(S \cup X \cup \{b\})$, so $r(S \cup X \cup \{a, b, c\}) = r(S \cup X) + 1$. Hence $\lambda_M(T \cap Y, S \cup X \cup \{a, b, c\}) = \lambda_{M'}(T \cap Y, S \cup X)$. □

Now $\lambda_{M'}(S \cap X) + \lambda_{M'}(T \cap Y) \leq \lambda_{M'}(S) + \lambda_{M'}(X) \leq= 2$. So by 2.13.7, $\lambda_{M'}(S \cap X) = 0$. By 2.13.4, we deduce that $|S \cap X| = 1$. Let $S \cap X = \{r\}$ and $A = \{a, b, c\}$. As $\lambda_{M'}(T \cup Y) = 0$, $r_M(T \cup Y) = r_{M'}(T \cup Y) = r(M') - 1 = r(M) - 2$. Thus $\lambda_M(T \cup Y) = r(A \cup \{r\}) - 2 \leq 2$, and

$(T \cup Y, A \cup \{r\})$ is a 3-separation of M. If $r \in \mathrm{cl}(A)$, then M must contain a 2-separation. So $r \notin \mathrm{cl}(A)$ and by Lemma 2.6, $r \in \mathrm{cl}^*(A)$, which implies that $\{a, b, c, r\}$ is a cosegment, which gives (iii). □

3. The Existence of Removable Elements

Lemma 3.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M, and let K denote the set of elements which are removable with respect to B. Suppose $b \in B - K$. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M. If $X \cap B \cap K = \emptyset$, then $|X \cap (E(M) - B) \cap K| \geq 2$.*

Proof. Recall that when we use the phrase "by closing Y" we refer to an application of Lemma 2.12 followed by an appropriate labelling. Firstly, we may assume; by closing Y, that $Y \cup \{b\}$ is closed. Note also that $X \cap B \neq \emptyset$ as Y is non-spanning. Suppose that $X \cap B \cap K = \emptyset$. Then for each $b_x \in X \cap B$, there exists a vertical 3-separation $(X_{b_x}, \{b_x\}, Y_{b_x})$ of M. If, for some $b_x \in X \cap B$, there is such a vertical 3-separation where either X_{b_x} or Y_{b_x} is contained in $X \cup \{b\}$, then, by switching X_{b_x} and Y_{b_x} if necessary, and closing Y_{b_x}, there exists such a vertical 3-separation where $X_{b_x} \subseteq X \cup \{b\}$ and $Y_{b_x} \cup \{b_x\}$ is closed. Then $X_{b_x} \subseteq (X - \{b_x\}) \cup \{b\}$. If equality holds here, then $Y_{b_x} = Y$, but then $b_x \in \mathrm{cl}(Y_{b_x}) = \mathrm{cl}(Y)$; a contradiction. Hence $X_{b_x} \subset (X - \{b_x\}) \cup \{b\}$. Now relabel so that $(X_{b_x}, \{b_x\}, Y_{b_x})$ becomes $(X, \{b\}, Y)$. By an iteration of this procedure, we eventually obtain a vertical 3-separation $(X, \{b\}, Y)$ of M with $Y \cup \{b\}$ closed such that if $(X_{b_x}, \{b_x\}, Y_{b_x})$ is a vertical 3-separation of M with $b_x \in X \cap B$, then neither X_{b_x} nor Y_{b_x} is contained in $X \cup \{b\}$. Moreover, we maintain the property that $X \cap B \cap K = \emptyset$.

Let b_x be an element of $X \cap B$, and let $(P, \{b_x\}, Q)$ be a vertical 3-separation of M. Without loss of generality, $b \in Q$. Moreover, by we may assume that $Q \cup \{b_x\}$ is closed by closing Q.

3.1.1. *$X \cap P, X \cap Q, Y \cap P$ and $Y \cap Q$ are all non-empty.*

Proof. If $X \cap P$ or $X \cap Q$ is empty, then P or Q is contained in $Y \cup \{b\}$ and so $b_x \in \mathrm{cl}(Y \cup \{b\})$; which contradicts the fact that $Y \cup \{b\}$ is closed. If $Y \cap P$ or $Y \cap Q$ is empty, then P or Q is contained in $X \cup \{b\}$; contradicting our construction of $(X, \{b\}, Y)$. □

3.1.2. *$X \cap P$ is 3-separating with $r((X \cap P) \cup \{b_x\}) = 2$.*

Proof. As $E(M) - (X \cup P) = (Y \cap Q) \cup \{b\}$, we have $|E(M) - (X \cup P)| \geq 2$, and thus, by uncrossing, $\lambda(X \cap P) \leq 2$. If $|X \cap P| = 1$, the second claim is immediate. So assume $|X \cap P| \geq 2$. Now $|E(M) - (X \cup (P \cup$

$\{b_x\}))| = |(Y \cap Q) \cup \{b\}| \geq 2$, so that, by uncrossing, $\lambda(X \cap (P \cup \{b_x\})) \leq 2$. We deduce that $\lambda(X \cap P) = \lambda((X \cap P) \cup \{b_x\}) = 2$. By Lemma 2.6, $b_x \in \mathrm{cl}^{(*)}(X \cap P)$. If $b_x \in \mathrm{cl}^*(X \cap P)$, then by Lemma 2.5, $b_x \notin \mathrm{cl}(Y \cup Q)$; a contradiction. So $b_x \in \mathrm{cl}(X \cap P)$. If $r((X \cap P) \cup \{b_x\}) \geq 3$, then $(X \cap P, \{b_x\}, Y \cup Q)$ is a vertical 3-separation of M; a contradiction to our earlier construction of $(X, \{b\}, Y)$. So $r((X \cap P) \cup \{b_x\}) = 2$. □

Suppose $|Y \cap P| = 1$. If $|X \cap P| = 2$, then P is a triad, which implies that $P \cup \{b_x\}$ is a contradictory 4-element fan by 3.1.2. So $|X \cap P| \geq 3$, and $(X \cap P) \cup \{b_x\}$ is a segment containing at least four elements. By Lemma 2.7 $|X \cap (E(M) - B) \cap K| \geq 2$ and the lemma holds. So assume that $|Y \cap P| \geq 2$.

3.1.3. $r((X \cap Q) \cup \{b, b_x\}) = 2$.

Proof. Since $|E - ((X \cup \{b\}) \cup (Q \cup \{b_x\}))| = |Y \cap P| \geq 2$, it follows by uncrossing that $\lambda((X \cap Q) \cup \{b, b_x\}) \leq 2$. But $|(X \cap Q) \cup \{b, b_x\}| \geq 3$ and so $\lambda((X \cap Q) \cup \{b, b_x\}) = 2$. Noting that $P \subseteq E - ((X \cap Q) \cup \{b, b_x\})$, we have $b_x \in \mathrm{cl}(E - ((X \cap Q) \cup \{b, b_x\}))$, and it follows from Lemmas 2.6 and 2.5 that $b_x \in \mathrm{cl}((X \cup \{b\}) \cap Q)$. If $r((X \cap Q) \cup \{b, b_x\}) \geq 3$, then $((X \cup \{b\}) \cap Q, \{b_x\}, E - ((X \cap Q) \cup \{b, b_x\}))$ is a vertical 3-separation of M; a contradiction. Therefore, $r((X \cap Q) \cup \{b, b_x\}) = 2$. □

Now let $L_1 = (X \cap Q) \cup \{b\}$ and $L_2 = (X \cap P) \cup \{b_x\}$. By 3.1.1 and 3.1.3, $|\mathrm{cl}(L_1)| \geq 3$. If $|\mathrm{cl}(L_1)| \geq 4$, then the result holds by Lemma 2.7. So assume $|\mathrm{cl}(L_1)| = 3$ and let $\mathrm{cl}(L_1) - \{b, b_x\} = \{a\}$. If $(L_2 - \{b_x\}) \cap B = \{b'\}$, then, as $\{a, b, b_x\}$ is a triangle of M/b', it follows by Lemma 2.10 that $\mathrm{si}(M/b')$ is 3-connected: a contradiction. So $(L_2 - \{b_x\}) \cap B = \emptyset$. If $|L_2| \geq 4$, then Lemma 2.7 again implies that $|X \cap (E(M) - B) \cap K| \geq 2$ as required. So assume $|L_2| \in \{2, 3\}$. Since $Y \cup \{b\}$ is closed and M contains no 4-element fans, it must be that $|L_2 - \{b_x\}| = 2$. Let $L_2 - \{b_x\} = \{x_1, x_2\} \subset E(M) - B$. An application of Lemma 2.8 now reveals that $\{x_1, x_2\} \subseteq X \cap (E(M) - B) \cap K$. □

Lemma 3.2. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M, and let K denote the set of elements which are removable with respect to B. Suppose $b \in B - K$. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M. Then either:*

(i) $|X \cap B \cap K| \geq 2$, *or*
(ii) $|X \cap (E(M) - B) \cap K| \geq 2$, *or*
(iii) $|X \cap B \cap K| \geq 1 \leq |X \cap (E(M) - B) \cap K|$, *or*
(iv) $X \cup \{b\}$ *is a circuit and there exists* $b_k, \gamma_1, \gamma_2 \in E(M)$ *such that* $X = \{b_k, \gamma_1, \gamma_2\}$ *with* $X \cap K = X \cap B = \{b_k\}$.

Proof. We shall assume that (i) does not hold. If $X \cap B \cap K = \emptyset$, then (ii) holds by Lemma 3.1. So we may assume that $|X \cap B \cap K| = 1$. Let $X \cap B \cap K = \{b_k\}$. Suppose firstly that $(X \cap B) - \{b_k\} \neq \emptyset$. Now if, for any $b_x \in (X \cap B) - \{b_k\}$, there is a vertical 3-separation $(X_{b_x}, \{b_x\}, Y_{b_x})$ of M such that $X_{b_x} \subseteq (X \cup \{b\}) - \{b_k\}$, then by Lemma 3.1, we can deduce that (ii) holds in addition to $\mathrm{si}(M/b_k)$ being 3-connected. So, taking some $b_x \in (X \cap B) - \{b_k\}$, and letting $(P, \{b_x\}, Q)$ be a vertical 3-separation of M, we may assume that neither P nor Q is contained in $(X \cup \{b\}) - \{b_k\}$. Without loss of generality, we may also assume that $b \in Q$ and $Q \cup \{b_x\}$ is closed. Now observe that 3.1.1, 3.1.2, and 3.1.3 of Lemma 3.1 all hold. We deduce that $b_k \in X \cap P$. Letting $L_1 = (X \cap Q) \cup \{b\}$ and $L_2 = (X \cap P) \cup \{b_x\}$ as per the argument following 3.1.3, we again deduce that $|\mathrm{cl}(L_1)| = |L_2| = 3$. Letting $L_2 - \{b_k, b_x\} = \{x_1\}$, it follows by an application of Lemma 2.8 that $\mathrm{co}(M\backslash x_1)$ is 3-connected. Thus (iii) is satisfied.

The final possibility to consider is when $(X \cap B) - \{b_k\} = \emptyset$. In this case, X is a cocircuit of rank 3. If $|X| \geq 4$, then Lemma 2.9 gives (iii). So assuming $|X| = 3$, X is a triad, and as M does not contain any 4-element fans, $X \cup \{b\}$ must be a circuit. Let $X - \{b_k\} = \{\gamma_1, \gamma_2\}$. If $\mathrm{co}(M\backslash\gamma_1)$ or $\mathrm{co}(M\backslash\gamma_2)$ is 3-connected, then (iii) holds. Otherwise, we have (iv). We conclude that the lemma holds. □

Lemma 3.3. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M, and let K denote the set of elements which are removable with respect to B. Suppose $b \in B - K$. Let $(X, \{b\}, Y)$ be a vertical 3-separation of M. Then either:*

(i) $|X \cap K| \geq 2$, *or*
(ii) $|X \cap K| = 1$, *and there exists $\gamma \in Y$ such that $X \cup \{\gamma\}$ is a 4-element cosegment.*

Proof. Assume that (i) does not hold. Then by Lemma 3.2, there exists $b_k, \gamma_1, \gamma_2 \in E(M)$ such that $X = \{b_k, \gamma_1, \gamma_2\}$ is a triad with $X \cap K = X \cap B = \{b_k\}$, while $X \cup \{b\}$ is a circuit. Now apply Lemma 2.13. As neither $\mathrm{co}(M\backslash\gamma_1)$ nor $\mathrm{co}(M\backslash\gamma_2)$ are 3-connected and M contains no 4-element fans, we are left to deduce the existence of some $\gamma \in Y$ such that $X \cup \{\gamma\}$ is a 4-element cosegment. □

Lemma 3.4. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M, and let K denote the set of elements which are removable with respect to B. For any $b \in B - K$, there exists a path of 3-separations $(X, \{b\}, Y)$ such that $|X \cap K| \geq 2 \leq |Y \cap K|$. Moreover, if $|K| = 4$, then $(X, \{b\}, Y)$ is a vertical 3-separation unless $M \in \Gamma_4$; in which case X is a 4-element cosegment.*

Proof. Take any $b \in B-K$ and let $(P,\{b\},Q)$ be a vertical 3-separation of M. If $|P \cap K| < 2$, then by Lemma 3.3, there exists $\gamma \in Q$ such that $P \cup \{\gamma\}$ is a 4-element cosegment. Now $|(P \cup \{\gamma\}) \cap K| \geq 2$ by the dual of Lemma 2.7. If $b \in \mathrm{cl}^*(X \cup \{\gamma\})$, then $X \cup \{b, \gamma\}$ is a 5-element cosegment, which; by Lemma 2.7, would imply that $\mathrm{si}(M/b)$ is 3-connected. So, by Lemma 2.5, it must be that $b \in \mathrm{cl}(Y - \{\gamma\})$. Now suppose $r(Q - \{\gamma\}) = 2$. If $|Q - \{\gamma\}| = 2$, then Q is a triad; so that $Q \cup \{b\}$ is a contradictory 4-element fan. So $|Q - \{\gamma\}| \geq 3$, hence $|(Q - \{\gamma\}) \cap K| \geq 2$ by Lemma 2.7. Here $(P \cup \{\gamma\}, \{b\}, Q - \{\gamma\})$ is a path of 3-separations which satisfies the requirements of the Lemma; in particular, if $|K| = 4$, then $M \in \Gamma_4$. Therefore we may assume that $r(Q - \{\gamma\}) \geq 3$, so that $(P \cup \{\gamma\}, \{b\}, Q - \{\gamma\})$ is a vertical 3-separation of M. If $|(Q - \{\gamma\}) \cap K| \geq 2$, the result is immediate, or else, by Lemma 3.3, there exists some $\gamma' \in P \cup \{\gamma\}$ such that $(Q - \{\gamma\}) \cup \{\gamma'\}$ is a 4-element cosegment. Certainly, $E(M) - \{b\}$ cannot be a 7-element cosegment by Lemma 2.7, so it must be that $P \cup \{\gamma\}$ and $(Q - \{\gamma\}) \cup \{\gamma'\}$ are two 4-element cosegments; each containing two basis elements, which intersect at exactly one point $\gamma' \in B$; which is absurd. Similarly if $|Q \cap K| < 2$. Otherwise, by Lemma 3.3, $|P \cap K| \geq 2 \leq |Q \cap K|$ as taken is a suitable path of 3-separations. □

Theorem 3.5. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M. Let K denote the set of elements which are removable with respect to B. If $|E(M)| \geq 4$, then $|K| \geq 4$.*

Proof. Suppose $|K| \leq 3$. Then by Lemma 3.4, together with its dual, $B - K = (E(M) - B) - K = \emptyset$. □

4. Arranging Removable Elements

For clarity of exposition, we begin this section by proving a weaker version of Proposition 4.2.

Proposition 4.1. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M, and let K denote the set of elements which are removable with respect to B. Suppose $|K| = 4$. If M contains a 4-element segment, then there exists a set $\{\alpha_1, \alpha_2\} \subset K$ such that for every $e \in E(M) - K$, there exists a path of 3-separations $(X, \{e\}, Y)$ with $X \cap K = \{\alpha_1, \alpha_2\}$.*

Proof. Let L be a 4-element segment and $\alpha_1, \alpha_2 \in L \cap K$. We begin by showing that the result holds for any element of the basis which is not removable with respect to B. Taking any $b \in B - K$, by Lemma 3.4, there exists a path of 3-separations $(P, \{b\}, Q)$ such that $|P \cap K| = |Q \cap K| = 2$. If $M \in \Gamma_4$, then Lemma 3.4 implies that $Q \cap K = \{\alpha_2, \alpha_2\}$

as required. So we may assume that $M \notin \Gamma_4$, hence that $(P, \{b\}, Q)$ is a vertical 3-separation of M. Without loss of generality, $|P \cap L| \geq 2$ and $P \cap L \cap K = \{\alpha_1\}$. Now $(P \cup \{\alpha_2\}, \{b\}, Q - \{\alpha_2\})$ is a vertical 3-separation of M by Lemma 2.12. Letting $(P \cap K) - \{\alpha_1, \alpha_2\} = \{\gamma\}$, Lemma 3.3 implies that $(Q - \{\alpha_2\}) \cup \{\gamma\}$ is a 4-element cosegment. As $b \notin K$, $b \notin \mathrm{cl}^*((Q - \{\alpha_2\}) \cup \{\gamma\})$, so $b \in \mathrm{cl}((P \cup \{\alpha_2\}) - \{\gamma\})$, and $((P \cup \{\alpha_2\}) - \{\gamma\}, \{b\}, (Q - \{\alpha_2\}) \cup \{\gamma\})$ is a path of 3-separations which satisfies the requirements of the lemma.

Now let c be any element of $(E(M) - B) - K$. By Lemma 3.4, there exists a path of 3-separations $(V, \{c\}, W)$ such that $|V \cap K| = |W \cap K| = 2$. If $M \in \Gamma_4$, the result is again immediate. So we may assume that $(V, \{c\}, W)$ is a cyclic 3-separation of M. Without loss of generality, $|V \cap L| \geq 2$ and $V \cap L \cap K = \{\alpha_1\}$. If $|W - \{\alpha\}| = 2$, then, as $(W - \{\alpha\}) \cup \{c\}$ is 3-separating by Lemma 2.6, it follows that $(W - \{\alpha\}) \cup \{c\}$ is a triad, whereas W is a triangle; however M does not contain any 4-element fans. So $|W - \{\alpha\}| \geq 3$. Lemma 2.6 now implies that $(V \cup \{\alpha_2\}, \{c\}, W - \{\alpha_2\})$ is a cyclic 3-separation of M unless $W - \{\alpha_2\}$ is independent; in which case $(W - \{\alpha_2\}) \cup \{c\}$ is a cosegment of M containing at least 4-elements; which, by Lemma 2.7 contradicts the fact that $|W \cap K| = 2$. Therefore $W - \{\alpha_2\}$ must contain a circuit, and by the dual of Lemma 3.3, there exists $\zeta \in (V \cup \{\alpha_2\}) \cap K$ such that $(W - \{\alpha_2\}) \cup \{\zeta\}$ is a 4-element segment. If $\zeta \in \{\alpha_1, \alpha_2\}$, then either $(V \cap L) \cup W$ is a 6-element segment containing exactly three elements which are removable with respect to B, or L and W are distinct 4-element segments, each containing two elements of B, whose intersection is $\{\zeta\} \in E(M) - B$; both possibilities being entirely incongruous. So $\zeta \notin \{\alpha_1, \alpha_2\}$. With appropriate use of Lemma 2.6, and by letting $X = (V - \{\zeta\}) \cup \{\alpha_2\}$, and $Y = (W - \{\alpha_2\}) \cup \{\zeta\}$, we arrive at a suitable path of 3-separations. □

Proposition 4.2. *Let M be a 3-connected matroid with no 4-element fans. Let B be a basis of M, and let K denote the set of elements which are removable with respect to B. Suppose $|K| = 4$. Then there exists a set $\{\alpha_1, \alpha_2\} \subset K$ such that for every $e \in E(M) - K$, there exists a path of 3-separations $(X, \{e\}, Y)$ with $X \cap K = \{\alpha_1, \alpha_2\}$.*

Proof. If M contains a 4-element segment or a 4-element cosegment, then the result follows immediately from Proposition 4.1 and its dual. So assume that M holds no such substructures. By Lemma 3.4, for each $e \in E(M) - K$, there exists a path of 3-separations $(X, \{e\}, Y)$ with $|X \cap K| = |Y \cap K| = 2$ which is either a vertical or a cyclic 3-separation according to whether or not $e \in B$. We shall proceed to show that in this case, for each element of $E(M) - K$, the paths of

3-separations obtained from Lemma 3.4 and its dual do indeed already satisfy the requirements of the current proposition.

Take any distinct $e_1, e_2 \in E(M) - K$ with corresponding paths of 3-separations $(X, \{e_1\}, Y)$ and $(P, \{e_2\}, Q)$ as stated. To prove the result, it suffices to show that $X \cap K \in \{P \cap K, Q \cap K\}$, so suppose this is not the case. Then $|X \cap P| = |X \cap Q| = |Y \cap P| = |Y \cap Q| = 1$, and without loss of generality, $e_1 \in Q, e_2 \in X$.

4.2.1. *For all $S \in \{X \cap P, X \cap Q, Y \cap P\}$, $\lambda(S) \leq 2$.*

Proof. As $|E - (Y \cup P)| \geq 2$, by uncrossing, $Y \cap P$ is 3-separating. Similarly, $\lambda(X \cap Q) \leq 2$. Suppose $|Y \cap Q| = 1$. Then $|Y \cap P| \geq 2$ and so $\lambda(Y \cap P) = \lambda((Y \cap P) \cup \{e_1\}) = 2$, which, by Lemma 2.6, means that $e_1 \in \mathrm{cl}^{(*)}(Y \cap P)$. If $|Y \cap P| \geq 3$ and $e_1 \in \mathrm{cl}(Y \cap P)$, then as M does not contain any 4-element segments, $r(Y \cap P) \geq 3$. But then $(X \cup Q, \{e_1\}, Y \cap P)$ is a vertical 3-separation of M with $|Y \cap P \cap K| = 1$; which by Lemma 3.3 implies the contradictory existence of a 4-element cosegment. Similarly it cannot be that $|Y \cap P| \geq 3$ and $e_1 \in \mathrm{cl}^*(Y \cap P)$. So $|Y \cap P| = 2$. Now either $(X, \{e_1\}, Y)$ is a vertical 3-separation of M, meaning that $(Y \cap P) \cup \{e_1\}$ is a triangle and Y is a triad, or $(X, \{e_1\}, Y)$ is a cyclic 3-separation, giving $(Y \cap P) \cup \{e_1\}$ as a triad and Y as a triangle. Each case revealing a 4-element fan in M. We conclude that $|Y \cap Q| = |E - (X \cup P)| \geq 2$, so that by uncrossing, $X \cap P$ is 3-separating. $\square$

We shall use 4.2.1 implicitly in what follows. Suppose now that $e_1, e_2 \in B$ so that $(X, \{e_1\}, Y)$ and $(P, \{e_2\}, Q)$ are vertical 3-separations of M. If $r(X \cap P) \geq 3$, then as $e_1 \in \mathrm{cl}(X \cap P)$ by Lemmas 2.6 and 2.5, it must be the case that $(X \cap P, \{e_1\}, Y \cup Q \cup \{e_2\})$ is a vertical 3-separation of M. This then contradicts Lemma 3.3, as M contains no 4-element cosegments. So $r(X \cap P) = 2$. If $|X \cap P| \geq 2$, then $\{e_1, e_2\} \in \mathrm{cl}(X \cap P)$ again by Lemmas 2.6 and 2.5, but such a 4-point segment cannot exist. Hence $|X \cap P| = 1$, and $(X \cap P) \cup \{e_1, e_2\}$ is a triangle. If $Y \cap P$ is then a singleton, P is a triad, giving a 4-element fan $P \cup \{e_2\}$ in M. So $|Y \cap P| \geq 2$. If $r(Y \cap P) \geq 3$, we again obtain a vertical 3-separation $(X \cup Q, \{e_1\}, Y \cap P)$ which contradicts Lemma 3.3. So $r(Y \cap P) = 2$, and to avoid a 4-element segment, $|Y \cap P| = 2$; so that $(Y \cap P) \cup \{e_1\}$ is a triangle. Let $X \cap P = \{k\}$ and $Y \cap P = \{z_1, z_2\}$. By Lemma 2.8, both $\mathrm{co}(M \backslash z_1)$ and $\mathrm{co}(M \backslash z_2)$ are 3-connected. But $|Y \cap P \cap K| = 1$, so we may assume that $z_1 \in B$. But then $\{e_1, e_2, k\}$ is a triangle of M/z_1, so that; by Lemma 2.10, $\mathrm{si}(M/z_1)$ is 3-connected; a contradiction. By duality, we arrive at the same contradiction if $e_1, e_2 \in E(M) - B$.

We now examine the case where $e_1 \in B$ and $e_2 \in E(M) - B$, so that $(X, \{e_1\}, Y)$ is a vertical 3-separation and $(P, \{e_2\}, Q)$ is cyclic. Provided $|X \cap P| \geq 2$, $e_1 \in \mathrm{cl}(X \cap P)$ and $e_2 \in \mathrm{cl}^*(X \cap P)$. If $|X \cap P| \geq 3$, then $r(X \cap P) = 2$ or else $(X \cap P, \{e_1\}, Y \cup Q \cup \{e_2\})$ is a vertical 3-separation of M with $|X \cap P \cap K| = 1$, contradicting Lemma 3.3. But then $X \cap P$ must be also be independent, otherwise $(X \cap P, \{e_2\}, Y \cup Q \cup \{e_1\})$ is a cyclic 3-separation of M which again contradicts Lemma 3.3. Hence $|X \cap P| \in \{1, 2\}$. If $|X \cap P| = 2$, then $(X \cap P) \cup \{e_1\}$ is a triangle, whereas $(X \cap P) \cup \{e_2\}$ is a triad, giving a 4-element fan. Finally, if $|X \cap P| = 1$ with $X \cap P = \{\psi\}$, then $\{e_1, e_2, \psi\}$ is 3-separating so is either a triad or a triangle. Using Lemma 2.5, the first possibility contradicts the fact that $e_1 \in \mathrm{cl}(Y)$, and the second, the fact that $e_2 \in \mathrm{cl}^*(Q)$. The case where $e_1 \in E(M) - B$ and $e_2 \in B$ is essentially identical. We conclude that $X \cap K \in \{P \cap K, Q \cap K\}$, and the result holds. □

5. Removable Elements and Path Width

We begin this section by noting a simple result on paths of 3-separations.

Lemma 5.1. *Let $\mathbb{P} = (P_0, ..., P_r)$ be a path of 3-separations in a matroid M. If, for some $i \in \{1, ..., r-1\}$, $|P_i| = 1$ with $P_i = \{e_i\}$, then either*

(i) $e_i \in \mathrm{cl}(P_0 \cup ... \cup P_{i-1}) \cap \mathrm{cl}(P_{i+1} \cup ... \cup P_r)$, *or*
(ii) $e_i \in \mathrm{cl}^*(P_0 \cup ... \cup P_{i-1}) \cap \mathrm{cl}^*(P_{i+1} \cup ... \cup P_r)$.

Lemma 5.2. *Let $\mathbb{P} = (P_0, ..., P_r)$ be a path of 3-separations in a matroid M. Suppose $i \in \{1, ..., r-1\}$, $e \in P_i$, and that there exists a path of 3-separations $(X, \{e\}, Y)$ of M with $P_0 \subseteq X$, $P_r \subseteq Y$ and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Then there exists a path of 3-separations $(X', \{e\}, Y')$ with $P_0 \cup ... \cup P_{i-1} \subseteq X'$, $P_{i+1} \cup ... \cup P_r \subseteq Y'$, and $e \in \mathrm{cl}(X') \cap \mathrm{cl}(Y')$.*

Proof. Let $S = P_0 \cup ... \cup P_{i-1}$ and $T = P_{i+1} \cup ... \cup P_r$.

5.2.1. *There exists a path of 3-separations $(X_0, \{e\}, Y_0)$ of M such that $S \subseteq X_0$, $P_r \subseteq Y_0$, and $e \in \mathrm{cl}(X_0) \cap \mathrm{cl}(Y_0)$.*

Proof. We have $\lambda(Y \cup \{e\}) = \lambda(P_i \cup T) = 2$. Note that $P_0 \subseteq X \cap S$, and P_0 contains at least two elements. Now

$$|E(M) - ((Y \cup \{e\}) \cup (P_i \cup T))| = |X \cap S| \geq |P_0| \geq 2$$

and it follows by uncrossing that $\lambda((Y \cup \{e\}) \cap (P_i \cup T)) \leq 2$. So $(X \cup S, (Y \cup \{e\}) \cap (P_i \cup T))$ is a 3-separating partition of M. We know that $|P_0| \geq 2$ and $|P_r| \geq 2$. Also, $P_0 \subseteq X \cup S$ and $P_r \subseteq (Y \cup \{e\}) \cap$

$(P_i \cup T)$. Therefore $(X \cup S, (Y \cup \{e\}) \cap (P_i \cup T))$ is a 3-separation of M. But $e \in \mathrm{cl}(X) \subseteq \mathrm{cl}(X \cup S)$, and it follows from Lemma 2.5 that $e \in \mathrm{cl}(((Y \cup \{e\}) \cap (P_i \cup T)) - \{e\})$ and that $(X \cup S, \{e\}, Y \cap (P_i \cup T))$ is a path of 3-separations. Letting $X_0 = X \cup S$ and $Y_0 = Y \cap (P_i \cup T)$, the result follows. □

Now $\lambda(X_0 \cup \{e\}) = \lambda(P_i \cup S) = 2$, and $|E(M) - ((X_0 \cup \{e\}) \cup (P_i \cup S))| = |Y_0 \cap T| \geq |P_r| \geq 2$, so that, by uncrossing, $\lambda((X_0 \cup \{e\}) \cap (P_i \cup S)) \leq 2$. As $P_0 \subseteq (X_0 \cup \{e\}) \cap (P_i \cup S)$, it must be that $\lambda((X_0 \cup \{e\}) \cap (P_i \cup S)) = 2$. Therefore $((X_0 \cup \{e\}) \cap (P_i \cup S), Y_0 \cup T)$ is an exact 3-separation of M. As $e \in \mathrm{cl}(Y_0 \cup T)$, it then follows from Lemma 2.5 that $e \in \mathrm{cl}(((X_0 \cup \{e\}) \cap (P_i \cup S)) - \{e\})$ and that $(X_0 \cap (P_i \cup S), \{e\}, Y_0 \cup T)$ is a path of 3-separations. Observing that $S \subseteq X_0 \cap (P_i \cup S)$ and $T \subseteq Y_0 \cup T$, we conclude that the lemma holds. □

The next result is an immediate consequence of Lemmas 5.1 and 5.2.

Corollary 5.3. *Let $\mathbb{P} = (P_0, ..., P_r)$ be a path of 3-separations in a matroid M. Suppose $i \in \{1, ..., r-1\}$, $e \in P_i$, and that there exists a path of 3-separations $(X, \{e\}, Y)$ of M with $P_0 \subseteq X$ and $P_r \subseteq Y$. Then $\mathbb{P}$ refines to a path $(P_0, ..., P_{i-1}, P_i', \{e\}, P_i'', P_{i+1}, ..., P_r)$ of 3-separations where $P_i' \cup \{e\} \cup P_i'' = P_i$.*

Lemma 5.4. *Let M be a 3-connected matroid with disjoint sets $\{a_1, a_2\} \subset E(M)$ and $\{z_1, z_2\} \subset E(M)$. Suppose that for every $e \in E(M) - \{a_1, a_2, z_1, z_2\}$, there exists a path of 3-separations $(X, \{e\}, Y)$ in M such that $\{a_1, a_2\} \subseteq X$ and $\{z_1, z_2\} \subseteq Y$. Then M has path-width 3.*

Proof. As M is 3-connected, $(\{a_1, a_2\}, E(M) - \{a_1, a_2, z_1, z_2\}, \{z_1, z_2\})$ is a path of 3-separations. If $E(M) = \{a_1, a_2, z_1, z_2\}$, the result is immediate. So suppose that $e \in E(M) - \{a_1, a_2, z_1, z_2\}$. Applying Corollary 5.3, we obtain a refinement of our original path of 3-separations. Now successively applying Corollary 5.3 to each of the other elements of $E(M) - \{a_1, a_2, z_1, z_2\}$; each time with respect to our new refined path of 3-separations, we eventually obtain a path of 3-separations $\mathbb{P} = (\{a_1, a_2\}, P_1, ..., P_q, \{z_1, z_2\})$ where, for all $i \in \{1, ..., q\}$, the step P_i is either a singleton or empty. Removing all empty steps from $\mathbb{P}$, we then obtain a path of 3-separations $\mathbb{P}' = (\{a_1, a_2\}, P_1', ..., P_n', \{z_1, z_2\})$ in which P_i is a singleton for all $i \in \{1, ..., n\}$. By Lemma 2.3, M has path-width 3. □

Now we are ready to prove our main result.

Proof of Theorem 1.1. Let K denote the set of elements which are removable with respect to B. By Theorem 3.5, $|K| \geq 4$. Suppose now that $|K| = 4$. By Proposition 4.2, there exists $k_1, k_2 \in K$ such that for each $e_i \in E(M) - K$, there exists a path of 3-separations $(X_i, \{e_i\}, Y_i)$ of M with $\{k_1, k_2\} \subset X_i$ and $K - \{k_1, k_2\} \subset Y_i$. By Lemma 5.4, M has path-width 3. □

6. Future Directions for Research

Let M be a 3-connected matroid with no 4-element fans. For the sake of this discussion, we shall say that M has minimal removability if there exists a basis B of M such that there exists exactly four elements of $E(M)$ which are removable with respect to B. In the present paper, we have shown that if M has minimal removability, then M necessarily has path-width 3. However, the class of matroids with minimal removability is certainly a proper subclass of the class of matroids with path-width 3. This knowledge, together with the fact that the class of matroids with path-width 3 is very well understood (see [1] and [2]), implies that one should be able to give a stronger and more explicit description of the class of matroids with minimal removability. For now, we leave this as an open problem.

An intriguing aspect of Theorem 1.1 is the fact that while the notion of being removable with respect to a basis is one which is relative to the choice of basis one makes, we are still able to deduce global structural information about the matroid itself. It is certainly possible that if M is a matroid and B is a basis of M which reveals that M has minimal removability, then upon switching to a different basis B' of M, we obtain more than four elements which are removable with respect to B'. We pose the following question. Does there exist a 3-connected matroid M with no 4-element fans such that for every basis B of M, there exists exactly four elements of $E(M)$ which are removable with respect to B? If so, what additional structure much such an M have?

Finally, while we have provided a new analogue of the Wheels and Whirls Theorem, we would very much like to see it extended to an analogue of the Splitter Theorem. We make the following conjecture:

Conjecture 6.1. *Let M be a 3-connected matroid with no 4-element fans, and let B be a basis of M. Let N be a 3-connected minor of M. If there exists some $b \in B$ such that M/b has an N-minor, or there exists some $c \in E(M) - B$ such that $M \backslash c$ has an N-minor, then there exists distinct $k_1, k_2 \in E(M)$ such that for $i \in \{1, 2\}$, either*

(i) *$k_i \in B$ and $\mathrm{si}(M/k_i)$ is 3-connected with an N-minor, or*
(ii) *$k_i \in E(M) - B$ and $\mathrm{co}(M \backslash k_i)$ is 3-connected with an N-minor.*

References

[1] Beavers, B., Oxley, J. Constructive characterizations of 3-connected matroids of path width three. *Europ. J. Combin.* **29** (2008) 1643-1661.

[2] Hall, R., Oxley, J., and Semple, C., The structure of 3-connected matroids of path width three. *Europ. J. Combin.* **28** (2007) 964-989.

[3] J. Oxley, C. Semple, and G. Whittle, Maintaining 3-connectivity relative to a fixed basis, *Adv. in Appl. Math*, **41** (2008), 1–9.

[4] W. T. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966), 1301–1324.

[5] G. Whittle, Stabilizers of Classes of Representable Matroids, *J. Comb. Theory Ser. B*, **77** (1999) 39?72

School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, Wellington, New Zealand

E-mail address: `geoff.whittle@vuw.ac.nz`

School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, Wellington, New Zealand

E-mail address: `alan.williams@msor.vuw.ac.nz`