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Is the missing axiom of matroid theory lost forever?

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Abstract

We conjecture that it is not possible to finitely axiomatize matroid representability in monadic second-order logic for matroids, and we describe some partial progress towards this conjecture. We present a collection of sentences in monadic second-order logic and show that it is possible to finitely axiomatize matroids using only sentences in this collection. Moreover, we can also axiomatize representability over any fixed finite field (assuming Rota's conjecture holds). We prove that it is not possible to finitely axiomatize representability, or representability over any fixed infinite field, using sentences from the collection.

1. *Introduction*

The problem of characterizing representable matroids is an old one. (When we say that a matroid is representable, we mean that it is representable over at least one field.) Whitney discusses the task of 'characterizing systems which represent matrices' in his foundational paper [**17**]. From the context, it seems likely that he means characterizing via a list of axioms. We believe that this task will never be completed. In other words, we conjecture that 'the missing axiom of matroid theory is lost forever'.

Conjecture 1.1. *It is not possible to finitely axiomatize representability for (finite) matroids, using the same logical language as the matroid axioms.*

Of course, this conjecture is not well-posed, unless we specify exactly what the language of matroid axioms is. Certainly, a logic powerful enough to express the existence of a matrix over a field whose columns have the required pattern of independence would suffice to axiomatize representability, but this logic would need to be much more powerful than the language typically used to axiomatize matroids. Conjecture 1.2 is an attempt to make Conjecture 1.1 more precise. In our main result (Theorem 1.3), we demonstrate that a weakened version of Conjecture 1.2 is true.

In Section 2 we develop *monadic second-order logic for matroids* (MSOL). Hliněný [**4**] introduced a logical language with the same name. It is easy to see that any sentence in Hliněný's language can be translated into a sentence in our language. In MSOL we are allowed to quantify over variables that are intended to represent elements or subsets of a ground set. We admit the function that takes a subset to its cardinality. We allow ourselves the relations of equality, element containment, set inclusion, and the 'less than or equal' order on integers. In addition, we also include a function, r, that takes subsets of the ground set to non-negative integers. This is intended to be interpreted as a rank function. As an example of the expressive capabilities of MSOL, a matroid is paving if and only if its rank function obeys the following sentence.

$$\forall X_1\ |X_1| < r(E) \rightarrow r(X_1) = |X_1|$$

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The matroid rank axioms can be stated as sentences in MSOL. (Throughout the article we consider a matroid to be a finite set equipped with a rank function.) Moreover, for any matroid N, we can construct a sentence in MSOL that will be true for a matroid M if and only if M has an N-minor (Proposition 3.2). This means that if Rota's conjecture is true, then GF(q)-representability can be finitely axiomatized in MSOL, for any prime power q (Lemma 3.1). We conjecture that it is impossible to finitely axiomatize representability in MSOL.

CONJECTURE 1.2. *There is no finite set of sentences, $\mathcal{K}$, in MSOL with the following property: a finite set, $E^{\mathcal{M}}$, equipped with a function $r^{\mathcal{M}}\colon \mathcal{P}(E^{\mathcal{M}}) \to \mathbb{Z}^{+} \cup \{0\}$, is a representable matroid if and only if $(E^{\mathcal{M}}, r^{\mathcal{M}})$ satisfies the rank axioms and every sentence in $\mathcal{K}$.*

Our main result (Theorem 1.3) shows that Conjecture 1.2 is true if we insist that the sentences in $\mathcal{K}$ must come from a restricted subset of MSOL. We use the terminology *M-logic* to describe a set of formulas in MSOL with constrained quantification. A formula in M-logic must have the following property: all variables representing subsets receive the same type of quantifier (universal or existential), and the same constraint applies to variables representing elements. M-logic is defined formally in Section 2.3.

If $\mathcal{F}$ is a collection of fields, let $M(\mathcal{F})$ be the set of matroids that are representable over at least one field in $\mathcal{F}$. Note that if $\mathcal{F}$ is the set of all fields, then $M(\mathcal{F})$ is the set of representable matroids.

THEOREM 1.3. *Let $\mathcal{F}$ be a set of fields that contains at least one infinite field. There does not exist a finite set, $\mathcal{K}$, of sentences in M-logic with the following property: a finite set, $E^{\mathcal{M}}$, equipped with a function $r^{\mathcal{M}}\colon \mathcal{P}(E^{\mathcal{M}}) \to \mathbb{Z}^{+} \cup \{0\}$, is a matroid in $M(\mathcal{F})$ if and only if $(E^{\mathcal{M}}, r^{\mathcal{M}})$ satisfies the rank axioms and every sentence in $\mathcal{K}$.*

We are interested in M-logic because it provides a separation between representability over finite fields and infinite fields. The axioms for matroid rank functions, independent sets, bases, and spanning sets can all be expressed using sentences in M-logic (Section 3.1). Moreover, if Rota's conjecture holds, then representability over a finite field can be finitely axiomatized using sentences in M-logic (Corollary 3.3). Theorem 1.3 shows this is not the case for any infinite field.

The reader may be puzzled by our titular question, since it is seemingly answered by a well-known article due to Vámos's [**14**]. His article has the dramatic title 'The missing axiom of matroid theory is lost forever'. When we examined the article, we were surprised to discover that the words 'matroid' and 'axiom' in his title were not used in the way we expected. Vamos's result has been interpreted as making a statement about finite matroids [**2**]; this is certainly what we anticipated. But in the title of his paper, 'matroid' refers to a potentially infinite object. Furthermore, it seems natural to use 'axiom' to mean a sentence constructed in the same language as the other matroid axioms, but Vámos uses it to mean a sentence in a language which we call V-logic. This logic is not capable of expressing the matroid axioms (as they are presented in [**10**, **16**, **17**]).

A *V-matroid* is a (possibly infinite) set E, along with a family, $\mathcal{I}$, of finite subsets such that: (**I1**) $\emptyset \in \mathcal{I}$, (**I2**) if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$, and (**I3**) if $I, I' \in \mathcal{I}$ and $|I| = |I'| + 1$, then there is an element $x \in I - I'$ such that $I' \cup x \in \mathcal{I}$. Note that every finite V-matroid is a matroid.

The first-order language that we call *V-logic* features, for every positive integer n, an n-ary predicate, I_n. The statement $I_n(x_1, \ldots, x_n)$ is designed to be interpreted as saying that

$\{x_1, \ldots, x_n\}$ is in $\mathcal{I}$. Then V-matroids can be axiomatized in V-logic. Let $\mathcal{A}$ be a set of sentences in V-logic that has the set of V-matroids as its models. For example, $\mathcal{A}$ might contain, for every n, the sentence

$$\forall x_1 \cdots \forall x_n \; I_n(x_1, \ldots, x_n) \to \bigwedge_{\sigma \in S_n} I_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

to ensure that $\mathcal{I}$ consists of unordered sets. It could also contain, for every n, the sentence

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_{n+1} \; I_n(x_1, \ldots, x_n) \wedge I_{n+1}(y_1, \ldots, y_{n+1}) \to \\ I_{n+1}(x_1, \ldots, x_n, y_1) \vee \cdots \vee I_{n+1}(x_1, \ldots, x_n, y_{n+1})$$

to ensure that (**I3**) holds.

Vámos declares a V-matroid, $(E, \mathcal{I})$, to be *representable* if there is a function from E to a vector space that preserves the rank of finite subsets. His theorem is as follows.

THEOREM 1.4 [**14**]. *There is no sentence, S, in V-logic, such that a V-matroid is representable if and only if it satisfies S.*

In our opinion, the most interesting questions about axiomatizing representability (including the question posed by our title) are left unanswered by Theorem 1.4. Here are three reasons why we hold this opinion.

V-matroids cannot be finitely axiomatized. We are accustomed to matroids being axiomatized with two, or three, or four sentences. A language that requires an infinite number of sentences to axiomatize matroids (or matroid-like objects) seems unnatural. Clearly $\mathcal{A}$ contains an infinite number of sentences. It is an easy exercise to show that no finite set of sentences in V-logic has the class of V-matroids as its set of models. Therefore V-matroids cannot be finitely axiomatized in V-logic.

Given that characterizing V-matroids requires infinitely many axioms in V-logic, we are not surprised to learn from Theorem 1.4 that representable V-matroids cannot be characterized with a single additional sentence. In fact, we would go further, and conjecture that no 'natural' class of V-matroids can be characterized by adding a single sentence to the list of V-matroid axioms. (We are being deliberately vague about the meaning of the word 'natural'.)

V-matroids are not matroids. V-matroids have not been studied nearly as much as (finite) matroids or independence spaces. An *independence space* is a possibly infinite set E, along with a family, $\mathcal{I}$, of possibly infinite subsets such that (**I1**) and (**I2**) hold, and (**I3**) holds when I and I' are finite sets. In addition, for every $X \subseteq E$, if all finite subsets of $X \subseteq E$ are in $\mathcal{I}$, then X is in $\mathcal{I}$ [**9**].

The fact that representability for V-matroids cannot be finitely axiomatized in V-logic does not tell us if the same statement applies for matroids or independence spaces. In fact, Vámos's proof strategy is intrinsically unable to prove that Theorem 1.4 holds for either of these two classes. The strategy relies upon the Compactness Theorem of first-order logic, so it can only be used to prove statements about classes that contain infinite objects. Furthermore, independence spaces cannot be axiomatized in V-logic, since first-order languages cannot distinguish between finite and infinite sets.

For a peculiar example of a V-matroid, consider an infinite set E, and let $\mathcal{I}$ be the collection of all finite subsets of E (c.f. [**9**, Example 3.1.1]). This is a V-matroid that has no maximal independent sets, and no minimal dependent sets. Perhaps examples such as this explain why V-matroids have not attracted much attention. In fact, so far as we are aware, Theorem 1.4 is the only result specifically about V-matroids that appears in the mathematical literature.

The proof is incomplete. Vámos's proof of Theorem 1.4 contains a gap. The proof depends on the statement that every non-representable V-matroid contains a finite restriction that is non-representable ([**14**, Lemma 0]). To support this claim, Vámos relies on [**12**], [**13**], and [**15**]. However these sources contain no results about V-matroids. All the statements they make are about independence spaces. As we have seen, not every V-matroid is an independence space.

We conclude this introduction by briefly describing the strategy for proving Theorem 1.3. The first step involves developing an infinite family of matroids, each of which is representable over all infinite fields (Section 4). Each matroid in the family has a number of circuit-hyperplanes, and relaxing any one produces a non-representable matroid, while relaxing two produces another matroid representable over all infinite fields. If there is a finite axiomatization of representability, then that set of axioms must be able to distinguish between these matroids. Roughly speaking, we obtain a contradiction by showing that, for large enough matroids in the family, the number of circuit-hyperplanes is so great that an axiom with a bounded number of variables cannot detect all the potential relaxations.

2. *A language for matroids*

In this section we develop monadic second-order logic for matroids, and we describe M-logic as a set of formulas in MSOL.

2.1. *Monadic second-order logic*

Monadic second-order logic for matroids is a formal language constructed from the following symbols: the variables $\{x_1, x_2, x_3, \ldots\}$ and $\{X_1, X_2, X_3, \ldots\}$; the constants $\{0, 1, 2, \ldots\}$, $\emptyset$, and E; the function symbols $|\cdot|$, $\{\cdot\}$, $\bar{\cdot}$, $r(\cdot)$, $+$, $\cup$, and $\cap$; the relation symbols $=$, $\in$, $\subseteq$, and $\leq$; and the logical symbols $\neg$, $\vee$, $\wedge$, $\exists$, and $\forall$.

Terms. We divide the terms in MSOL into three classes, $\mathcal{E}$, $\mathcal{S}$, and $\mathcal{N}$. Let $\mathcal{E}$ be the infinite set of variables $\{x_1, x_2, x_3, \ldots\}$. The terms in $\mathcal{E}$ are intended to represent elements of a ground set.

The set of terms in $\mathcal{S}$ is the smallest collection of expressions satisfying:

(1) the constants E and $\emptyset$ are terms in $\mathcal{S}$,
(2) every variable X_i is a term in $\mathcal{S}$,
(3) if x_i is a variable in $\mathcal{E}$, then $\{x_i\}$ is a term in $\mathcal{S}$,
(4) if X and Y are terms in $\mathcal{S}$, then so are $\overline{X}$, $X \cup Y$, and $X \cap Y$.

The terms in $\mathcal{S}$ are intended to represent subsets of a ground set.

Finally, we define the terms in $\mathcal{N}$. These are intended to represent non-negative integers. The set of terms in $\mathcal{N}$ is the smallest set of expressions satisfying:

(1) every constant in $\{0, 1, 2, \ldots\}$ is a term in $\mathcal{N}$,
(2) if X is a term in $\mathcal{S}$, then $|X|$ and $r(X)$ are terms in $\mathcal{N}$,
(3) if p and q are terms in $\mathcal{N}$, then $p + q$ is a term in $\mathcal{N}$.

If T is a term, then we recursively define $\mathrm{Var}(T)$ to be the set of variables in T:

(1) $\mathrm{Var}(E)$ and $\mathrm{Var}(\emptyset)$ are empty, and so is $\mathrm{Var}(p)$, for any constant $p \in \{0, 1, 2, \ldots\}$,
(2) $\mathrm{Var}(X_i) = \{X_i\}$,
(3) $\mathrm{Var}(x_i) = \mathrm{Var}(\{x_i\}) = \{x_i\}$,
(4) $\mathrm{Var}(\overline{X}) = \mathrm{Var}(|X|) = \mathrm{Var}(r(X)) = \mathrm{Var}(X)$, for any term $X \in \mathcal{S}$,
(5) $\mathrm{Var}(X \cup Y) = \mathrm{Var}(X \cap Y) = \mathrm{Var}(X) \cup \mathrm{Var}(Y)$, for any terms $X, Y \in \mathcal{S}$,
(6) $\mathrm{Var}(p + q) = \mathrm{Var}(p) \cup \mathrm{Var}(q)$, for any terms $p, q \in \mathcal{N}$.

Formulas. Now we recursively define formulas in MSOL, and simultaneously define their sets of variables. An *atomic formula* is one of the following expressions:

(1) if $x, y \in \mathcal{E}$, then $x = y$ is an atomic formula, and $\mathrm{Var}(x = y) = \{x, y\}$.
(2) if $X, Y \in \mathcal{S}$, then $X = Y$ and $X \subseteq Y$ are atomic formulas, and $\mathrm{Var}(X = Y) = \mathrm{Var}(X \subseteq Y) = \mathrm{Var}(X) \cup \mathrm{Var}(Y)$,
(3) if $p, q \in \mathcal{N}$, then $p = q$ and $p \leq q$ are atomic formulas, and $\mathrm{Var}(p = q) = \mathrm{Var}(p \leq q) = \mathrm{Var}(p) \cup \mathrm{Var}(q)$,
(4) if $x \in \mathcal{E}$ and $X \in \mathcal{S}$, then $x \in X$ is an atomic formula, and $\mathrm{Var}(x \in X) = \mathrm{Var}(X) \cup \{x\}$,

A *formula* is an expression generated by a finite application of the following rules. Every formula has an associated set of variables and *free variables*:

(1) every atomic formula P is a formula, and $\mathrm{Fr}(P) = \mathrm{Var}(P)$,
(2) if P is a formula and $X_i \in \mathrm{Fr}(P)$, then $\exists X_i P$ and $\forall X_i P$ are formulas, and $\mathrm{Var}(\exists X_i P) = \mathrm{Var}(\forall X_i P) = \mathrm{Var}(P)$, while $\mathrm{Fr}(\exists X_i P) = \mathrm{Fr}(\forall X_i P) = \mathrm{Fr}(P) - \{X_i\}$,
(3) if P is a formula and $x_i \in \mathrm{Fr}(P)$, then $\exists x_i P$ and $\forall x_i P$ are formulas, and $\mathrm{Var}(\exists x_i P) = \mathrm{Var}(\forall x_i P) = \mathrm{Var}(P)$, while $\mathrm{Fr}(\exists x_i P) = \mathrm{Fr}(\forall x_i P) = \mathrm{Fr}(P) - \{x_i\}$,
(4) if P is a formula, then $\neg P$ is a formula, and $\mathrm{Var}(\neg P) = \mathrm{Var}(P)$ while $\mathrm{Fr}(\neg P) = \mathrm{Fr}(P)$,
(5) if P and Q are formulas, and $\mathrm{Fr}(P) \cap (\mathrm{Var}(Q) - \mathrm{Fr}(Q)) = \emptyset = (\mathrm{Var}(P) - \mathrm{Fr}(P)) \cap \mathrm{Fr}(Q)$, then $P \vee Q$ and $P \wedge Q$ are formulas, and $\mathrm{Var}(P \vee Q) = \mathrm{Var}(P \wedge Q) = \mathrm{Var}(P) \cup \mathrm{Var}(Q)$, while $\mathrm{Fr}(P \vee Q) = \mathrm{Fr}(P \wedge Q) = \mathrm{Fr}(P) \cup \mathrm{Fr}(Q)$.

A *sentence* in MSOL is a formula P satisfying $\mathrm{Fr}(P) = \emptyset$.

REMARK 1. In (5) we insist that no variable is free in exactly one of P and Q when we construct the formulas $P \vee Q$ and $P \wedge Q$. This is standard (see, for example, [**7**, p. 10]) and imposes no real difficulties, since a variable that is not free can always be relabeled. For example, $(X_1 = X_2) \wedge (\exists X_1\, |X_1| = 1)$ is not a formula, but we can rewrite it as $(X_1 = X_2) \wedge (\exists X_3\, |X_3| = 1)$.

Abbreviations. We allow several standard shorthands. If P and Q are formulas then $P \to Q$ is a shorthand for $\neg P \vee Q$. If $x \in \mathcal{E}$ and $X \in \mathcal{S}$, then $x \notin X$ is shorthand for $\neg(x \in X)$. If $p, q \in \mathcal{N}$, then $p < q$ is shorthand for $p \leq q \wedge \neg(p = q)$. If $X, Y \in \mathcal{S}$, then $X - Y$ is shorthand for the term $X \cap \overline{Y}$, and $X \nsubseteq Y$ is shorthand for the formula $\neg(X \subseteq Y)$. In addition, we are casual with the use of parentheses, inserting them freely to reduce ambiguity, and omitting them when this will cause no confusion.

2.2. *Structures and satisfiability*

We have constructed MSOL as a collection of formally defined strings. In this section we are going to consider how to interpret these strings as statements about a set system. A *structure*, $\mathcal{M}$, consists of a pair $(E^{\mathcal{M}}, r^{\mathcal{M}})$, where $E^{\mathcal{M}}$ is a finite set and $r^{\mathcal{M}}$ is a function from $\mathcal{P}(E^{\mathcal{M}})$, the power set of $E^{\mathcal{M}}$, to the non-negative integers.

Let $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ be a structure, and let P be a formula in MSOL. Let $\phi_{\mathcal{S}}$ be a function from $\mathrm{Fr}(P) \cap \mathcal{S}$ to $\mathcal{P}(E^{\mathcal{M}})$ and let $\phi_{\mathcal{E}}$ be a function from $\mathrm{Fr}(P) \cap \mathcal{E}$ to $E^{\mathcal{M}}$. We call the pair $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ an *interpretation* of P. Note that an interpretation of a sentence necessarily consists of two empty functions. We are going to recursively define what it means for the structure $\mathcal{M}$ to satisfy $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$.

First, we create a correspondence between terms in $\mathcal{E}$, $\mathcal{S}$, and $\mathcal{N}$, and elements of $E^{\mathcal{M}}$, subsets of $E^{\mathcal{M}}$, and non-negative integers. If x_i is a term in $\mathcal{E}$, and x_i is in the domain of

$\phi_{\mathcal{E}}$, then the notation† $x_i^{\mathcal{M}}$ stands for $\phi_{\mathcal{E}}(x_i)$. Similarly, if X is a term in $\mathcal{S}$, and $\mathrm{Var}(X) \subseteq \mathrm{Dom}(\phi_{\mathcal{S}}) \cup \mathrm{Dom}(\phi_{\mathcal{E}})$, then $X^{\mathcal{M}}$ is the corresponding subset of $E^{\mathcal{M}}$, recursively defined as follows:

(1) if $X = E$, then $X^{\mathcal{M}} = E^{\mathcal{M}}$, and if $X = \emptyset$, then $X^{\mathcal{M}}$ is the empty subset,
(2) if X is the variable X_i, then $X_i^{\mathcal{M}} = \phi_{\mathcal{S}}(X_i)$,
(3) if $X = \{x_i\}$ for some variable x_i, then $X^{\mathcal{M}} = \{\phi_{\mathcal{E}}(x_i)\}$,
(4) if $X = \overline{Y}$, for some $Y \in \mathcal{S}$, then $X^{\mathcal{M}} = E^{\mathcal{M}} - Y^{\mathcal{M}}$, and if X is equal, respectively, to $Y \cup Z$ or $Y \cap Z$, where $Y, Z \in \mathcal{S}$, then $X^{\mathcal{M}}$ is, respectively, $Y^{\mathcal{M}} \cup Z^{\mathcal{M}}$ or $Y^{\mathcal{M}} \cap Z^{\mathcal{M}}$.

Now let p be a term in $\mathcal{N}$ such that $\mathrm{Var}(p) \subseteq \mathrm{Dom}(\phi_{\mathcal{S}}) \cup \mathrm{Dom}(\phi_{\mathcal{E}})$. Then $p^{\mathcal{M}}$ is the corresponding non-negative integer, defined as follows:

(1) if p is a constant in $\mathcal{N}$, then $p^{\mathcal{M}}$ is the corresponding non-negative integer,
(2) if p is $|X|$ or $r(X)$, where X is a term in $\mathcal{S}$, then $p^{\mathcal{M}}$ is, respectively, $|X^{\mathcal{M}}|$, or $r^{\mathcal{M}}(X^{\mathcal{M}})$,
(3) if p is $q + r$, for some terms $q, r \in \mathcal{N}$, then $p^{\mathcal{M}}$ is $q^{\mathcal{M}} + r^{\mathcal{M}}$.

Now we are able to recursively define when $\mathcal{M}$ *satisfies* $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$. First we consider the case that P is an atomic formula:

(1) if P is $x = y$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if $x^{\mathcal{M}} = y^{\mathcal{M}}$,
(2) if P is, respectively, $X = Y$ or $X \subseteq Y$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if, respectively, $X^{\mathcal{M}} = Y^{\mathcal{M}}$ or $X^{\mathcal{M}} \subseteq Y^{\mathcal{M}}$,
(3) if P is, respectively, $p = q$ or $p \leq q$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if, respectively, $p^{\mathcal{M}} = q^{\mathcal{M}}$ or $p^{\mathcal{M}} \leq q^{\mathcal{M}}$,
(4) if P is $x \in X$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if $x^{\mathcal{M}} \in X^{\mathcal{M}}$.

Next we consider the case that P is not atomic:

(1) if $P = \exists X_i Q$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if there is some subset $X_i' \subseteq E^{\mathcal{M}}$ such that $Q(\phi_{\mathcal{S}} \cup (X_i, X_i'), \phi_{\mathcal{E}})$ is satisfied; and if $P = \forall X_i Q$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if $Q(\phi_{\mathcal{S}} \cup (X_i, X_i'), \phi_{\mathcal{E}})$ is satisfied for every subset $X_i' \subseteq E^{\mathcal{M}}$,
(2) if $P = \exists x_i Q$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if there is some element $x_i' \in E^{\mathcal{M}}$ such that $Q(\phi_{\mathcal{S}}, \phi_{\mathcal{E}} \cup (x_i, x_i'))$ is satisfied; and if $P = \forall x_i Q$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if $Q(\phi_{\mathcal{S}}, \phi_{\mathcal{E}} \cup (x_i, x_i'))$ is satisfied for every element $x_i' \in E^{\mathcal{M}}$,
(3) if $P = \neg Q$ is a formula, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if $Q(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is not satisfied,
(4) if $P = Q \vee R$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if either $Q(\phi_{\mathcal{S}}|_{\mathrm{Fr}(Q)\cap\mathcal{S}}, \phi_{\mathcal{E}}|_{\mathrm{Fr}(Q)\cap\mathcal{E}})$ or $R(\phi_{\mathcal{S}}|_{\mathrm{Fr}(R)\cap\mathcal{S}}, \phi_{\mathcal{E}}|_{\mathrm{Fr}(R)\cap\mathcal{E}})$ is satisfied; and if $P = Q \wedge R$, then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied if both $Q(\phi_{\mathcal{S}}|_{\mathrm{Fr}(Q)\cap\mathcal{S}}, \phi_{\mathcal{E}}|_{\mathrm{Fr}(Q)\cap\mathcal{E}})$ and $R(\phi_{\mathcal{S}}|_{\mathrm{Fr}(R)\cap\mathcal{S}}, \phi_{\mathcal{E}}|_{\mathrm{Fr}(R)\cap\mathcal{E}})$ are satisfied.

Let $\mathcal{M}$ be a structure, and let P be a sentence in MSOL. We say that $\mathcal{M}$ satisfies P if it satisfies $P(\emptyset, \emptyset)$; that is, if it satisfies P under the empty interpretation. If $\mathcal{T}$ is a set of sentences, then $\mathcal{M}$ *satisfies* $\mathcal{T}$ if it satisfies every sentence in $\mathcal{T}$.

2.3. *M-logic*

Now we describe M-logic as a set of formulas from MSOL. Let a be a variable. Note that $\neg\exists a P$ is equivalent to $\forall a \neg P$, in the sense that a structure satisfies one of these formulas if and only if it satisfies both. Similarly, $\neg\forall a P$ is equivalent to $\exists a \neg P$. Now suppose that $P \vee (\exists a Q)$ is a formula. Then a is not free in P, and $P \vee (\exists a Q)$ is equivalent to $\exists a(P \vee Q)$. Similarly, $P \wedge (\forall a Q)$ is equivalent to $\forall a(P \wedge Q)$. This discussion means that every formula in MSOL is equivalent to a formula of the form $Q_1 a_1 \cdots Q_t a_t P$, where each Q_i is in $\{\exists, \forall\}$, each a_i is a variable, and P is a formula that contains no quantifiers.

A formula of the form $\exists x Q_1 a_1 \cdots Q_t a_t P$, where x is a variable in $\mathcal{E}$, is equivalent to

$$\exists X Q_1 a_1 \cdots Q_t a_t \forall x (X = \{x\}) \rightarrow P$$

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†Technically, we should write $x_i^{(\mathcal{M}, \phi_{\mathcal{E}})}$, since the element corresponding to x_i depends on the interpretation as well as the structure.

where X is a new variable in $\mathcal{S}$. Similarly, $\forall x Q_1 a_1 \cdots Q_t a_t P$ is equivalent to $\forall X Q_1 a_1 \cdots Q_t a_t \forall x (X = \{x\}) \to P$. From this discussion we see that every formula in MSOL is equivalent to a formula of the form

$$Q_{i_1} X_{i_1} \cdots Q_{i_m} X_{i_m} Q_{j_1} x_{j_1} \cdots Q_{j_n} x_{j_n} P$$

where $X_{i_1}, \ldots, X_{i_m}$ and $x_{j_1}, \ldots, x_{j_n}$ are variables in $\mathcal{S}$ and $\mathcal{E}$ respectively, where each Q_k is in $\{\exists, \forall\}$, and $\mathrm{Var}(P) = \mathrm{Fr}(P)$ (c.f. [**1**, p. 39]). We say that this formula is in *M-logic* if $\{Q_{i_1}, \ldots, Q_{i_m}\}$ is either $\{\exists\}$ or $\{\forall\}$, and similarly $\{Q_{j_1}, \ldots, Q_{j_n}\}$ is either $\{\exists\}$ or $\{\forall\}$. That is, M-logic is the collection of formulas in MSOL that are equivalent to a formula of the form $Q_{i_1} X_{i_1} \cdots Q_{i_m} X_{i_m} Q_{j_1} x_{j_1} \cdots Q_{j_n} x_{j_n} P$, where P is quantifier-free, and $Q_k = Q_l$ for all $k, l \in \{i_1, \ldots, i_m\}$ and all $k, l \in \{j_1, \ldots, j_n\}$.

3. *Matroid axioms*

In this section we show that M-logic is expressive enough to make natural statements about matroids. Some common axiom schemes for matroids can be expressed using sentences in M-logic. Furthermore, if N is a fixed matroid, then there is a sentence in M-logic that characterizes having a minor isomorphic to N. Throughout the section, we will let $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ be a structure (recall this implies $E^{\mathcal{M}}$ is finite).

3.1. *Axioms*

We consider a matroid to be a finite set equipped with a function obeying the rank axioms. Thus $(E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid if and only if $\mathcal{M}$ satisfies the following sentences in M-logic.

R1 $\forall X_1\ r(X_1) \leq |X_1|$

R2 $\forall X_1 \forall X_2\ X_1 \subseteq X_2 \to r(X_1) \leq r(X_2)$

R3 $\forall X_1 \forall X_2\ r(X_1 \cup X_2) + r(X_1 \cap X_2) \leq r(X_1) + r(X_2)$

Let $I(X)$ be shorthand for $r(X) = |X|$, where $X \in \mathcal{S}$. Then $(E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid with $\{X \subseteq E^{\mathcal{M}} \mid r^{\mathcal{M}}(X) = |X|\}$ as its family of independent sets if and only if $\mathcal{M}$ satisfies the following sentences.

I1 $I(\emptyset)$

I2 $\forall X_1 \forall X_2\ I(X_2) \wedge X_1 \subseteq X_2 \to I(X_1)$

I3 $\forall X_1 \forall X_2 \exists x_1\ I(X_1) \wedge I(X_2) \wedge |X_1| < |X_2| \to x_1 \notin X_1 \wedge x_1 \in X_2 \wedge I(X_1 \cup \{x_1\})$

Let $B(X)$ be shorthand for $r(X) = |X| \wedge r(X) = r(E)$, where $X \in \mathcal{S}$. Then $(E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid with $\{X \subseteq E^{\mathcal{M}} \mid r^{\mathcal{M}}(X) = |X| = r^{\mathcal{M}}(E^{\mathcal{M}})\}$ as its family of bases if and only if $\mathcal{M}$ satisfies the following sentences.

B1 $\exists X_1\ B(X_1)$

B2 $\forall X_1 \forall X_2 \forall X_3 \exists x_1\ B(X_1) \wedge B(X_2) \wedge |X_3| = 1 \wedge X_3 \subseteq X_1 \wedge X_3 \nsubseteq X_2 \to$
$x_1 \notin X_1 \wedge x_1 \in X_2 \wedge B((X_1 - X_3) \cup \{x_1\})$

Note that the natural form of the basis-exchange axiom is

> 'for every basis B, and for every basis B', and for every element $x \in B - B'$, there exists an element $y \in B' - B$ such that ...'

This statement cannot be expressed directly in M-logic. We sidestep this problem by using the set variable, X_3, to represent the single element x.

Let $S(X)$ be shorthand for $r(X) = r(E)$. Then $(E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid with $\{X \subseteq E^{\mathcal{M}} \mid r^{\mathcal{M}}(X) = r^{\mathcal{M}}(E^{\mathcal{M}})\}$ as its set of spanning sets if and only if $\mathcal{M}$ satisfies the following sentences.

S1 $\exists X_1\ S(X_1)$

S2 $\forall X_1 \forall X_2\ S(X_1) \wedge X_1 \subseteq X_2 \to S(X_2)$

S3 $\forall X_1 \forall X_2 \exists x_1\ S(X_1) \wedge S(X_2) \wedge |X_1| < |X_2| \to x_1 \notin X_1 \wedge x_1 \in X_2 \wedge S(X_2 - \{x_1\})$

3.2. *Axiomatising* GF(q)*-representability*

M-logic is strong enough so that representability over any finite field can be axiomatized with a finite number of sentences, assuming that Rota's conjecture is true. This assumption implies that there is a finite number of excluded minors for GF(q)-representability, for any prime power q. In this section we prove the following result.

LEMMA 3.1. *Assume that Rota's conjecture is true. For every finite field* GF(q)*, there is a finite set of sentences, $\mathcal{Q}$, in M-logic, with the following property: the structure $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ is a* GF(q)*-representable matroid if and only if $\mathcal{M}$ satisfies $\{\mathbf{R1}, \mathbf{R2}, \mathbf{R3}\} \cup \mathcal{Q}$.*

Indeed, any minor-closed class with finitely many excluded minors can be finitely axiomatized in M-logic (Corollary 3.3). However, the converse is not obviously true. There may be a minor-closed class with infinitely many excluded minors that can be finitely axiomatized in M-logic. Theorem 1.3 shows that this is not the case with the class of matroids representable over an infinite field. Any such class has an infinite number of excluded minors [**10**, Theorem 6.5.17], and by Theorem 1.3, any such class is impossible to finitely axiomatize using M-logic.

Lemma 3.1 follows immediately from the next two results.

PROPOSITION 3.2. *Let N be a matroid. There is a sentence, $\mathbf{S}_N$, in M-logic, such that the structure $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid with an N-minor if and only if $\mathcal{M}$ satisfies $\{\mathbf{R1}, \mathbf{R2}, \mathbf{R3}, \mathbf{S}_N\}$.*

Proof. Let the ground set of N be $T = \{1, \ldots, m\}$. For every subset $S \subseteq T$, let $r_N(S)$ denote the rank of S in N. Let P_N be the formula

$$(r(X_1) = |X_1|) \wedge \left(X_1 \cap \bigcup_{i=1}^{m} \{x_i\} = \emptyset\right) \wedge \left(\left|\bigcup_{i=1}^{m} \{x_i\}\right| = m\right) \wedge \bigwedge_{S \subseteq T} r\left(X_1 \cup \bigcup_{i \in S} \{x_i\}\right) = r(X_1) + r_N(S).$$

Assume that $\mathcal{M}$ satisfies $\{\mathbf{R1}, \mathbf{R2}, \mathbf{R3}\}$, so that $(E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid. Then $\mathcal{M}$ satisfies $P_N(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ if and only if $\phi_{\mathcal{S}}(X_1)$ is independent, the set $\{\phi_{\mathcal{E}}(x_1), \ldots, \phi_{\mathcal{E}}(x_m)\}$ contains m distinct elements and is disjoint from $\phi_{\mathcal{S}}(X_1)$, and the matroid produced by contracting $\phi_{\mathcal{S}}(X_1)$ and restricting to $\{\phi_{\mathcal{E}}(x_1), \ldots, \phi_{\mathcal{E}}(x_m)\}$ has the same rank function as N. Thus $\mathbf{S}_N = \exists X_1 \exists x_1 \cdots \exists x_m \ P_N$ is the desired sentence. □

COROLLARY 3.3. *If $\mathcal{N}$ is a minor-closed class of matroids with a finite number of excluded minors, then there is a finite set of sentences, $\mathbf{S}(\mathcal{N})$, in M-logic, with the following property: the structure $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid in $\mathcal{N}$ if and only if $\mathcal{M}$ satisfies $\{\mathbf{R1}, \mathbf{R2}, \mathbf{R3}\} \cup \mathbf{S}(\mathcal{N})$.*

Proof. Let $N_1, \ldots, N_t$ be the list of excluded minors for $\mathcal{M}$. Notice that the negation of a sentence in M-logic is equivalent to another sentence in M-logic. We let $\mathbf{S}(\mathcal{N}) = \{\neg \mathbf{S}_{N_1}, \ldots, \neg \mathbf{S}_{N_t}\}$. □

4. *Kinser matroids*

In this section we construct an infinite family of matroids, which we call *Kinser matroids*. Let $r \geq 4$ be an integer. Then $\mathrm{Kin}(r)$ is a rank-r matroid with $r^2 - 3r + 4$ elements. For our purposes, the most important property of Kinser matroids is that they are representable over any infinite field, but can be made non-representable by relaxing a single circuit-hyperplane. To prove this fact, we are going to use the family of inequalities discovered by Kinser [**6**].

LEMMA 4.1. *Let M be a matroid that is representable over a field. If $X_1, \ldots, X_n$ is any collection of subsets of $E(M)$, where $n \geq 4$, then*

$$r(X_1 \cup X_2) + r(X_1 \cup X_3 \cup X_n) + r(X_3) + \sum_{i=4}^{n}(r(X_i) + r(X_2 \cup X_{i-1} \cup X_i)) \leq$$
$$r(X_1 \cup X_3) + r(X_1 \cup X_n) + r(X_2 \cup X_3) + \sum_{i=4}^{n}(r(X_2 \cup X_i) + r(X_{i-1} \cup X_i)).$$

Proof. Since M is representable, there is a function, ϕ, from $E(M)$ to some vector space, such that $r_M(S) = \dim\langle\{\phi(s) \mid x \in S\}\rangle$ for all subsets $S \subseteq E(M)$. If $S_1, \ldots, S_t$ is some collection of subsets of $E(M)$, then

$$r_M(S_1 \cup \cdots \cup S_t) = \dim\langle\{\phi(s) \mid s \in S_1 \cup \cdots \cup S_t\}\rangle = \dim \sum_{i=1}^{t}\langle\{\phi(s) \mid s \in S_i\}\rangle$$

where the sum in the last expression is the direct sum of vector subspaces. Now the result follows immediately from [**6**, Theorem 1]. □

We note here that if $n = 4$, then the inequality in Lemma 4.1 is identical to Ingleton's inequality for representable matroids [**5**].

As an intermediate step for constructing $\mathrm{Kin}(r)$, we define a rank-$(r+1)$ transversal matroid, M_{r+1}. The transversal system that describes M_{r+1} contains $r+1$ sets: $\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}, \mathcal{A}, \mathcal{A}'$. Let $H_1, \ldots, H_r$ be pairwise disjoint sets such that

$$|H_1| = \cdots = |H_{r-1}| = r - 2$$

and $H_r = \{e, f\}$. The ground set of M_{r+1} is $H_1 \cup \cdots \cup H_r$. Let $\mathcal{A} = E(M_{r+1})$, and let $\mathcal{A}' = H_r$. For $i \in \{1, \ldots, r-1\}$, let

$$\mathcal{A}_i = (H_1 \cup \cdots \cup H_{r-1}) - (H_{i-1} \cup H_i)$$

(when appropriate we interpret subscripts modulo $r - 1$). Then M_{r+1} is the transversal matroid $M[\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}, \mathcal{A}, \mathcal{A}']$. We define $\mathrm{Kin}(r)$ to be the truncation, $T(M_{r+1})$, of M_{r+1}.

Thus, for example, $\mathrm{Kin}(4)$ is a rank-4 matroid with 8 elements, and its non-spanning circuits are all the 4-element subsets of the form $H_i \cup H_j$, where $i \neq j$. In fact, $\mathrm{Kin}(4)$ is also known as the rank-4 *tipless free spike* (see [**3**, page 136]).

We will use the next result in our proof that Kinser matroids are representable over infinite fields.

PROPOSITION 4.2. *Let $r \geq 3$ be an integer. Let P be the projective geometry $\mathrm{PG}(r-1, \mathbb{K})$, where $\mathbb{K}$ is an infinite field, and let $S_1, \ldots, S_t$ be a finite collection of proper subspaces of P. If S is a subspace of P that is not contained in any of $S_1, \ldots, S_t$, then S is not contained in $S_1 \cup \cdots \cup S_t$.*

Proof. Assume that the result is false, and that $S_1, \ldots, S_t$ have been chosen so that none of these subsets contains S, and yet $S_1 \cup \cdots \cup S_t$ does. Assume also that $S_1, \ldots, S_t$ has been chosen so that t is as small as possible. The hypotheses imply that $t > 1$. The minimality of t means that there is a point, p, in $S - (S_2 \cup \cdots \cup S_t)$ and another point, p', in $S - (S_1 \cup \cdots \cup S_{t-1})$. Let l be the line spanned by p and p'. Then l is contained in S, but every subspace S_i contains at most one point of l, for otherwise S_i contains l, and hence contains p and p'. Therefore l contains at most t points, contradicting the fact that $\mathbb{K}$ is infinite. □

PROPOSITION 4.3. *Let $\mathbb{K}$ be an infinite field. Then $\mathrm{Kin}(r)$ is $\mathbb{K}$-representable for any $r \geq 4$.*

Proof. Certainly M_{r+1} is $\mathbb{K}$-representable, as it is transversal (see [**10**, Corollary 11.2.17]). Consider a $\mathbb{K}$-representation of M_{r+1} as an embedding of $E(M_{r+1})$ in the projective space $\mathrm{PG}(r, \mathbb{K})$.

The non-spanning subsets of M_{r+1} span a finite number of proper subspaces of $\mathrm{PG}(r, \mathbb{K})$. We let $S = \mathrm{PG}(r, \mathbb{K})$, and apply Proposition 4.2. Thus there is a point $p \in \mathrm{PG}(r, \mathbb{K})$ that is not spanned by any non-spanning subset of $E(M_{r+1})$. Consider the $\mathbb{K}$-representable matroid, M'_{r+1}, represented by the subset $E(M_{r+1}) \cup p$ of $\mathrm{PG}(r, \mathbb{K})$. Then M'_{r+1} is a free extension of M_{r+1}; that is, the only circuits that contain p are spanning circuits. Contracting p produces the truncation $T(M_{r+1}) = \mathrm{Kin}(r)$. Since $M'_{r+1}/p = \mathrm{Kin}(r)$ is $\mathbb{K}$-representable, the proof is complete. □

PROPOSITION 4.4. *Let $r \geq 4$ be an integer. Then $H_s \cup H_r$ is a circuit-hyperplane of $\mathrm{Kin}(r)$ for any $s \in \{1, \ldots, r-1\}$.*

Proof. Let G be the bipartite graph that corresponds to the transversal system $(\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}, \mathcal{A}, \mathcal{A}')$. In G, the $r-2$ vertices in H_s are each adjacent to the $r-2$ vertices

$$\{\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}, \mathcal{A}\} - \{\mathcal{A}_s, \mathcal{A}_{s+1}\},$$

while the two vertices in H_r are adjacent only to $\mathcal{A}$ and $\mathcal{A}'$. Thus $H_s \cup H_r$ contains r vertices and has a neighbourhood set of $r-1$ vertices. Therefore $H_s \cup H_r$ is dependent, and in fact it is very easy to confirm that it is a circuit of M_{r+1}. Since it has cardinality r, it is also a circuit in $T(M_{r+1}) = \mathrm{Kin}(r)$.

Let x be an element in $E(M_{r+1}) - (H_s \cup H_r)$. Then x is adjacent to either $\mathcal{A}_s$ or $\mathcal{A}_{s+1}$ in G. Thus the vertices in $H_s \cup H_r \cup x$ are adjacent to r vertices, so

$$r_{M_{r+1}}(H_s \cup H_r \cup x) > r_{M_{r+1}}(H_s \cup H_r).$$

This shows that $H_s \cup H_r$ is a flat in M_{r+1}. As $r_{M_{r+1}}(H_s \cup H_r) = r - 1 = r(M_{r+1}) - 2$, it follows that $H_s \cup H_r$ is also a flat in $T(M_{r+1}) = \mathrm{Kin}(r)$, and is therefore a hyperplane of this matroid. Thus $H_s \cup H_r$ is a circuit-hyperplane of $\mathrm{Kin}(r)$. □

PROPOSITION 4.5. *Let $r \geq 4$ be an integer, and let s be in $\{1, \ldots, r-1\}$. The matroid obtained from $\mathrm{Kin}(r)$ by relaxing the circuit-hyperplane $H_s \cup H_r$ is not representable over any field.*

Proof. By relabeling $\mathcal{A}_i$ and H_i as $\mathcal{A}_{i-s+1}$ and H_{i-s+1} (modulo $r-1$) for each $i \in \{1, \ldots, r-1\}$, we can assume that $s = 1$. Let M be the matroid obtained from $\mathrm{Kin}(r)$ by relaxing $H_1 \cup H_r$. We prove that M is non-representable by setting $n = r$ and setting

$$(X_1, X_2, X_3, \ldots, X_n) = (H_1, H_r, H_2, \ldots, H_{r-1}),$$

and then applying Lemma 4.1 to M. Since $X_1 \cup X_2$ is a relaxed circuit-hyperplane in M, it follows that $r(X_1 \cup X_2) = r$. Note that $H_i \cup H_r$ is a circuit-hyperplane of M for any $i \in \{2, \ldots, r-1\}$. Thus, any set X_i, where $i \in \{3, \ldots, n\}$, is a $(r-2)$-element subset of a circuit-hyperplane. This means that $r(X_i) = r-2$. In particular, $r(X_3) = r(H_2) = r-2$. In the bipartite graph G, the vertices in H_2 are adjacent to the $r-2$ vertices $\mathcal{A}_1, \mathcal{A}_4, \ldots, \mathcal{A}_{r-1}, \mathcal{A}$. Each vertex in H_1 is adjacent to $\mathcal{A}_3$, while every vertex in H_{r-1} is adjacent to $\mathcal{A}_2$. These considerations imply that $H_1 \cup H_2 \cup H_{r-1}$ has rank at least r in M_{r+1}, and hence in M. Thus $r(X_1 \cup X_3 \cup X_n) = r$. For $i \in \{4, \ldots, n\}$, the set $X_2 \cup X_{i-1} = H_r \cup H_{i-2}$ is a circuit-hyperplane of M. It follows that $r(X_2 \cup X_{i-1} \cup X_i) = r$. Now the left-hand side of the inequality in Lemma 4.1 evaluates to

$$r + r + (r-2) + (r-3)[(r-2) + r] = 2r^2 - 5r + 4.$$

On the other hand, if $i \in \{1, \ldots, r-1\}$, then the neighbourhood in G of $H_i \cup H_{i+1}$ contains the $r-1$ vertices $\{\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}, \mathcal{A}\} - \mathcal{A}_{i+1}$. Thus $H_i \cup H_{i+1}$ has rank at most $r-1$ in M_{r+1}. In fact it has rank exactly $r-1$, as H_i as rank $r-2$, and any vertex in H_{i+1} is adjacent to $\mathcal{A}_i$, while no vertex in H_i is. Thus $H_i \cup H_{i+1}$ has rank $r-1$ in M. This shows that $r(X_1 \cup X_3)$, $r(X_1 \cup X_n)$, and $r(X_{i-1} \cup X_i)$ for $i \in \{4, \ldots, n\}$ are all equal to $r-1$. Furthermore, $X_2 \cup X_i$ is a circuit-hyperplane for all $i \in \{3, \ldots, n\}$, so has rank $r-1$. Now every term in the right-hand side of the inequality in Lemma 4.1 is equal to $r-1$, so this side evaluates to $(2(r-3)+3)(r-1) = 2r^2 - 5r + 3$. Thus the inequality in Lemma 4.1 does not hold, so M is not representable over any field. □

If $r \geq 4$ is an integer, then we define $\mathrm{Kin}(r)^-$ to be the matroid obtained from $\mathrm{Kin}(r)$ by relaxing the circuit-hyperplane $H_1 \cup H_r$. The previous result shows that $\mathrm{Kin}(r)^-$ is non-representable. Since $\mathrm{Kin}(4)$ is isomorphic to the rank-4 tipless free spike, it is easy to see that $\mathrm{Kin}(4)^-$ is the *Vámos matroid* (see [**10**, page 84] or [**11**]). In fact, we can think of $\mathrm{Kin}(n)^-$ as exemplifying matroids that fail the inequality in Lemma 4.1, in exactly the same way that the Vámos matroid exemplifies matroids that fail the Ingleton inequality [**5**].

Relaxing a single circuit-hyperplane in $\mathrm{Kin}(r)$ produces a non-representable matroid. We show in the next result that by relaxing two, we can recover representability over any infinite field.

Lemma 4.6. *Let $\mathbb{K}$ be an infinite field, let $r \geq 4$ be an integer, and let s and t be distinct members of $\{1, \ldots, r-1\}$. The matroid that is obtained from $\mathrm{Kin}(r)$ by relaxing the circuit-hyperplanes $H_s \cup H_r$ and $H_t \cup H_r$ is $\mathbb{K}$-representable.*

Proof. We assume that $s < t$. By relabeling $\mathcal{A}_i$ and H_i as $\mathcal{A}_{i-t+r-1}$ and $H_{i-t+r-1}$ for every $i \in \{1, \ldots, r-1\}$, we can assume that $t = r-1$. Relabel $s - t + r - 1$ as s. Let M be the matroid obtained from $\mathrm{Kin}(r)$ by relaxing $H_s \cup H_r$ and $H_{r-1} \cup H_r$. We aim to show that M is $\mathbb{K}$-representable.

We start by constructing a rank-r transversal matroid, M', on the ground set

$$E(M_{r+1} \backslash \{e, f\}) \cup \{p, p'\},$$

where p and p' are distinct elements, neither of which is in $E(M_{r+1})$. Let $\mathcal{A}_0$ be $E(M_{r+1} \backslash \{e, f\}) \cup \{p, p'\}$. For $i \in \{1, \ldots, s\}$, let $\mathcal{A}'_i$ be $\mathcal{A}_i \cup p$. For $i \in \{s+1, \ldots, r-1\}$, let $\mathcal{A}'_i$ be $\mathcal{A}_i \cup p'$. Let M' be the transversal matroid $M[\mathcal{A}'_1, \ldots, \mathcal{A}'_{r-1}, \mathcal{A}_0]$.

It is clear that $M'\backslash\{p,p'\} = M_{r+1}\backslash\{e,f\}$. Moreover, it is straightforward to verify that $\{e,f\}$ is a series pair in M_{r+1}, and from this it follows easily that

$$M_{r+1}\backslash\{e,f\} = T(M_{r+1})\backslash\{e,f\} = \mathrm{Kin}(r)\backslash\{e,f\}.$$

Thus $M'\backslash\{p,p'\} = \mathrm{Kin}(r)\backslash\{e,f\}$.

Since M' is transversal, it is $\mathbb{K}$-representable. We consider it as a subset of points in the projective space $P = \mathrm{PG}(r-1,\mathbb{K})$. Let l be the line of P that is spanned by p and p'.

4.6.1. Let X be a subset of $E(M_{r+1}\backslash\{e,f\})$ that is non-spanning in M'. Then X does not span l.

Proof. Assume otherwise. Then there is a subset of $E(M_{r+1}\backslash\{e,f\})$ that spans l and is independent and non-spanning in M'. Let X be such a subset. Thus X spans p and p'. Let $C \subseteq X \cup p$ be a circuit that contains p. Let c be an element in $C - p$. Then in G', the bipartite graph corresponding to the system $(\mathcal{A}'_1,\ldots,\mathcal{A}'_{r-1},\mathcal{A}_0)$, the vertex c has $r-2$ neighbours. Since the neighbourhood set of C is one element smaller than C, this means that $|C| \geq r-1$. Let $C' \subseteq X \cup p'$ be a circuit that contains p'. The same argument shows that $|C'| \geq r-1$. Since

$$r > |X| \geq |(C-p)\cup(C'-p')| \geq (2r-4) - |(C-p)\cap(C'-p')|$$

and $r \geq 4$, this means that there is an element, x, in $(C-p)\cap(C'-p')$. Assume that x is in one of $H_1,\ldots,H_{s-1}$. As p is adjacent to the vertices $\mathcal{A}_0,\mathcal{A}'_1,\ldots,\mathcal{A}'_s$, and x is adjacent to all vertices, other than two in $\mathcal{A}'_1,\ldots,\mathcal{A}'_s$, it follows that the neighbourhood set of C contains r vertices. This means that $|C| \geq r+1$, which is impossible as X is non-spanning. Similarly, if x is in one of $H_{s+1},\ldots,H_{r-2}$, then, as p' is adjacent to $\mathcal{A}'_{s+1},\ldots,\mathcal{A}'_{r-1}$, and x is adjacent to every vertex other than two in $\mathcal{A}'_{s+1},\ldots,\mathcal{A}'_{r-1}$, we deduce that $|C'| \geq r+1$. This contradiction means that $(C-p)\cap(C'-p)$ is contained in $H_s \cup H_{r-1}$. If $(C-p)\cap(C'-p')$ contains elements from both H_s and H_{r-1}, then the neighbourhood set of either C or C' contains all r vertices, and this leads to the same contradiction as before. Thus $(C-p)\cap(C'-p')$ is contained in either H_s or H_{r-1}. Thus the neighbourhood set of C includes every vertex other than either $\mathcal{A}'_{s+1}$ or $\mathcal{A}'_{r-1}$, meaning that $|C| \geq r$, and hence $|C| = r$. Similarly, the neighbourhood set of C' contains every vertex other than either $\mathcal{A}'_1$ or $\mathcal{A}'_s$, so $|C'| = r$. As $r > |X|$ and $X \supseteq (C-p)\cup(C'-p')$, we deduce that $C - p = C' - p'$. Our earlier arguments show that C is contained in either $H_s \cup p$ or $H_{r-1} \cup p$. But this means that $|C| \leq r-1$, and the neighbourhood set of C contains all of the r vertices other than either $\mathcal{A}'_{s+1}$ or $\mathcal{A}'_{r-1}$. This contradicts the fact that C is a circuit, and completes the proof of the claim. □

Consider all the subspaces of P that are spanned by non-spanning subsets of $E(M_{r+1}\backslash\{e,f\})$. This is a finite collection of subspaces, and the previous claim says that none of them contains l. By Proposition 4.2, there is a point, f, on l that is not spanned by any non-spanning subset of $E(M_{r+1}\backslash\{e,f\})$. We can apply the same argument, augmenting the collection of subspaces with $\langle\{f\}\rangle$, and find another, distinct, point, e, on l that is not spanned by any non-spanning subset of $E(M_{r+1}\backslash\{e,f\})$. Consider the $\mathbb{K}$-representable matroid corresponding to the subset $H_1 \cup \cdots \cup H_{r-1} \cup \{e,f\}$ of P. Let this matroid be N. We will show that $N = M$, and this will complete the proof of Lemma 4.6.

Certainly $N\backslash\{e,f\} = M'\backslash\{p,p'\}$, and we deduced earlier that $M'\backslash\{p,p'\} = \mathrm{Kin}(r)\backslash\{e,f\}$. As e and f are contained in the circuit-hyperplanes $H_s \cup H_r$ and $H_{r-1} \cup H_r$, deleting them from M effectively undoes the relaxations that produced M (see [**10**, Proposition 3.3.5]); that is, $M\backslash\{e,f\} = \mathrm{Kin}(r)\backslash\{e,f\}$. Now we have shown that $N\backslash\{e,f\} = M\backslash\{e,f\}$. Moreover, in $N\backslash e$, the element f is freely placed by construction, so $N\backslash e$ is a free extension of $N\backslash\{e,f\}$.

On the other hand, as e is in $H_s \cup H_r$ and $H_{r-1} \cup H_r$, it follows that

$$M\backslash e = \mathrm{Kin}(r)\backslash e = T(M_{r+1})\backslash e = T(M_{r+1}\backslash e).$$

But f is a coloop in $M_{r+1}\backslash e$, so it is freely placed in $T(M_{r+1}\backslash e) = M\backslash e$. Therefore $M\backslash e$ is a free extension of $M\backslash\{e, f\}$. As $N\backslash\{e, f\} = M\backslash\{e, f\}$, this means that $N\backslash e = M\backslash e$.

Assume that $N \neq M$. Then there is a set, X, which is a non-spanning circuit in one of $\{M, N\}$, and independent in the other. As $N\backslash e = M\backslash e$, it follows that e is in X. We will show that f is also in X. If X is a non-spanning circuit of N, then $f \in X$, for otherwise $X - e$ is a non-spanning subset of $E(M_{r+1}\backslash\{e, f\})$ that spans e, and N was constructed so that no such subset exists. Therefore assume that X is a non-spanning circuit in M. Then X is also a non-spanning circuit in $\mathrm{Kin}(r) = T(M_{r+1})$, and hence in M_{r+1}. But $\{e, f\}$ is a series pair in M_{r+1}, so any circuit that contains e also contains f. Thus X contains $\{e, f\}$ in either case.

First we assume that X is a non-spanning circuit of M, and hence of $\mathrm{Kin}(r)$ and M_{r+1}. Since $|X| \leq r$, the neighbourhood set of X in G, the bipartite graph corresponding to $(\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}, \mathcal{A}, \mathcal{A}')$, has at most $r - 1$ vertices. If $X - \{e, f\}$ contains elements from two distinct sets in $\{H_1, \ldots, H_{r-1}\}$, then the neighbourhood set of these two elements contains all but at most one vertex from $\{\mathcal{A}_1, \ldots, \mathcal{A}_{r-1}\}$. As e and f are adjacent to $\mathcal{A}$ and $\mathcal{A}'$, this means that X has a neighbourhood set containing r vertices. It follows that $X - \{e, f\}$ is contained in one of $H_1, \ldots, H_{r-1}$. Thus the neighbourhood set of X contains exactly $r - 1$ vertices. Thus X has cardinality r, so $X = H_i \cup \{e, f\}$, for some $i \in \{1, \ldots, r - 1\}$. However, i is not s or $r - 1$, as $H_s \cup \{e, f\}$ and $H_{r-1} \cup \{e, f\}$ are bases in M. If $i \in \{1, \ldots, s - 1\}$, then in the bipartite graph G', the $r - 1$ vertices in $H_i \cup p'$ are adjacent to the $r - 2$ vertices in

$$\{\mathcal{A}'_1, \ldots, \mathcal{A}'_{r-1}, \mathcal{A}_0\} - \{\mathcal{A}'_i, \mathcal{A}'_{i+1}\}.$$

Thus $H_i \cup p'$ is dependent in M'. But $\{e, f, p'\}$ is dependent in $P|E(M') \cup \{e, f\}$. It follows easily that $H_i \cup \{e, f\} = X$ is dependent in $P|E(M') \cup \{e, f\}$, and hence in N. Similarly, if $i \in \{s + 1, \ldots, r - 2\}$, then the neighbourhood set of $H_i \cup p$ is

$$\{\mathcal{A}'_1, \ldots, \mathcal{A}'_{r-1}, \mathcal{A}_0\} - \{\mathcal{A}'_i, \mathcal{A}'_{i+1}\},$$

so $H_i \cup p$ and $\{e, f, p\}$ are dependent. This leads to the contradiction that X is dependent in N. Hence we now assume that X is a non-spanning circuit of N.

Note that X and $\{e, p, p'\}$ are both circuits of $P|E(N) \cup \{p, p'\}$. We apply strong-circuit elimination, and deduce that there is a circuit, C, contained in $(X - e) \cup \{p, p'\}$ that contains p. If $f \in C$, then we apply strong circuit-elimination to C and $\{f, p, p'\}$, and find a circuit that contains p but not f. Thus we lose no generality in assuming that $C \subseteq (X - \{e, f\}) \cup \{p, p'\}$ is a circuit of M' that contains p. If p' is in C, then the neighbourhood set of C in G' contains all r vertices $\mathcal{A}'_1, \ldots, \mathcal{A}'_{r-1}, \mathcal{A}_0$. Thus $|X| \geq |C| \geq r + 1$, which is impossible. Hence $p' \notin C$. If C contains an element from $H_1 \cup \cdots \cup H_s$ or H_{r-1}, then the neighbourhood set of C in G' contains all but at most one vertex from $\mathcal{A}'_1, \ldots, \mathcal{A}'_{r-1}, \mathcal{A}_0$. Thus $|C| \geq r$. As $p' \notin X$ implies $|X| \geq |C| + 1$, this leads to a contradiction. Therefore $C - p$ is contained in $H_{s+1} \cup \cdots \cup H_{r-2}$. If C contains elements from two of $H_{s+1}, \ldots, H_{r-2}$, then its neighbourhood set again contains at least $r - 1$ elements, leading to a contradiction. Therefore $C - p$ is contained in one of $H_{s+1}, \ldots, H_{r-2}$, so the neighbourhood set of C contains $r - 2$ elements. It follows that $|C| = r - 1$, so $C = H_i \cup p$ for some $i \in \{s + 1, \ldots, r - 2\}$. As $|X| \leq r$, this implies that $X = H_i \cup \{e, f\}$. But then X is a circuit-hyperplane in M, contradicting the fact that it is independent in this matroid.

We conclude that $N = M$, so M is $\mathbb{K}$-representable, as desired. □

Recall that $\mathrm{Kin}(r)^-$ is the matroid obtained from $\mathrm{Kin}(r)$ by relaxing $H_1 \cup H_r$. If $r \in \{2, \ldots, r - 1\}$, then we will let $\mathrm{Kin}(r)^{=}_i$ be the matroid obtained from $\mathrm{Kin}(r)^-$ by relaxing $H_i \cup H_r$. Thus the results in this section show that $\mathrm{Kin}(r)$ and $\mathrm{Kin}(r)^{=}_i$ are representable over any infinite field, and that $\mathrm{Kin}(r)^-$ is representable over no field.

5. *Proof of the main theorem*

In this section we prove our main theorem. Theorem 5.1 is a restatement of Theorem 1.3 that uses slightly different language. If $\mathcal{F}$ is a set of fields, then define $M(\mathcal{F})$ to be

$$\bigcup_{F\in\mathcal{F}} \{M \mid M \text{ is an } F\text{-representable matroid}\}.$$

THEOREM 5.1. *Let $\mathcal{F}$ be a set of fields that contains at least one infinite field. There does not exist a finite set, $\mathcal{K}$, of sentences in M-logic with the following property: if $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ is a structure, then $(E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid in $M(\mathcal{F})$ if and only if $\mathcal{M}$ satisfies $\{\mathbf{R1}, \mathbf{R2}, \mathbf{R3}\} \cup \mathcal{K}$.*

Before we prove this theorem, we discuss some preliminaries. Assume that $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ is a structure. For every function, ϕ, into $\mathcal{P}(E^{\mathcal{M}})$, there is an induced family of subsets of $E^{\mathcal{M}}$ that we call *definable* subsets (relative to ϕ). Let us say that a *minterm* is a subset of $E^{\mathcal{M}}$ that can be expressed in the form

$$\bigcap_{X\in\mathrm{Dom}(\phi)} f(X),$$

where $f(X)$ is either $\phi(X)$ or $E^{\mathcal{M}} - \phi(X)$, and the intersection ranges over the domain of ϕ. Note that distinct minterms are disjoint, and that every element of $E^{\mathcal{M}}$ is in a minterm. We say that a subset of $E^{\mathcal{M}}$ is *definable* if it is a union of minterms. Note that if the domain of ϕ has size m, then there are at most 2^m possible minterms, and hence at most 2^{2^m} definable subsets.

Now assume that $\{X_i\}_{i\in I}$ and $\{x_j\}_{j\in J}$ are sets of variables in $\mathcal{S}$ and $\mathcal{E}$ respectively, and that $\phi_{\mathcal{S}}\colon \{X_i\}_{i\in I} \to \mathcal{P}(E^{\mathcal{M}})$ and $\phi_{\mathcal{E}}\colon \{x_j\}_{j\in J} \to E^{\mathcal{M}}$ are assignments of set and element variables to subsets and elements of $E^{\mathcal{M}}$. We say that a set is definable relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ if it is definable relative to the function that takes X_i to $\phi_{\mathcal{S}}(X_i)$ for every $i \in I$, and x_j to $\{\phi_{\mathcal{E}}(x_j)\}$ for every $j \in J$.

Let P be a formula in M-logic such that $\mathrm{Var}(P) = \mathrm{Fr}(P)$. Let $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ be an interpretation of P. Observe that any set $\phi_{\mathcal{S}}(X_i)$ is definable relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$, since it is the union of all minterms in which $f(X_i) = \phi_{\mathcal{S}}(X_i)$. Similarly, any set $\{\phi_{\mathcal{E}}(x_j)\}$ is definable. Both $E^{\mathcal{M}}$ and the empty set are definable (the former as the union of all possible minterms, the latter as the empty union). Furthermore, if X and Y are definable sets, then $E^{\mathcal{M}} - X$, $X \cup Y$, and $X \cap Y$ are also definable. It follows that, if $X \in \mathcal{S}$ is a term that appears in P, then $X^{\mathcal{M}}$ is one of the

$$2^{2^{|\mathrm{Var}(P)|}}$$

definable subsets of $E^{\mathcal{M}}$.

PROPOSITION 5.2. *Let P be a formula in M-logic such that* $\mathrm{Var}(P) = \mathrm{Fr}(P)$. *Let $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ be an interpretation of P, and let $T = \phi_{\mathcal{E}}(\mathrm{Var}(P) \cap \mathcal{E})$ be the image of $\phi_{\mathcal{E}}$. Every definable set relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ can be expressed in the form $(A - T) \cup B$, where A is definable relative to $\phi_{\mathcal{S}}$, and B is a subset of T.*

Proof. Consider a minterm

$$Z = \bigcap_{x_j\in\mathrm{Var}(P)\cap\mathcal{E}} f(x_j),$$

relative to the function that takes every variable x_j to $\{\phi_{\mathcal{E}}(x_j)\}$. If $f(x_{j_1}) = \{\phi_{\mathcal{E}}(x_{j_1})\}$ and $f(x_{j_2}) = \{\phi_{\mathcal{E}}(x_{j_2})\}$, for variables x_{j_1} and x_{j_2} such that $\phi_{\mathcal{E}}(x_{j_1}) \neq \phi_{\mathcal{E}}(x_{j_2})$, then $Z = \emptyset$. If all the variables x_j satisfying $f(x_j) = \{\phi_{\mathcal{E}}(x_j)\}$, have the same image under $\phi_{\mathcal{E}}$, then Z is either the empty set, or a singleton subset of T. Finally, if $f(x_j) = E^{\mathcal{M}} - \{\phi_{\mathcal{E}}(x_j)\}$ for every variable x_j, then $Z = E^{\mathcal{M}} - T$.

Every minterm relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is the intersection of a minterm relative to $\phi_{\mathcal{S}}$ with a minterm relative to the function $x_j \mapsto \{\phi_{\mathcal{E}}(x_j)\}$. Thus every minterm relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is either the empty set, a singleton subset of T, or the intersection of $E^{\mathcal{M}} - T$ with a minterm relative to $\phi_{\mathcal{S}}$. Now it is clear that any union of such minterms is the union of a subset of T, and the intersection of A, a definable subset relative to $\phi_{\mathcal{S}}$, with $E^{\mathcal{M}} - T$. Thus the proposition holds. □

Proof of Theorem 5.1. We assume for a contradiction that $\mathcal{K}$ is a finite set of sentences in M-logic having the property that $\mathcal{M} = (E^{\mathcal{M}}, r^{\mathcal{M}})$ is a matroid in $M(\mathcal{F})$ if and only if $\mathcal{M}$ satisfies $\{\mathbf{R1}, \mathbf{R2}, \mathbf{R3}\} \cup \mathcal{K}$.

Let L be an integer such that $|\mathrm{Var}(S)| \leq L$ for every sentence $S \in \mathcal{K}$. Let

$$N = 2^{2^L} + 3,$$

and let $E^{\mathcal{M}} = E(\mathrm{Kin}(N))$.

Since $\mathrm{Kin}(N)^-$ is not representable, by Proposition 4.5, there is a sentence in $\mathcal{K}$ that is not satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^-$. Let S be such a sentence. We can assume S is a formula with one of the following forms:

(1) $\exists X_{i_1} \cdots \exists X_{i_m} \exists x_{j_1} \cdots \exists x_{j_n} P$
(2) $\exists X_{i_1} \cdots \exists X_{i_m} \forall x_{j_1} \cdots \forall x_{j_n} P$
(3) $\forall X_{i_1} \cdots \forall X_{i_m} \forall x_{j_1} \cdots \forall x_{j_n} P$
(4) $\forall X_{i_1} \cdots \forall X_{i_m} \exists x_{j_1} \cdots \exists x_{j_n} P$

where P is a formula such that $\mathrm{Var}(P) \cap \mathcal{S} = \mathrm{Fr}(P) \cap \mathcal{S} = \{X_{i_1}, \ldots, X_{i_m}\}$ and $\mathrm{Var}(P) \cap \mathcal{E} = \mathrm{Fr}(P) \cap \mathcal{E} = \{x_{j_1}, \ldots, x_{j_n}\}$. Let I and J be the index sets $\{i_1, \ldots, i_m\}$ and $\{j_1, \ldots, j_n\}$. Note that $m + n \leq L$.

Case 1. We first assume that S has the form

$$\exists X_{i_1} \cdots \exists X_{i_m} \exists x_{j_1} \cdots \exists x_{j_n} P.$$

Since $\mathrm{Kin}(N)$ is representable over at least one field in $\mathcal{F}$ (Proposition 4.3), there is an interpretation, $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$, such that $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)$. Consider the definable subsets relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$. There are at most $2^{2^{m+n}} \leq 2^{2^L}$ such subsets. As

$$N - 1 = 2^{2^L} + 2$$

is greater than the number of definable subsets, there is an index $s \in \{1, \ldots, N-1\}$ such that $H_s \cup H_N$ is not definable. Let M be the matroid obtained from $\mathrm{Kin}(N)$ by relaxing $H_s \cup H_N$. The rank functions of M and $\mathrm{Kin}(N)$ differ only on the set $H_s \cup H_N$. Since this set is not definable, we see that if X is any set term appearing in P, then the rank of $X^{\mathcal{M}}$ in M is the same as its rank in $\mathrm{Kin}(N)$. Thus $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied when $r^{\mathcal{M}}$ is the rank function of M. For $k \in \{1, \ldots, N-1\}$, let p_k be an arbitrary bijection from H_k to H_{k-s+1}. These bijections clearly induce an isomorphism from M to $\mathrm{Kin}(N)^-$. By composing this isomorphism with $\phi_{\mathcal{S}}$ and $\phi_{\mathcal{E}}$, we obtain an interpretation that satisfies P when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^-$. Thus S is satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^-$, contrary to our assumption.

Case 2. Next we assume that S has the form

$$\exists X_{i_1} \cdots \exists X_{i_m} \forall x_{j_1} \cdots \forall x_{j_n} P.$$

As $\mathrm{Kin}(N)$ is representable over a field in $\mathcal{F}$, there is some function

$$\phi_{\mathcal{S}} \colon \{X_i\}_{i \in I} \to \mathcal{P}(E^{\mathcal{M}}),$$

such that for every possible function

$$\phi_{\mathcal{E}}\colon \{x_j\}_{j\in J} \to E^{\mathcal{M}},$$

$P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)$.

For every $k \in \{1, \ldots, N-1\}$, let

$$\mathcal{H}_k = \{(H_k \cup H_N) \triangle Z \mid Z \subseteq E^{\mathcal{M}},\ |Z| \leq 2n\},$$

where $\triangle$ denotes symmetric difference. If some subset of $E^{\mathcal{M}}$ is contained in $\mathcal{H}_{k_1}$ and $\mathcal{H}_{k_2}$, where $k_1 \neq k_2$, then $(H_{k_1} \cup H_N) \triangle Z_1 = (H_{k_2} \cup H_k) \triangle Z_2$, for some sets Z_1 and Z_2 satisfying $|Z_1|, |Z_2| \leq 2n$. Thus

$$\emptyset = ((H_{k_1} \cup H_N) \triangle Z_1) \triangle ((H_{k_2} \cup H_N) \triangle Z_2) = (H_{k_1} \triangle H_{k_2}) \triangle (Z_1 \triangle Z_2).$$

But $H_{k_1} \triangle H_{k_2} = H_{k_1} \cup H_{k_2}$, and this set has cardinality $2N-4$. Thus

$$2^{2^L+1} + 2 = 2N - 4 = |Z_1 \triangle Z_2| \leq |Z_1 \cup Z_2| \leq |Z_1| + |Z_2| \leq 4n,$$

and this is impossible as $n \leq L$. This shows that no subset of $E^{\mathcal{M}}$ lies in two distinct families in $\mathcal{H}_1, \ldots, \mathcal{H}_{N-1}$. The number of definable subsets relative to $\phi_{\mathcal{S}}$ is 2^{2^m}, which is less than $N-1$. Let s be an index in $\{1, \ldots, N-1\}$ such that no member of $\mathcal{H}_s$ is definable relative to $\phi_{\mathcal{S}}$.

Let $\phi_{\mathcal{E}}\colon \{x_j\}_{j\in J} \to E^{\mathcal{M}}$ be an arbitrary assignment of element variables. Proposition 5.2 tells us that a definable set relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is obtained from a definable set relative to $\phi_{\mathcal{S}}$ by removing at most n elements and then adding at most n elements. That is, a definable set relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is the symmetric difference of a definable set relative to $\phi_{\mathcal{S}}$, and a set of cardinality at most $2n$. Thus $H_s \cup H_N$ is not definable in $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$, for any choice of the assignment $\phi_{\mathcal{E}}$, or else some definable set relative to $\phi_{\mathcal{S}}$ would be in $\mathcal{H}_s$.

Let M be the matroid obtained from $\mathrm{Kin}(N)$ by relaxing the circuit-hyperplane $H_s \cup H_N$. Then the rank functions of $\mathrm{Kin}(N)$ and M differ only in $H_s \cup H_N$. Since $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is satisfied by $\mathrm{Kin}(N)$ for any choice of the function $\phi_{\mathcal{E}}$, it follows that $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is also satisfied by M for any assignment $\phi_{\mathcal{E}}$. Thus S is satisfied when $r^{\mathcal{M}}$ is the rank function of M. Clearly M is isomorphic to $\mathrm{Kin}(N)^-$, and it is easy to show that S is satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^-$, contradicting our assumption.

Case 3. Assume that S has the form

$$\forall X_{i_1} \cdots \forall X_{i_m} \forall x_{j_1} \cdots \forall x_{j_n} P.$$

Then there are functions, $\phi_{\mathcal{S}}$ and $\phi_{\mathcal{E}}$, such that $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is not satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^-$. Choose $s \in \{2, \ldots, N-1\}$ so that $H_s \cup H_N$ is not definable relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$. Then $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is also not satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^=_s$. This means that S is not satisfied by $\mathrm{Kin}(N)^=_s$, and this is a contradiction as Lemma 4.6 implies that $\mathrm{Kin}(N)^=_s$ is representable over at least one field in $\mathcal{F}$.

Case 4. In the final case, we assume that S has the form

$$\forall X_{i_1} \cdots \forall X_{i_m} \exists x_{j_1} \cdots \exists x_{j_n} P.$$

Let

$$\phi_{\mathcal{S}}\colon \{X_i\}_{i\in I} \to \mathcal{P}(E^{\mathcal{M}}),$$

be an assignment so that $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is not satisfied by $\mathrm{Kin}(N)^-$ for every choice of assignment

$$\phi_{\mathcal{E}}\colon \{x_j\}_{j\in J} \to E^{\mathcal{M}}.$$

For $k \in \{2, \ldots, N-1\}$, we define $\mathcal{H}_k$ exactly as we did in Case 2. Choose the index $s \in \{2, \ldots, N-1\}$ so that no subset in $\mathcal{H}_s$ is definable relative to $\phi_{\mathcal{S}}$. Then $H_s \cup H_N$ is not definable relative to $(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$, for any choice of assignment $\phi_{\mathcal{E}}$. This means that $P(\phi_{\mathcal{S}}, \phi_{\mathcal{E}})$ is not satisfied by $\mathrm{Kin}(N)^{=}_{s}$, for the assignment $\phi_{\mathcal{S}}$ and any choice of assignment $\phi_{\mathcal{E}}$. Thus S is not satisfied when $r^{\mathcal{M}}$ is the rank function of $\mathrm{Kin}(N)^{=}_{s}$, and as this matroid is in $M(\mathcal{F})$, we have reached a contradiction that completes the proof of Theorem 1.3. $\square$

Remark 2. We developed MSOL using the function r, which has an intended interpretation as a rank function. If we add a unary independence predicate, I, for set terms, it is still not possible to finitely axiomatize representability over any infinite field, using sentences in M-logic. To see this, note that if there were such an axiomatization, we could simply replace every occurrence of $I(X)$ with the predicate $r(X) = |X|$. Then we would have a contradiction to Theorem 1.3. The same comment applies when we add a predicate for bases or spanning sets.

Remark 3. The authors of [**8**] conjecture that if $\mathcal{F}$ is a collection of finite fields, then $M(\mathcal{F})$ has a finite number of excluded minors. This would imply that membership in $M(\mathcal{F})$ can always be finitely axiomatized using sentences in M-logic when $\mathcal{F}$ contains no infinite field. In other words, if the conjecture is true, then the constraint in Theorem 1.3 that $\mathcal{F}$ contains an infinite field is always necessary.

We conclude with a conjecture that strengthens Conjecture 1.2.

Conjecture 5.3. *Theorem* 5.1 *holds even if we replace the words 'M-logic' with 'MSOL'.*

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