EXPOSING 3-SEPARATIONS IN 3-CONNECTED MATROIDS

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Abstract. Let M be a 3-connected matroid other than a wheel or a whirl. In the next paper in this series, we prove that there is an element whose deletion from M or M^* is 3-connected and whose only 3-separations are equivalent to those induced by M. The strategy used to prove this theorem involves showing that we can remove some element from a leaf of the tree of 3-separations of M. The main result of this paper is designed to allow us to do this.

1. Introduction

This is the second in a series of three papers—the others are [11, 13]—in which we address the question of when it is possible to find an element that can be deleted or contracted from a 3-connected matroid in such a way as to remain 3-connected and avoid creating new unwanted 3-separations. Such 3-separations are called *exposed* 3-separations. The formal definition of "exposed" require some preparation and is given in Section 2. In [13] we prove that it is almost always possible to find such an element.

Theorem 1.1. *Let M be a 3-connected matroid other than a wheel or whirl. Then M has an element e whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

In [11], we considered the special case of triangles and determined the structure that arises when no element of a triangle can be deleted without either losing 3-connectivity or exposing a 3-separation. In this paper, we consider another important special case. The following is our main result.

Theorem 1.2. *Let (A, B) be a non-sequential 3-separation in a 3-connected matroid M. Suppose that B is fully closed, A meets no triangle or triad of M, and if (X, Y) is a non-sequential 3-separation of M, then either $A \subseteq \mathrm{fcl}(X)$ or $A \subseteq \mathrm{fcl}(Y)$. Then A contains an element whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

While technical, Theorem 1.2 is a key ingredient in the proof of Theorem 1.1. The proof is surprisingly long. In particular, Section 7 occupies much of the space. This deals with a bounded-size case check on the 3-separator A of Theorem 1.2. This case check is essential to verify Theorem 1.2 and, while it could possibly be slightly streamlined, we see no way of avoiding the bulk of the work.

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A key tool in the proof of Theorem 1.2 involves taking a 3-separation (A, B) in a 3-connected matroid, adding two elements α and β freely on the guts of (A, B), and then deleting B. In the resulting matroid, α and β are clones that we can think of as replacing the set B. We call this process clonal replacement and define it formally in Section 4. The main result of that section, Lemma 4.13, shows that, by imposing some natural conditions on (A, B), we can ensure that the clonal replacement is 4-connected.

2. Preliminaries

Our terminology will follow Oxley [8] except that the simplification and cosimplification of a matroid N will be denoted by $\mathrm{si}(N)$ and $\mathrm{co}(N)$, respectively. We write $x \in \mathrm{cl}^{(*)}(Y)$ to mean that $x \in \mathrm{cl}(Y)$ or $x \in \mathrm{cl}^*(Y)$. A *quad* is a 4-element set in a matroid that is both a circuit and a cocircuit. The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* if $|X|, |E - X| \geq k$. A k-separating set X, or a k-separating partition $(X, E-X)$, or a k-separation $(X, E-X)$ is *exact* if $\lambda_M(X) = k-1$. A k-separation $(X, E - X)$ is *minimal* if $\min\{|X|, |E - X|\} = k$.

A set X in a matroid M is *fully closed* if it is closed in both M and M^*, that is, $\mathrm{cl}(X) = X$ and $\mathrm{cl}^*(X) = X$. The *full closure* of X, denoted $\mathrm{fcl}(X)$, is the intersection of all fully closed sets that contain X. Two exactly 3-separating partitions (A_1, B_1) and (A_2, B_2) of M are *equivalent*, written $(A_1, B_1) \cong (A_2, B_2)$, if $\mathrm{fcl}(A_1) = \mathrm{fcl}(A_2)$ and $\mathrm{fcl}(B_1) = \mathrm{fcl}(B_2)$. If $\mathrm{fcl}(A_1)$ or $\mathrm{fcl}(B_1)$ is $E(M)$, then (A_1, B_1) is *sequential*. A 3-connected matroid M is *sequentially 4-connected* if it has no non-sequential 3-separations.

Let e be an element of a matroid M such that both M and $M\backslash e$ are 3-connected. A 3-separation (X, Y) of $M\backslash e$ is *well blocked* by e if, for all exactly 3-separating partitions (X', Y') equivalent to (X, Y), neither $(X' \cup e, Y')$ nor $(X', Y' \cup e)$ is exactly 3-separating in M. An element f of M *exposes* a 3-separation (U, V) if (U, V) is a 3-separation of $M\backslash f$ that is well blocked by f. Although (U, V) is actually a 3-separation of $M\backslash f$, we often say that *f exposes a 3-separation (U, V) in M*. Evidently, if e exposes an exactly 3-separating partition (E_1, E_2), then e exposes all exactly 3-separating partitions (E'_1, E'_2) that are equivalent to (E_1, E_2). We remark that implicit in the assertion that an element f exposes a 3-separation in M is the requirement that $M\backslash f$ is 3-connected.

Let X be an exactly 3-separating set in a matroid M. If there is an ordering $(x_1, x_2, \dots, x_n)$ of X such that $\{x_1, x_2, \dots, x_i\}$ is 3-separating for all i in $[n]$, then X is *sequential* and $(x_1, x_2, \dots, x_n)$ is a *sequential ordering* of X. An exactly 3-separating partition (X, Y) of M is *sequential* if X or Y is a sequential 3-separating set. In a 3-connected matroid M, a 3-*sequence* is an ordered partition $(A, x_1, x_2, \dots, x_n, B)$ of $E(M)$ such that $|A|, |B| \geq 2$ and $(A \cup \{x_1, x_2, \dots, x_i\}, \{x_{i+1}, x_{i+2}, \dots, x_n\} \cup B)$ is exactly 3-separating for all i in $\{0, 1, \dots, n\}$. If M has a 3-sequence in which $|A| = |B| = 2$, then M is *sequential*.

Let S be a subset of the ground set of a matroid M with $|S| \geq 3$. Then S is a *segment* if every 3-element subset of S is a triangle; and S is a *cosegment* if every 3-element subset of S is a triad.

Let k be an integer exceeding one. A matroid M is $(4, k)$-*connected* if M is 3-connected and, whenever (X, Y) is a 3-separating partition of $E(M)$, either $|X| \leq k$ or $|Y| \leq k$. Hall [4] called such a matroid 4-*connected up to separators of size* k. Matroids that are $(4, 3)$-connected and $(4, 4)$-connected are also called *internally* 4-*connected* and *weakly* 4-*connected* respectively. A 3-connected matroid M is $(4, k, S)$-*connected* if M is both $(4, k)$-connected and sequentially 4-connected.

The next two lemmas are elementary properties of matroids. The second is a restatement of the Mac Lane-Steinitz exchange property.

Lemma 2.1. *Let e be an element of a matroid M, and X and Y be disjoint sets whose union is $E(M) - e$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 2.2. *Let e and f be elements of a matroid M and let X be a subset of $E(M) - \{e, f\}$. If $e \notin \mathrm{cl}(X \cup f)$ and $f \notin \mathrm{cl}(X)$, then $f \notin \mathrm{cl}_{M/e}(X)$.*

The following lemma [2, Lemma 4.1], an important tool in the proof of the main result of [2], will also be useful here.

Lemma 2.3. *Let M be a 4-connected matroid and z be an element of M. Then $M\backslash z$ or M/z is $(4, 4)$-connected.*

The connectivity function λ_M of a matroid M has many attractive properties. Clearly $\lambda_M(X) = \lambda_M(E - X)$. Moreover, one easily checks that $\lambda_M(X) = r(X) + r^*(X) - |X|$ for all subsets X of $E(M)$. Hence $\lambda_M(X) = \lambda_{M^*}(X)$. We often abbreviate λ_M as λ. This function is submodular, that is, $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E(M)$. The next lemma is a consequence of this. We make frequent use of it here and write *by uncrossing* to mean "by an application of Lemma 2.4."

Lemma 2.4. *Let M be a 3-connected matroid, and let X and Y be 3-separating subsets of $E(M)$.*

(i) *If* $|X \cap Y| \geq 2$, *then* $X \cup Y$ *is* 3*-separating.*
(ii) *If* $|E(M) - (X \cup Y)| \geq 2$, *then* $X \cap Y$ *is* 3*-separating.*

Another consequence of the submodularity of λ is the following very useful result for 3-connected matroids, known as Bixby's Lemma [1].

Lemma 2.5. *Let* e *be an element of a* 3*-connected matroid* M. *Then either* $M\backslash e$ *or* M/e *has no non-minimal* 2*-separations. Moreover, in the first case,* $\mathrm{co}(M\backslash e)$ *is* 3*-connected while, in the second case,* $\mathrm{si}(M/e)$ *is* 3*-connected.*

A useful companion function to the connectivity function is the *local connectivity*, $\sqcap(X, Y)$, defined for sets X and Y in a matroid M, by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Evidently, $\sqcap(X, E - X) = \lambda_M(X)$. For a field $\mathbb{F}$, when M is $\mathbb{F}$-representable and hence essentially viewable as a subset of the vector space $V(r(M), \mathbb{F})$, the local connectivity $\sqcap(X, Y)$ is precisely the dimension of the intersection of those subspaces in $V(r(M), \mathbb{F})$ that are spanned by X and Y.

An attractive link between connectivity and local connectivity is provided by the next result [9, Lemma 2.6], which follows immediately by substitution.

Lemma 2.6. *Let* X *and* Y *be disjoint sets in a matroid* M, *then*

$$\lambda_M(X \cup Y) = \lambda_M(X) + \lambda_M(Y) - \sqcap_M(X, Y) - \sqcap_{M^*}(X, Y).$$

The first part of the next lemma [9, Lemma 2.3] just restates [8, Lemma 8.2.10]. The second part, which follows from the first, is the well-known fact that the connectivity function is monotone under taking minors.

Lemma 2.7. *Let* M *be a matroid.*

(i) *Let* X_1, X_2, Y_1 *and* Y_2 *be subsets of* $E(M)$. *If* $X_1 \subseteq Y_1$ *and* $X_2 \subseteq Y_2$, *then* $\sqcap(X_1, X_2) \leq \sqcap(Y_1, Y_2)$.
(ii) *If* N *is a minor of* M *and* $X \subseteq E(M)$, *then*

$$\lambda_N(X \cap E(N)) \leq \lambda_M(X).$$

We shall use the following result of Lemos [6, Theorem 1] several times.

Lemma 2.8. *Let* M *be a* 3*-connected matroid and* C^* *be a cocircuit of* M *such that* M/e *is not* 3*-connected for all* e *in* C^*. *Then* C^* *meets at least two triangles of* M.

The following elementary lemma [9, Lemma 3.1] will be used repeatedly.

Lemma 2.9. *For a positive integer* k, *let* (A, B) *be an exactly* k*-separating partition in a matroid* M.

(i) *For e in $E(M)$, the partition $(A \cup e, B - e)$ is k-separating if and only if $e \in \mathrm{cl}^{(*)}(A)$.*
(ii) *For e in B, the partition $(A \cup e, B - e)$ is exactly k-separating if and only if e is in exactly one of $\mathrm{cl}(A) \cap \mathrm{cl}(B - e)$ and $\mathrm{cl}^*(A) \cap \mathrm{cl}^*(B - e)$.*
(iii) *The elements of $\mathrm{fcl}(A) - A$ can be ordered $b_1, b_2, \ldots, b_n$ so that $A \cup \{b_1, b_2, \ldots, b_i\}$ is k-separating for all i in $[n]$.*

The next lemma is a consequence of Lemma 2.9.

Lemma 2.10. *Let M be a 3-connected matroid.*

(i) *If (X, e, Y) is a 3-sequence of M and $e \in \mathrm{cl}^*(X)$, then $\sqcap(X, Y) = 1$.*
(ii) *If (X, e, f, Y) is a 3-sequence of M, where $e \in \mathrm{cl}^*(X)$ and $f \in \mathrm{cl}^*(X \cup e)$, then $\sqcap(X, Y) = 0$.*

Proof. We prove (ii). The proof of (i) is similar. Since $f \in \mathrm{cl}^*(X \cup e)$, it follows by Lemma 2.9, that $f \in \mathrm{cl}^*(Y)$ and so

$$r(X \cup e) + r(Y \cup f) - r(M) = r(X) + 1 + r(Y) + 1 - r(M).$$

Therefore, as $(X \cup e, Y \cup f)$ is a 3-separation, $r(X) + r(Y) = r(M)$. Since M is 3-connected, $r(X \cup Y) = r(M)$, so $\sqcap(X, Y) = 0$. □

Lemma 2.11. *Let $(X, \{z\}, Y)$ be a partition of the ground set of a 3-connected matroid M. Assume that $(X, z \cup Y)$ and $(X \cup z, Y)$ are 3-separations of M. Then exactly one of the following holds:*

(i) *$z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$ and $\mathrm{co}(M \backslash z)$ is 3-connected; or*
(ii) *$z \in \mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y)$ and $\mathrm{si}(M/z)$ is 3-connected.*

Proof. The fact that z is in exactly one of $\mathrm{cl}(X) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X) \cap \mathrm{cl}^*(Y)$ follows by (ii) of Lemma 2.9. By duality, we may suppose that $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. As M is 3-connected, $M \backslash z$ is 2-connected. By Lemma 2.5, we need only show that $M \backslash z$ has no non-minimal 2-separations.

Let (A, B) be a non-minimal 2-separation of $M \backslash z$. Neither $(A \cup z, B)$ nor $(A, B \cup z)$ is a 2-separation of M so each is a 3-separation. Hence z is in neither $\mathrm{cl}(A)$ nor $\mathrm{cl}(B)$, so, by Lemma 2.1, z is in both $\mathrm{cl}^*(B)$ and $\mathrm{cl}^*(A)$.

As $z \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$ but z is in neither $\mathrm{cl}(A)$ nor $\mathrm{cl}(B)$, all of the sets $X \cap A, X \cap B, Y \cap A$, and $Y \cap B$ are non-empty. As A has at least three elements, $X \cap A$ or $Y \cap A$ has at least two elements. Without loss of generality, assume the former. If $|Y \cap B| = 1$, then both $|X \cap B|$ and $|Y \cap A|$ exceed one. Thus, we have that either

(a) $|X \cap A| \geq 2$ and $|Y \cap B| \geq 2$; or

(a) $|X \cap B| \geq 2$ and $|Y \cap A| \geq 2$.

By symmetry, we may assume the former. Then, by uncrossing, both $X \cup A$ and $X \cup A \cup z$ are 3-separating in M. Hence both $(Y \cap B) \cup z$ and $Y \cap B$ are 3-separating in M, so z is in $\text{cl}(Y \cap B)$ or z is in $\text{cl}^*(Y \cap B)$. Both possibilities yield contradictions to orthogonality since $z \in \text{cl}^*(A) \subseteq \text{cl}^*(X \cup A)$ and $z \in \text{cl}(X) \subseteq \text{cl}(X \cup A)$. □

For two 3-separations (X_1, X_2) and (Y_1, Y_2) of a 3-connected matroid M, one easily checks that $\text{cl}(X_1) = \text{cl}(Y_1)$ if and only if $\text{cl}(X_2) = \text{cl}(Y_2)$. When $\text{cl}(X_i) = \text{cl}(Y_i)$ for some i, we call (X_1, X_2) and (Y_1, Y_2) *closure-equivalent*.

Distinct elements α and β of a matroid M are *clones* if M has an automorphism that interchanges α and β and fixes every other element. When α and β are *clones* in M, we call $\{\alpha, \beta\}$ a *clonal pair* in M. Evidently if $\{\alpha, \beta\}$ is a clonal pair in M, and N is a minor of M with $\{\alpha, \beta\} \subseteq E(N)$, then $\{\alpha, \beta\}$ is a clonal pair in N.

Lemma 2.12. *Let M be a 3-connected matroid, and let $\{\alpha, \beta\}$ be a clonal pair in M. If M is not sequentially 4-connected, then M has a non-sequential 3-separation (U, V) such that $\{\alpha, \beta\} \subseteq U$ or $\{\alpha, \beta\} \subseteq V$.*

Proof. Assume the lemma fails and let (X, Y) be a non-sequential 3-separation of M. Then $|X|, |Y| \geq 4$. As neither X nor Y contains $\{\alpha, \beta\}$, we may assume that $\alpha \in X$ and $\beta \in Y$. If $\alpha \in \text{cl}_M(X - \alpha)$, then $\beta \in \text{cl}_M(X - \alpha)$ and so $(X \cup \beta, Y - \beta)$ is a 3-separation in M. Moreover, as (X, Y) is non-sequential, so is $(X \cup \beta, Y - \beta)$; a contradiction. Thus $\alpha \notin \text{cl}_M(X - \alpha)$. Then, by Lemma 2.1, $\alpha \in \text{cl}^*_M(Y)$. Hence $(X - \alpha, Y \cup \alpha)$ is a non-sequential 3-separation of M; a contradiction. □

Lemma 2.13. *Let M be a 3-connected matroid with no triangles. Let $\{z_1, z_2, z_3, z_4\}$ be a circuit of M that contains a cocircuit C^*. If $z_i \in C^*$, then M/z_i is 3-connected.*

Proof. Suppose that M/z_i is not 3-connected. Then M/z_i has a 2-separation (X, Y). Since M has no triangles, $r_{M/z_i}(X), r_{M/z_i}(Y) \geq 2$. Thus, as $\{z_1, z_2, z_3, z_4\} - z_i$ is a triangle in M/z_i, we may assume without loss of generality that $\{z_1, z_2, z_3, z_4\} - z_i \subseteq X$. Since z_i is in a cocircuit of M contained in $\{z_1, z_2, z_3, z_4\}$, it follows that $r_{M/z_i}(Y) = r_M(Y)$. Therefore

$$\begin{aligned} r_M(X \cup z_i) + r_M(Y) - r(M) &= r_{M/z_i}(X) + 1 + r_{M/z_i}(Y) - (r(M/z_i) + 1) \\ &= 1, \end{aligned}$$

contradicting the fact that M is 3-connected. □

Lemma 2.14. *Let M be a 3-connected matroid with no triangles, and let $\{\alpha,\beta\}$ be a clonal pair in M. If $|E(M)| \geq 4$ and $\{\alpha,\beta,z\}$ is a triad of M, then M/z is 3-connected.*

Proof. If M/z is not 3-connected, then M/z has a 2-separation (X,Y). Since M has no triangles, $r_{M/z}(X), r_{M/z}(Y) \geq 2$. Furthermore, $|X|,|Y| \geq 3$; otherwise X or Y is a 2-cocircuit in M/z. If $\alpha,\beta \in X$, then $r_{M/z}(Y) = r_M(Y)$, and so

$$r_M(X \cup z) + r_M(Y) - r(M) = r_{M/z}(X) + 1 + r_{M/z}(Y) - (r(M/z)+1) = 1;$$

a contradiction. It follows that we may assume that M/z has no 2-separation in which $\{\alpha,\beta\} \subseteq X$ or $\{\alpha,\beta\} \subseteq Y$ and hence that $\alpha \in X$ and $\beta \in Y$. If $\alpha \in \mathrm{cl}_{M/z}(X-\alpha)$, then $\beta \in \mathrm{cl}_{M/z}(X-\alpha)$ and so $(X \cup \beta, Y-\beta)$ is a 2-separation of M/z; a contradiction. Thus $\alpha \notin \mathrm{cl}_{M/z}(X-\alpha)$ so $(X-\alpha, Y\cup\alpha)$ is a 2-separation of M/z; a contradiction. Hence M/z is 3-connected. □

The next lemma is from [11, Lemma 2.4].

Lemma 2.15. *Let M be a 3-connected matroid. If f exposes a 3-separation (U,V) in M, then (U,V) is non-sequential. In particular, $|U|,|V| \geq 4$. Moreover, if $|V| = 4$, then V is a quad of $M\backslash f$.*

Lemma 2.16. *Let $\{\alpha,\beta,a,b\}$ be a sequential 3-separating set in a 3-connected matroid M. Suppose α and β are clones. Then (α,β,x,y) is a sequential ordering of $\{\alpha,\beta,a,b\}$ for some permutation (x,y) of $\{a,b\}$.*

Proof. Let (e_1,e_2,e_3,e_4) be a sequential ordering of $\{\alpha,\beta,a,b\}$. If $\{\alpha,\beta\} \subseteq \{e_1,e_2,e_3\}$, then we can reorder e_1, e_2, and e_3 so that the sequence begins (α,β). We may now assume that $e_4 \in \{\alpha,\beta\}$. As α and β are clones, we may suppose $e_4 = \alpha$. By reordering (e_1,e_2,e_3), we may assume $e_3 = \beta$. By duality, we may assume $\{e_1,e_2,\beta\}$ is a triangle. Thus so is $\{e_1,e_2,\alpha\}$. Then $r(\{e_1,e_2,\alpha,\beta\}) = 2$ and (α,β,a,b) is a sequential ordering of $\{\alpha,\beta,a,b\}$. □

For a 3-connected matroid N, we shall be interested in 3-separations of N that show that it is not $(4,k,S)$-connected. We call a 3-separation (X,Y) of N a $(4,k,S)$-*violator* if either

(i) $|X|,|Y| \geq k+1$; or
(ii) (X,Y) is non-sequential.

Observe that, when $k=3$, condition (ii) implies condition (i). Hence (X,Y) is a $(4,3,S)$-violator of N if and only if $|X|,|Y| \geq 4$.

The next lemma [12, Lemma 2.11] is used in proving the subsequent result.

Lemma 2.17. *Let N be a 3-connected matroid. Then (X, Y) is a $(4, 4, S)$-violator if and only if*

(i) $|X|, |Y| \geq 5$*; or*
(ii) X *and* Y *are non-sequential and at least one is a quad.*

Lemma 2.18. *Let M be a 4-connected matroid with a 5-point rank-3 set P. If $e \in \mathrm{cl}^*(P) - P$, then M/e is $(4, 4, S)$-connected.*

Proof. Certainly M/e is 3-connected. Let (R, G) be a $(4, 4, S)$-violator of it. Without loss of generality, we may assume that $|R \cap P| \geq 3$. Thus $R \cap P$ spans P in M/e. Hence the 3-separating partition $(R \cup P, G - P)$ of M/e is equivalent to (R, G). Now, by Lemma 2.17, either $|G| \geq 5$ or G is non-sequential. In the first case, $|G - P| \geq 3$; in the second, $G - P$ is non-sequential so $|G - P| \geq 4$. Hence, in both cases, $(R \cup P, G - P)$ is a 3-separation of M/e. But $e \in \mathrm{cl}^*(P)$, so $e \in \mathrm{cl}^*(R \cup P)$. Hence $(R \cup P \cup e, G - P)$ is a 3-separation of the 4-connected matroid M; a contradiction. □

Lemma 2.19. *A 3-connected matroid M of rank at most three is sequentially 4-connected.*

Proof. Let (X, Y) be a 3-separation of M. Then $r(X) + r(Y) = r(M) + 2$ and $|X|, |Y| \geq 3$. Thus $r(X), r(Y) \geq 2$. But $r(M) \leq 3$. Hence X or Y spans M, so (X, Y) is sequential. □

Lemma 2.20. *Let Q be a quad in a 3-connected matroid M with $|E(M)| \geq 7$. If $\{\alpha, \beta\}$ is a clonal pair in M that meets Q, then $\{\alpha, \beta\} \subseteq Q$.*

Proof. We may assume that $\alpha \in Q$ and $\beta \notin Q$. As Q is a quad of M and $\{\alpha, \beta\}$ is a clonal pair, $(Q - \alpha) \cup \beta$ is a quad of M. Hence

$$r(Q \cup \beta) + r^*(Q \cup \beta) - |Q \cup \beta| \leq (5 - 2) + (5 - 2) - 5 = 1.$$

Since $|E(M)| \geq 7$, this contradicts the fact that M is 3-connected. □

The next two lemmas are repeatedly used in the last section of the paper.

Lemma 2.21. *Let M be a 3-connected matroid and let (X, Y) be a 3-separation of M. If $M \backslash e$ is 3-connected, then $e \in \mathrm{cl}(X - e)$ or $e \in \mathrm{cl}(Y - e)$.*

Proof. Since (X, Y) is a 3-separation of M, we have $|X|, |Y| \geq 3$. Therefore, as $M \backslash e$ is 3-connected, $r_{M \backslash e}(X - e) + r_{M \backslash e}(Y - e) - r(M \backslash e) = 2$. As $r(M) = r(M \backslash e)$, it follows that

$$r_M(X) + r_M(Y) = 2 + r(M) = r_{M \backslash e}(X - e) + r_{M \backslash e}(Y - e).$$

If $e \notin Y$, then $r_M(Y) = r_{M\backslash e}(Y - e)$ and so $e \in \mathrm{cl}(X - e)$. Similarly, if $e \notin X$, then $e \in \mathrm{cl}(Y - e)$. □

Lemma 2.22. *Let N be a 4-connected matroid, and let a and e be distinct elements of $E(N)$. Let (X, Y) be a 3-separation of N/a and suppose that $N/a\backslash e$ is 3-connected. If $X - e$ contains a triad of $N/a\backslash e$, then $e \in \mathrm{cl}_{N/a}(X - e)$, but $e \notin \mathrm{cl}_{N/a}(Y - e)$. In particular, there are no two triads T_X and T_Y in $N/a\backslash e$ such that $T_X \subseteq X - e$ and $T_Y \subseteq Y - e$.*

Proof. Since $N/a\backslash e$ is 3-connected, it follows by Lemma 2.21 that either $e \in \mathrm{cl}_{N/a}(X - e)$ or $e \in \mathrm{cl}_{N/a}(Y - e)$. Suppose that T is a triad of $N/a\backslash e$ such that $T \subseteq X - e$. If $e \in \mathrm{cl}_{N/a}(Y - e)$, then T is a triad in N/a. But then T is a triad in N, contradicting the fact that N is 4-connected. Therefore $e \notin \mathrm{cl}_{N/a}(Y - e)$ and $e \in \mathrm{cl}_{N/a}(X - e)$. The second part of the lemma is an immediate consequence of the first part. □

The following theorem [12, Theorem 1.6] is used a number of times in the last two sections.

Theorem 2.23. *Let M be a 4-connected matroid with $|E(M)| \geq 11$. Let $\{a, b, c, d, e\}$ be a rank-3 subset of $E(M)$. Then there are at least two elements x of $\{a, b, c, d, e\}$ such that $M\backslash x$ is internally 4-connected.*

We end this section with a brief outline of the strategy that we use in the proof of Theorem 1.2. We extend M by a clonal pair of elements, α and β, which are freely placed so that, in the resulting extension of M, these elements lie in the intersection of the closures of A and B. We then delete the elements of B and denote the resulting matroid by N, calling it the *clonal replacement* of B by $\{\alpha, \beta\}$. We show in Lemma 4.13 that N is 4-connected. We then show that N has an element e not in $\{\alpha, \beta\}$ such that the deletion of e from M or M^* is 3-connected but does not expose any 3-separations. For N having at least 13 elements, this is done in Section 6, while for N having at most twelve elements this is done in Section 7.

3. Flowers

In this section, we recall some essential definitions from [9, 10]. Let $(P_1, P_2, \dots, P_n)$ be a *flower* Φ in a 3-connected matroid M, that is, $(P_1, P_2, \dots, P_n)$ is an ordered partition of $E(M)$ such that $\lambda_M(P_i) = 2 = \lambda_M(P_i \cup P_{i+1})$ for all i in $\{1, 2, \dots, n\}$, where all subscripts are interpreted modulo n. The sets $P_1, P_2, \dots, P_n$ are the *petals* of Φ. Each has at least two elements. It is shown in [9, Theorem 4.1] that every flower in a 3-connected matroid is either an *anemone* or a *daisy*. In the first case, all unions of petals are 3-separating; in the second, a union of petals is 3-separating if and only if the petals are consecutive in the cyclic ordering $(P_1, P_2, \dots, P_n)$.

The classes of anemones and daisies can be further refined using local connectivity. Let $(P_1, P_2, \dots, P_n)$ be a flower Φ with $n \geq 3$. If Φ is an anemone, then $\sqcap(P_i, P_j)$ takes a fixed value k in $\{0, 1, 2\}$ for all distinct i, j in $[n]$. We call Φ a *paddle* if $k = 2$, a *copaddle* if $k = 0$, and a *spike-like* flower if $k = 1$ and $n \geq 4$. Similarly, if Φ is a daisy, then $\sqcap(P_i, P_j) = 1$ for all consecutive i and j. We say Φ is *swirl-like* if $n \geq 4$ and $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and Φ is *Vámos-like* if $n = 4$ and $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$. An element e of M is *loose* in Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ; otherwise e is *tight*.

If (P_1, P_2, P_3) is a flower Φ and $\sqcap(P_i, P_j) = 1$ for all distinct i and j, we call Φ *ambiguous* if it has no loose elements, *spike-like* if there is an element in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $\mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, and *swirl-like* otherwise. Every flower with at least three petals is of one of these six *types*: a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or ambiguous [9].

Lemma 3.1. *Let Φ be a flower $(\{\alpha, \beta\}, P_1, P_2)$ in a sequentially 4-connected matroid M, where $\{\alpha, \beta\}$ is a clonal pair. If Φ is a paddle or a copaddle, then P_1 or P_2 is sequential. Moreover, if*

$$\sqcap(\{\alpha, \beta\}, P_1) = \sqcap(P_1, P_2) = \sqcap(P_2, \{\alpha, \beta\}) = 1$$

and $P_1, P_2 \not\subseteq \mathrm{fcl}(\{\alpha, \beta\})$, then both P_1 and P_2 are sequential.

Proof. First suppose that Φ is a paddle. If P_2 is not sequential, then, as M is sequentially 4-connected, $\{\alpha, \beta\} \cup P_1$ is sequential. Choose a sequential ordering $(z_1, z_2, \dots, z_k)$ of $\{\alpha, \beta\} \cup P_1$ with the greatest j such that $\{\alpha, \beta\} \subseteq \{z_1, z_2, \dots, z_j\}$. We may assume that $\{\alpha, \beta\} = \{z_{j-1}, z_j\}$. If $j = k$, then $(z_1, z_2, \dots, z_{j-2})$ is a sequential ordering of P_1 and so P_1 is sequential. Therefore, we may assume that $j < k$. Since Φ is a paddle, $\sqcap(P_2, \{\alpha, \beta\}) = 2$ and so $\alpha, \beta \in \mathrm{cl}(P_2)$. It now follows by two applications of Lemma 2.9 that $(z_1, z_2, \dots, z_{j-2}, z_{j+1}, \alpha, \beta, z_{j+2}, \dots, z_k)$ is a sequential ordering of $\{\alpha, \beta\} \cup P_1$, contradicting the maximality of the choice of j. Hence P_1 is sequential. Dually, if Φ is a copaddle, then P_1 or P_2 is sequential.

Now suppose that

$$\sqcap(\{\alpha, \beta\}, P_1) = \sqcap(P_1, P_2) = \sqcap(P_2, \{\alpha, \beta\}) = 1$$

and $P_1, P_2 \not\subseteq \mathrm{fcl}(\{\alpha, \beta\})$. Assume that P_2 is not sequential. Then both P_1 and $P_1 \cup \{\alpha, \beta\}$ are sequential. Let $(z_1, z_2, \dots, z_k)$ be a sequential ordering of $P_1 \cup \{\alpha, \beta\}$. Then, by repeated applications of uncrossing (see [5, Lemma 4.3]), we may assume that $(z_{k-1}, z_k) = (\alpha, \beta)$. Then $P_1 \cup \alpha$ is 3-separating. As α and β are clones and $\sqcap(\{\alpha, \beta\}, P_1) = 1$, we have $\alpha \notin \mathrm{cl}(P_1)$ and

$\beta \notin \mathrm{cl}(P_1)$. Therefore,

$$\begin{aligned} 2 &= r(P_1 \cup \alpha) + r(P_2 \cup \beta) - r(M) \\ &= r(P_1) + 1 + r(P_2) + 1 - r(M) \\ &= r(\{\alpha, \beta\} \cup P_1) + r(P_2) + 1 - r(M). \end{aligned}$$

But then $r(\{\alpha, \beta\} \cup P_1) + r(P_2) - r(M) = 1$ and so $(\{\alpha, \beta\} \cup P_1, P_2)$ is a 2-separation of M; a contradiction as M is 3-connected. Therefore P_2 is sequential. By symmetry, so too is P_1. □

4. Clonal replacements

Let M be a 3-connected matroid and (A, B) be a 3-separation of M. We want to add a clonal pair $\{\alpha, \beta\}$ on the guts of (A, B) and then delete B, thereby replacing B by the clonal pair $\{\alpha, \beta\}$. In this section, we give a formal description of this process and derive some of its properties. These are then used to prove Lemma 4.13, the main result of the section. That lemma shows that, under certain natural conditions, the matroid one derives from this construction is 4-connected.

The construction proceeds as follows. Use the principal modular cut of M generated by $\{\mathrm{cl}_M(B)\}$ to adjoin the element α to M. In the resulting matroid M_α, use the principal modular cut generated by $\{\mathrm{cl}_{M_\alpha}(B)\}$ to adjoin the element β and get the matroid M^+.

The next lemma can be proved by determining all the flats of M^+, which can be done using [8, Corollary 7.2.4]. We omit the straightforward details.

Lemma 4.1. *The elements α and β are clones in M^+.*

For disjoint sets X and Y of the ground set E of a matroid M', let $\kappa_{M'}(X, Y) = \min\{\lambda_{M'}(S) : X \subseteq S \subseteq E - Y\}$. By Geelen, Gerards, and Whittle's extension of Tutte's Linking Theorem [3, Theorem 4.2], M^+ has a minor N with ground set $A \cup \{\alpha, \beta\}$ and with $\kappa_N(A, \{\alpha, \beta\}) = \kappa_{M^+}(A, \{\alpha, \beta\})$ such that $N|\{\alpha, \beta\} = M^+|\{\alpha, \beta\}$ and $N|A = M^+|A = M|A$. Now M is 3-connected and α and β are not added as loops, coloops, or parallel elements, so M^+ is 3-connected. Moreover, $\{\alpha, \beta\} \subseteq \mathrm{cl}_{M^+}(B)$, so

$$2 \leq \kappa_{M^+}(A, \{\alpha, \beta\}) \leq \lambda_{M^+}(A) = \lambda_M(A) = 2.$$

Because α and β are clones in M^+, they are also clones in N.

The matroid N is called the *clonal replacement* of B by $\{\alpha, \beta\}$. We shall show below that N is unique. Since $N|A = M^+|A$ and α and β are clones in N, to determine N, we need only specify $r_N(X \cup \alpha), r_N(X \cup \beta)$, and $r_N(X \cup \{\alpha, \beta\})$ for all subsets X of A. Lemmas 4.3 and 4.4 do this.

Observe that, since $\kappa_N(A, \{\alpha, \beta\}) = \kappa_{M^+}(A, \{\alpha, \beta\}) = 2$ and $r_N(\{\alpha, \beta\}) = r_{M^+}(\{\alpha, \beta\}) = 2$, we have

$$r(N) = r_N(A \cup \{\alpha, \beta\}) = r_N(A) = r_{M^+}(A) = r_M(A).$$

Since N is a minor of M^+, there is an independent set C_N and a coindependent set D_N in M^+ such that $N = M^+/C_N \backslash D_N$.

We omit the straightforward proof of the next result.

Lemma 4.2. *The set C_N is a basis of M/A.*

Lemma 4.3. *The following are equivalent for a subset X of A.*

(i) $\mathrm{cl}_N(X) \cap \{\alpha, \beta\} \neq \emptyset$;
(ii) $\{\alpha, \beta\} \subseteq \mathrm{cl}_N(X)$;
(iii) $\sqcap_M(B, X) = 2$;
(iv) $\sqcap_M(B, X) \geq 2$.

Proof. Since α and β are clones in N, (i) and (ii) are equivalent. Moreover, by Lemma 2.7, $\sqcap_M(B, X) \leq \sqcap_M(B, A) = 2$, so (iii) and (iv) are equivalent.

As $\{\alpha, \beta\}$ is independent in N, the set $C_N \cup \{\alpha, \beta\}$ spans $B \cup \{\alpha, \beta\}$ in M^+. Now $\{\alpha, \beta\} \subseteq \mathrm{cl}_N(X)$ if and only if $r_N(X \cup \{\alpha, \beta\}) = r_N(X) = r_M(X)$. But

$$\begin{aligned} r_N(X \cup \{\alpha, \beta\}) &= r_{M^+/C_N \backslash D_N}(X \cup \{\alpha, \beta\}) \\ &= r_{M^+}(X \cup \{\alpha, \beta\} \cup C_N) - r_{M^+}(C_N) \\ &= r_{M^+}(X \cup B) - (r_M(B) - 2) \\ &= r_M(X \cup B) - r_M(B) + 2 \\ &= r_M(X) + r_M(B) - \sqcap_M(X, B) - r_M(B) + 2 \\ &= r_M(X) + (2 - \sqcap_M(X, B)). \end{aligned}$$

We conclude that $r_N(X \cup \{\alpha, \beta\}) = r_M(X)$ if and only if $2 = \sqcap_M(X, B)$. □

Lemma 4.4. *For a subset X of A,*

(i) $\alpha \in \mathrm{cl}_N(X \cup \beta) - \mathrm{cl}_N(X)$ *if and only if* $\sqcap_M(B, X) = 1$; *and*
(ii) $\alpha \notin \mathrm{cl}_N(X \cup \beta)$ *if and only if* $\sqcap_M(B, X) = 0$.

Proof. For (i), we have that the following statements are equivalent, where we note that, by Lemma 4.2, $|C_N| = r_M(B) - 2$.

(a) $\alpha \in \mathrm{cl}_N(X \cup \beta) - \mathrm{cl}_N(X)$;
(b) $r_N(X \cup \beta \cup \alpha) = r_N(X) + 1$;
(c) $r_{M^+}(X \cup C_N \cup \beta \cup \alpha) = r_{M^+}(X \cup C_N) + 1$;
(d) $r_M(X \cup B) = r_{M^+}(X) + |C_N| + 1$;

(e) $r_M(X) + r_M(B) - \sqcap_M(B, X) = r_M(X) + (r_M(B) - 2) + 1$;
(f) $\sqcap_M(B, X) = 1$.

The equivalence of (a) and (b) follows because α and β are clones. The equivalence of (c) and (d) relies on the fact that

$$r_{M^+}(X) = r_M(X) = r_N(X) = r_{M^+}(X \cup C_N) - |C_N|.$$

We conclude that (i) holds.

To prove (ii), note that, by Lemma 2.7,

$$0 \leq \sqcap_M(B, X) \leq \sqcap_M(B, A) \leq 2.$$

By the previous lemma, $\sqcap_M(B, X) = 2$ if and only if $\alpha \in \mathrm{cl}_N(X)$. By (i), $\sqcap_M(B, X) = 1$ if and only if $\alpha \in \mathrm{cl}_N(X \cup \beta) - \mathrm{cl}_N(X)$. The remaining possibility, that $\sqcap_M(B, X) = 0$, must occur if and only if $\alpha \notin \mathrm{cl}_N(X \cup \beta)$. $\square$

We now know that clonal replacement is uniquely defined. Next we use the last two lemmas to give a more useful description of N.

Lemma 4.5. *Let Z be an arbitrary basis of M/A and $Y = B - Z$. Then $N = M^+/Z\backslash Y$.*

Proof. We shall prove that the rank functions of N and $M^+/Z\backslash Y$ coincide. Let $X \subseteq A$. Then

$$r_{M^+/Z\backslash Y}(X) = r_{M^+}(X \cup Z) - |Z| = r_M(X \cup Z) - |Z|. \tag{1}$$

But $|Z| = r_{M/A}(Z) = r_M(A \cup Z) - r_M(A)$, so $r_M(A \cup Z) = r_M(A) + |Z| = r_M(A) + r_M(Z)$. Hence $r_M(X \cup Z) = r_M(X) + r_M(Z)$ as $X \subseteq A$. Thus, from (1), $r_{M^+/Z\backslash Y}(X) = r_M(X)$, so $(M^+/Z\backslash Y)|A = M|A = N|A$.

Now $\sqcap_M(B, X) \in \{0, 1, 2\}$. Suppose $\sqcap_M(B, X) = 2$. Then, by Lemma 4.3, $\{\alpha, \beta\} \subseteq \mathrm{cl}_N(X)$. We have

$$r_{M^+}(B \cup X) = r_{M^+}(B) + r_{M^+}(X) - 2. \tag{2}$$

Since Z is a basis of M/A, the set Z is independent in $M^+|B$ and has $r(B)-2$ elements. Thus $Z \cup \{\alpha, \beta\}$ has $r(B)$ elements and this set is independent since α and β were freely added to B. Hence $Z \cup \{\alpha, \beta\}$ spans B in M^+. Therefore, from (2), $r_{M^+}(Z \cup \{\alpha, \beta\} \cup X) = r_M(B) + r_M(X) - 2$, so

$$r_{M^+/Z\backslash Y}(X \cup \{\alpha, \beta\}) = r_M(B) + r_M(X) - 2 - r_M(Z) = r_{M^+}(X) = r_N(X).$$

Thus $r_{M^+/Z\backslash Y}(X \cup \{\alpha, \beta\}) = r_N(X)$ and so $r_{M^+/Z\backslash Y}(X \cup \alpha) = r_N(X \cup \alpha)$ and $r_{M^+/Z\backslash Y}(X \cup \beta) = r_N(X \cup \beta)$.

Next suppose that $\sqcap_M(B, X) = 0$. Then, by Lemma 4.4, $\alpha \notin \mathrm{cl}_N(X \cup \beta)$. Thus $\beta \notin \mathrm{cl}_N(X)$ as α and β are clones. Hence $r_N(X \cup \{\alpha, \beta\}) = r_N(X) + 2$.

Now $r_M(B \cup X) = r_M(B) + r_M(X)$, so $r_{M^+}(Z \cup \{\alpha,\beta\} \cup X) = r_M(B) + r_M(X)$. Hence

$$\begin{aligned} r_{M^+/Z\backslash Y}(X \cup \{\alpha,\beta\}) &= r_M(B) + r_M(X) - r_M(Z) \\ &= r_M(X) + 2 \\ &= r_{M^+/Z\backslash Y}(X) + 2. \end{aligned}$$

Thus $r_N(X \cup \{\alpha,\beta\}) = r_{M^+/Z\backslash Y}(X \cup \{\alpha,\beta\})$ and $r_N(X \cup \gamma) = r_{M^+/Z\backslash Y}(X \cup \gamma)$ for each γ in $\{\alpha,\beta\}$.

Finally, suppose that $\sqcap_M(B, X) = 1$. Then, by Lemma 4.4, $r_N(X \cup \{\alpha,\beta\}) = r_N(X \cup \alpha) = r_N(X \cup \beta) = r_N(X) + 1$. Now

$$r_{M^+}(X \cup Z \cup \beta) \geq r_{M^+}(X \cup Z) = r_{M^+}(X) + r_{M^+}(Z).$$

On the other hand,

$$\begin{aligned} r_{M^+}(X \cup Z \cup \beta) &\leq r_{M^+}(X \cup Z \cup \beta \cup \alpha) \\ &= r_{M^+}(X \cup B) \\ &= r_{M^+}(X) + r_{M^+}(B) - 1 \\ &= r_{M^+}(X) + r_{M^+}(Z) + 1. \end{aligned}$$

If $r_{M^+}(X \cup Z \cup \beta) = r_{M^+}(X \cup Z)$, then, as α and β are clones, $r_{M^+}(X \cup Z \cup \beta \cup \alpha) = r_{M^+}(X \cup Z)$; a contradiction. Thus $r_{M^+}(X \cup Z \cup \beta) > r_{M^+}(X \cup Z)$, so $r_{M^+}(X \cup Z \cup \beta) = r_{M^+}(X) + r_{M^+}(Z) + 1$. Hence $r_{M^+/Z\backslash Y}(X \cup \beta) = r_{M^+}(X) + 1 = r_{M^+/Z\backslash Y}(X) + 1$. We conclude that the rank functions of N and $M^+/Z\backslash Y$ do indeed coincide, so these two matroids are equal. □

Lemma 4.6. *Suppose $Z \subseteq E(M)$.*

(i) *If $Z \supseteq B$, then*

$$r_M(Z) = r_N((Z - B) \cup \{\alpha,\beta\}) + r_M(B) - 2.$$

(ii) *If $Z \subseteq A$, then $r_M(Z) = r_N(Z)$.*

Proof. Part (ii) follows immediately from the fact that $N|A = M|A$. For (i), we note that $r_M(Z) = r_{M^+}(Z \cup \{\alpha,\beta\})$. Recall that $N = M^+/C_N\backslash D_N$ where $|C_N| = r_M(B) - 2$ and $C_N \cup \{\alpha,\beta\}$ spans $B \cup \{\alpha,\beta\}$ in M^+. Thus

$$\begin{aligned} r_N((Z - B) \cup \{\alpha,\beta\}) &= r_{M^+}((Z - B) \cup C_N \cup \{\alpha,\beta\}) - r_M(B) + 2 \\ &= r_{M^+}(Z \cup \{\alpha,\beta\}) - r_M(B) + 2 \\ &= r_M(Z) - r_M(B) + 2. \end{aligned}$$

Hence (i) holds. □

The next result is a straightforward consequence of the last lemma and we omit the proof.

Corollary 4.7. *Let (A, B) be a 3-separation in a 3-connected matroid M. Let N be the clonal replacement of B by $\{\alpha, \beta\}$. If $(X \cup \{\alpha, \beta\}, Y)$ is a 3-separation of N with $X \cap \{\alpha, \beta\} = \emptyset$, then $(X \cup B, Y)$ is a 3-separation of M.*

Lemma 4.8. *Let (A, B) be a 3-separation in a 3-connected matroid M. Let N be the clonal replacement of B by $\{\alpha, \beta\}$. Suppose $X \subseteq A$ and $y \in A - X$. Then*

(i) *$y \in \mathrm{cl}_M(X)$ if and only if $y \in \mathrm{cl}_N(X)$; and*
(ii) *$y \in \mathrm{cl}^*{}_M(X)$ if and only if $y \in \mathrm{cl}^*{}_N(X)$.*

Proof. Since $M|(X \cup y) = N|(X \cup y)$, part (i) is immediate. For (ii), we note that $y \in \mathrm{cl}^*{}_M(X)$ if and only if $y \notin \mathrm{cl}_M(E(M) - (X \cup y))$. The latter holds if and only if $y \notin \mathrm{cl}_{M^+}(E(M^+) - (X \cup y))$.

Now

$$\begin{aligned} \mathrm{cl}^*{}_N(X) &= \mathrm{cl}_{N^*}(X) \\ &= \mathrm{cl}_{(M^+/C_N\backslash D_N)^*}(X) \\ &= \mathrm{cl}_{(M^+)^*\backslash C_N/D_N}(X) \\ &= \mathrm{cl}_{(M^+)^*}(X \cup D_N) - (C_N \cup D_N). \end{aligned}$$

Since $y \in A - X$, we have $y \in \mathrm{cl}^*{}_N(X)$ if and only if $y \in \mathrm{cl}^*{}_{M^+}(X \cup D_N)$. The latter holds if and only if $y \notin \mathrm{cl}_{M^+}(E(M^+) - (X \cup D_N \cup y))$. But $C_N \cup \{\alpha, \beta\}$ spans D_N, so $\mathrm{cl}_{M^+}(E(M^+) - (X \cup D_N \cup y)) = \mathrm{cl}_{M^+}(E(M^+) - (X \cup y))$. Hence $y \in \mathrm{cl}^*{}_N(X)$ if and only if $y \notin \mathrm{cl}_{M^+}(E(M^+) - (X \cup y))$. By the first paragraph, the latter holds if and only if $y \in \mathrm{cl}^*{}_M(X)$. Thus (ii) holds. □

Lemma 4.9. *Let (A, B) be a 3-separation in a 3-connected matroid M. Let N be the clonal replacement of B by $\{\alpha, \beta\}$. If $e \in A$ and $Z \subseteq A - e$, then $\lambda_{N\backslash e}(Z) = \lambda_{M\backslash e}(Z)$.*

Proof. We have $\lambda_{N\backslash e}(Z) = r_{N\backslash e}(Z) + r_{N\backslash e}(E(N\backslash e) - Z) - r(N\backslash e)$, so

$$\begin{aligned} \lambda_{N\backslash e}(Z) &= r_{M\backslash e}(Z) + r_{M^+\backslash e/C_N\backslash D_N}(E(N\backslash e) - Z) - r(N) \\ &= r_{M\backslash e}(Z) + r_{M^+\backslash e}((E(N\backslash e) \cup C_N) - Z) - |C_N| - r(M) + |C_N| \\ &= r_{M\backslash e}(Z) + r_{M\backslash e}(E(M\backslash e) - Z) - r(M\backslash e) \\ &= \lambda_{M\backslash e}(Z) \end{aligned}$$

where the second-last equality holds because $(E(N\backslash e) - Z) \cup C_N$ contains $C_N \cup \{\alpha, \beta\}$, which spans $C_N \cup D_N \cup \{\alpha, \beta\}$ in $M^+\backslash e$. □

Corollary 4.10. *Let (A, B) be a 3-separation in a 3-connected matroid M. Let N be the clonal replacement of B by $\{\alpha, \beta\}$. If $e \in A$ and $Z \subseteq A - e$, then Z is sequential in $N\backslash e$ if and only if Z is sequential in $M\backslash e$.*

Proof. Suppose that Z is sequential in $N\backslash e$. Then there is a sequential ordering $(z_1, z_2, \ldots, z_n)$ of Z in $N\backslash e$. Thus $\lambda_{N\backslash e}(\{z_1, z_2, \ldots, z_i\}) = 2$ for all i in $\{2, 3, \ldots, n\}$. Hence $\lambda_{M\backslash e}(\{z_1, z_2, \ldots, z_i\}) = 2$ for all such i, and Z is sequential in $M\backslash e$. The proof of the converse is similar. □

Lemma 4.11. *Let (A, B) be a 3-separation in a 3-connected matroid M. Let N_1 and N_2 be the clonal replacements of B by $\{\alpha, \beta\}$ in M and M^*, respectively. Then $N_1^* = N_2$.*

Proof. Let $X, \{y\}, Z$ be disjoint sets whose union is $A \cup \{\alpha, \beta\}$. The matroids N_1 and N_2 are dual to each other if and only if, for every such collection of sets, $y \in \text{cl}_{N_1}(X)$ if and only if $y \notin \text{cl}_{N_2}(Z)$.

Suppose first that $X \subseteq A$ and $y \in A$. By Lemma 4.8, the following statements are equivalent:

(a) $y \in \text{cl}_{N_1}(X)$;
(b) $y \in \text{cl}_M(X)$;
(c) $y \in \text{cl}^*{}_{M^*}(X)$;
(d) $y \in \text{cl}^*{}_{N_2}(X)$;
(e) $y \in \text{cl}_{N_2^*}(X)$;
(f) $y \notin \text{cl}_{N_2}(Z)$.

Next assume that $X \subseteq A$ and $y = \alpha$. Then the following are equivalent.

(a) $\alpha \in \text{cl}_{N_1}(X)$;
(b) $\sqcap_M(B, X) = 2$;
(c) $\sqcap_M(B, Z - \beta) + \lambda_M(X) - \lambda_M(Z - \beta) = 2$;
(d) $\sqcap_M(B, Z - \beta) + \lambda_M(B \cup (Z - \beta)) - \lambda_M(Z - \beta) = 2$;
(e) $\lambda_M(B) - \sqcap_{M^*}(B, Z - \beta) = 2$;
(f) $\sqcap_{M^*}(B, Z - \beta) = 0$;
(g) $\alpha \notin \text{cl}_{N_2}(Z)$.

The equivalence of (a) and (b) follows from Lemma 4.3; for (b) and (c), use [9, Lemma 2.4(iv)]; for (d) and (e), use Lemma 2.6; for (f) and (g), use Lemma 4.4.

Next suppose that $\{\alpha, \beta\} \subseteq X$. Then $Z \subseteq A$ and this case is symmetric to the case when $X \subseteq A$ and $y \in A$. Likewise, the case when $\alpha \in X$ and $y = \beta$ is symmetric to the case when $X \subseteq A$ and $y = \alpha$.

By symmetry, the only remaining case is when $\alpha \in X$ and $\beta \in Z$. Suppose $y \in \text{cl}_{N_1}(X)$. In particular, suppose $y \in \text{cl}_{N_1}(X - \alpha)$. Then, from above, $y \in \text{cl}_{N_2^*}(X - \alpha)$, so $y \notin \text{cl}_{N_2}(Z \cup \alpha)$. Hence $y \notin \text{cl}_{N_2}(Z)$. Now suppose $y \notin \text{cl}_{N_1}(X - \alpha)$. Then, by the Mac Lane-Steinitz condition, $\alpha \in \text{cl}_{N_1}((X -$

$\alpha)\cup y)$. Thus $\alpha \in \mathrm{cl}_{N_2^*}((X-\alpha)\cup y)$. If $\alpha \notin \mathrm{cl}_{N_2^*}(X-\alpha)$, then $y \in \mathrm{cl}_{N_2^*}((X-\alpha)\cup\alpha) = \mathrm{cl}_{N_2^*}(X)$, so $y \notin \mathrm{cl}_{N_2}(Z)$. If $\alpha \in \mathrm{cl}_{N_2^*}(X-\alpha)$, then $\alpha \in \mathrm{cl}_{N_1}(X-\alpha)$, so $\mathrm{cl}_{N_1}(X) = \mathrm{cl}_{N_1}(X-\alpha)$ and $y \in \mathrm{cl}_{N_1}(X-\alpha)$; a contradiction. We conclude that, when $\alpha \in X$ and $\beta \in Z$, if $y \in \mathrm{cl}_{N_1}(X)$, then $y \notin \mathrm{cl}_{N_2}(Z)$.

Finally, when $\alpha \in X$ and $\beta \in Z$, assume that $y \notin \mathrm{cl}_{N_1}(X)$. Then $y \notin \mathrm{cl}_{N_1}(X-\alpha)$, so $y \notin \mathrm{cl}_{N_2^*}(X-\alpha)$. Hence $y \in \mathrm{cl}_{N_2}(Z\cup\alpha)$. If $y \in \mathrm{cl}_{N_2}(Z)$, we have the desired result, so assume that $y \notin \mathrm{cl}_{N_2}(Z)$. Then $\alpha \in \mathrm{cl}_{N_2}(Z\cup y)$. Moreover, $\alpha \notin \mathrm{cl}_{N_2}(Z)$ otherwise $\mathrm{cl}_{N_2}(Z\cup\alpha) = \mathrm{cl}_{N_2}(Z)$. Thus $\alpha \in \mathrm{cl}_{N_2^*}((X-\alpha)\cup y)$, so, from above, $\alpha \in \mathrm{cl}_{N_1}((X-\alpha)\cup y)$. Then $\alpha \in \mathrm{cl}_{N_1}(X-\alpha)$ otherwise $y \in \mathrm{cl}_{N_1}(X)$. Thus $\alpha \in \mathrm{cl}_{N_2^*}(X-\alpha)$ so $\alpha \notin \mathrm{cl}_{N_2}(Z\cup y)$; a contradiction. □

Having developed this theory of clonal replacements, we are now ready to use it to prove the main result of the section, Lemma 4.13. We shall require one more preliminary result.

Lemma 4.12. *Let $(S, E(M)-S)$ be a non-sequential 3-separation in a 3-connected matroid M. Suppose that, for every non-sequential 3-separation (U,V) of M, either $S \subseteq \mathrm{fcl}(U)$ or $S \subseteq \mathrm{fcl}(V)$. If X is a non-sequential 3-separating set that is contained in S, then $(X, E(M)-X) \cong (S, E(M)-S)$.*

Proof. Assume that $(X, E(M)-X) \ncong (S, E(M)-S)$. Since $E(M)-X \supseteq E(M)-S$ and the latter is non-sequential, so too is the former. Thus $(X, E(M)-X)$ is non-sequential. Therefore either $S \subseteq \mathrm{fcl}(X)$ or $S \subseteq \mathrm{fcl}(E(M)-X)$. Thus either $\mathrm{fcl}(X) \subseteq \mathrm{fcl}(S) \subseteq \mathrm{fcl}(X)$, or $\mathrm{fcl}(X) \subseteq \mathrm{fcl}(S) \subseteq \mathrm{fcl}(E(M)-X)$. The latter case implies that $X \subseteq \mathrm{fcl}(E(M)-X)$, so X is sequential; a contradiction. Hence $\mathrm{fcl}(X) = \mathrm{fcl}(S)$. Since $(S, E(M)-S)$ is non-sequential, it follows, by [9, Lemma 3.3], that $(S, E(M)-S) \cong (X, E(M)-X)$; a contradiction. □

Lemma 4.13. *Let $(S, E(M)-S)$ be a non-sequential 3-separation in a 3-connected matroid M. Suppose that $E(M)-S$ is fully closed, and that, for every non-sequential 3-separation (U,V) of M, either $S \subseteq \mathrm{fcl}(U)$ or $S \subseteq \mathrm{fcl}(V)$. If S contains no triangles or triads of M, then the clonal replacement, N, of $E(M)-S$ by $\{\alpha,\beta\}$ is 4-connected.*

Proof. We have $r_N(S\cup\{\alpha,\beta\}) = r_N(S)$. Let (X,Y) be a k-separation of N for some k in $\{1,2,3\}$. We shall show that we can choose (X,Y) so that $\{\alpha,\beta\} \subseteq X$ or $\{\alpha,\beta\} \subseteq Y$. Suppose, instead, that $\alpha \in X$ and $\beta \in Y$. Then $|X|,|Y| \geq 2$. We may suppose that $|Y| \geq |X|$. Assume that $\alpha \in \mathrm{cl}_N(X-\alpha)$. Then $\beta \in \mathrm{cl}_N(X-\alpha)$ and $(X\cup\beta, Y-\beta)$ is a k-separating partition of N that is a k-separation unless $|Y| = k$. In the exceptional case, $|E(N)| = 2k$. But $|E(N)| = |S|+2$ and S is non-sequential, so $|E(N)| \geq 6$. Thus $k = 3$ and Y is a triangle of N and hence of M; a contradiction. We may now suppose that $\alpha \notin \mathrm{cl}_N(X-\alpha)$. Then $(X-\alpha, Y\cup\alpha)$ is a k-separating partition of

N and $\alpha \in {\rm cl}^*{}_N(X-\alpha)$, so $\beta \in {\rm cl}^*{}_N(X-\alpha)$. Hence $(X \cup \beta, Y-\beta)$ is a k-separation of N unless $|E(N)| = 6$, and X and Y are triads of N. Since α and β are clones in N, it is straightforward to show that $N \cong U_{4,6}$. Hence S contains a triad of M; a contradiction. Thus we may assume that the k-separation (X,Y) of N is chosen so that $\{\alpha,\beta\} \subseteq X$.

Let $R = E(M) - S$. Then, since $N = M^+/C_N \backslash D_N$ where $|C_N| = r_M(R) - 2$ and $C_N \cup \{\alpha,\beta\}$ spans R in M^+, we have

$$\begin{aligned}
&r_M((X-\{\alpha,\beta\}) \cup R) + r_M(Y) - r(M)\\
&= r_{M^+}(X \cup R) + r_N(Y) - r(N) - |C_N|\\
&= r_N(X) + |C_N| + r_N(Y) - r(N) - |C_N| = \lambda_N(X).
\end{aligned}$$

Since M is 3-connected, we deduce that $\lambda_N(X) = 2$, that is, (X,Y) is a 3-separation of M.

If Y is sequential in N, then, by Lemma 4.8, Y is sequential in M, so Y contains a triangle or triad of M; a contradiction. We conclude that Y is non-sequential. Now suppose that X is sequential in N having $(x_1, x_2, \ldots, x_k)$ as a sequential ordering. As α and β are clones, we may assume that $(\alpha,\beta) = (x_i, x_{i+1})$ for some i. If $i > 3$, then $\{x_1, x_2, x_3\}$ is sequential in N and hence in M, so $\{x_1, x_2, x_3\}$ is a triangle or triad of M; a contradiction. Thus $i \le 3$. If $i \le 2$, we may relabel so that $i = 1$. If $i = 3$, then $\{x_1, x_2, \alpha, \beta\}$ has rank 2 in N or N^* and again we may relabel so that $i = 1$. But, when $i = 1$, we have $x_3 \in {\rm cl}_N(\{\alpha,\beta\})$ or $x_3 \in {\rm cl}_{N^*}(\{\alpha,\beta\})$. In the first case, by Lemma 4.4(i), $x_3 \in {\rm cl}_M(R)$ contradicting the fact that R is fully closed in M. By Lemma 4.11, N^* can be constructed from M^* by the clonal replacement of S by $\{\alpha,\beta\}$. Hence, when $x_3 \in {\rm cl}_{N^*}(\{\alpha,\beta\})$, we also obtain a contradiction. We deduce that X is non-sequential.

We now know that (X,Y) is non-sequential and that X contains $\{\alpha,\beta\}$. Suppose that $\{\alpha,\beta\} \subseteq {\rm fcl}_N(Y)$. Then, for some subset X' of X, there is a 3-sequence $(X', \alpha, \beta, z_3, \ldots, z_m, Y)$ in N. As X is non-sequential in N, so too is X'. Thus, by Lemma 4.8, X' is non-sequential in M. Let $Y' = Y \cup \{z_3, z_4, \ldots, z_m\}$. Then $E(M) - X' = Y' \cup R$. As $(X' \cup \{\alpha,\beta\}, Y')$ is a 3-separation of N, it follows by Corollary 4.7 that $(X' \cup R, Y')$ is a 3-separation of M.

We show next that $Y' \cup R$ is non-sequential in M. Assume it is sequential having $(y_1, y_2, \ldots, y_k)$ as a sequential ordering. Consider $\{y_1, y_2, y_3\} \cap R$. As R is fully closed, if this intersection has at least two elements, it contains three elements and, in that case, by uncrossing, we may assume that the sequential ordering of $Y' \cup R$ begins with all the elements of R. But this contradicts the fact that R is fully closed. We deduce that $|\{y_1, y_2, y_3\} \cap R| \le$

1, so we can reorder if necessary to get that $\{y_1, y_2\} \subseteq Y'$. Then, since $X' \cup R$ is 3-separating in M, it follows by uncrossing that $Y' \cup R$ has a sequential ordering that begins with the elements of Y'. As $|Y'| \geq 3$ and $Y' \subseteq S$, we get the contradiction that S contains a triangle or triad of M. We deduce that $E(M) - X'$ is indeed non-sequential in M.

As $X' \subseteq S$, it follows by Lemma 4.12 that $(X', E(M) - X') \cong (S, E(M) - S)$ in M. Thus $\text{fcl}_M(X') = \text{fcl}_M(S)$. Since S is 3-separating, there is a 3-sequence $(X', u_1, u_2, \ldots, u_p, E(M) - S)$ in M. Thus, for all j in $[p]$, the set $X' \cup \{u_1, u_2, \ldots, u_j\}$ is 3-separating in N. Therefore $Y' \cap (X' \cup \{u_1, u_2, \ldots, u_j\})$ is 3-separating in N. But $Y' = \{u_1, u_2, \ldots, u_p\}$, so Y' is sequential in N. Hence so is Y; a contradiction.

We may now assume that $\{\alpha, \beta\} \not\subseteq \text{fcl}_N(Y)$. Let $\text{fcl}_N(Y) = Z$. Then $Z \subseteq S$ because α and β are clones in N. As Z is non-sequential in N, it is non-sequential in M. By Lemma 4.12, $\text{fcl}_M(Z) = \text{fcl}_M(S)$. Thus $\text{fcl}_N(Y) \supseteq S$, so $\text{fcl}_N(Y) \supseteq \{\alpha, \beta\}$; a contradiction.

We conclude that N has no 3-separations, so N is 4-connected. □

5. Some technical results

Geelen and Whittle [2, Theorem 5.1] proved that a 4-connected matroid has an element z whose deletion or contraction is sequentially 4-connected. In this section, we shall extend this result by showing that the element z can be chosen to avoid a specified clonal pair. In order to prove this extension, we shall first extend Lemmas 5.3 and 5.4 from [2]. Our proofs of these results will very closely follow the original proofs. We also close a small gap in the original proof of [2, Lemma 5.3].

We shall use the following result [2, Lemma 5.2].

Lemma 5.1. *Let $\{t_1, t_2, t_3, a_1, a_2, a_3, b_1, b_2, b_3\}$ be distinct elements of a 4-connected matroid M. Suppose, for each k in $\{1, 2, 3\}$, that $M \backslash t_k$ is $(4,4)$-connected and that $\{t_1, t_2, t_3, a_k, b_k\} - \{t_k\}$ is a quad of $M \backslash t_k$. Then M/t_1 is sequentially 4–connected.*

Lemma 5.2. *Let M be a 4-connected matroid with at most 11 elements and let $x, a, p, b_1, b_2, c_1, c_2$ be distinct elements of M. Suppose that $M \backslash x$ is $(4,4)$-connected with a quad $\{a, p, b_1, b_2\}$, and that $\{b_1, b_2, c_1, c_2\}$ is a quad of $M \backslash p$. Suppose that M/b_1 is not sequentially 4-connected. Then*

(i) *M/b_1 has a non-sequential 3-separation (R, G) with $|R - \{x, a, p, b_2, c_1, c_2\}| = 2$;*
(ii) *$M/c_1, M/c_2, M \backslash c_1$, or $M \backslash c_2$ is sequentially 4-connected; and*

(iii) *if none of* $M/b_2, M\backslash b_1$, *or* $M\backslash b_2$ *is sequentially* 4*-connected and* $\{c_1, c_2\}$ *is a clonal pair of* M*, then,*
  (a) *for some permutation* (i, j) *of* $\{1, 2\}$*, there are elements* e_1 *and* e_2 *of* $E(M) - \{x, a, p, b_1, b_2, c_1, c_2\}$ *such that* $\{b_j, p, e_1, e_2\}$ *is a quad of* $M\backslash b_i$*; and*
  (b) $M/p, M/e_1, M/e_2, M\backslash e_1$*, or* $M\backslash e_2$ *is sequentially* 4*-connected.*

Proof. Let $D = E(M) - \{x, a, p, b_1, b_2, c_1, c_2\}$, let $P = \{a, p, b_1, b_2\}$, and let $Q = \{b_1, b_2, c_1, c_2\}$. The quads Q and P imply that

$$\lambda_{M\backslash x}(Q \cup P) = r_{M\backslash x}(Q \cup P) + r^*_{M\backslash x}(Q \cup P) - |Q \cup P| \leq 4 + 4 - 6 = 2.$$

Thus equality holds here, so $r(Q \cup P) = 4$. But $\lambda_{M\backslash x}(Q \cup P) = r(D) + r(Q \cup P) - r(M)$, so $r(D) = r(M) - 2$. Now D is 3-separating in $M\backslash x$ but not in M, so $x \notin \text{cl}(D)$. The cocircuits $P \cup x$ and $Q \cup p$ of M imply that $r(D \cup a) = r(D) + 1$ and $r(D \cup \{a, x\}) < r(M)$. Hence $r(D \cup \{a, x\}) = r(D \cup a) = r(D \cup x)$, so $x \in \text{cl}(D \cup a)$ and $a \in \text{cl}(D \cup x)$.

Since M/b_1 is not sequentially 4-connected, it has a non-sequential 3-separation (R, G). As M is 4-connected, $b_1 \in \text{cl}_M(R) \cap \text{cl}_M(G)$. Since $\{a, p, b_2\}$ is a triangle of M/b_1, we may assume that $\{a, p, b_2\} \subseteq R$. Since $\{b_2, c_1, c_2\}$ is also a triangle of M/b_1, we may also assume that either $\{c_1, c_2\} \subseteq G$, or $\{c_1, c_2\} \subseteq R$. In the latter case, $G \subseteq D \cup x$. But the cocircuit $Q \cup p$ of M implies that $b_1 \notin \text{cl}(D \cup x)$, so $b_1 \notin \text{cl}(G)$; a contradiction. We conclude that $\{c_1, c_2\} \subseteq G$. Moreover, as $b_1 \in \text{cl}(G)$, we must have that $x \in G$. Thus $|R \cap D| \geq 2$ otherwise (R, G) is sequential since R is a 4-element 3-separating set containing a triangle.

Next we show that $|R \cap D| = 2$, that is, that (i) holds. Suppose not. Then R contains at least three elements of D. Then, as $|G| \geq 4$, it follows that $|D| = 4$, that $|G| = 4$, and that R contains exactly three elements of D. Thus G is a quad of M/b_1 so, by orthogonality, D is not a circuit of M/b_1. Moreover, $b_1 \notin \text{cl}_M(D)$, so D is independent in M/b_1 and hence in $M\backslash x$. As D is 3-separating in $M\backslash x$, we deduce, since $r_{M\backslash x}(D) = |D|$, that $r^*_{M\backslash x}(D) = 2$. Thus $R \cap D$ is a triad of $M\backslash x$. The circuit $G \cup b_1$ of M implies that $x \in \text{cl}((G \cup b_1) - x)$, so $R \cap D$ is a triad of M; a contradiction. We conclude that $|R \cap D| = 2$.

We now show the following.

5.2.1. *If* $r(M) \geq 5$*, then* $r(M) = 5$ *and* $r(D) = 3$*.*

Let $R \cap D = \{d_1, d_2\}$. As $b_2 \in \text{cl}_{M/b_1}(G)$, the 3-separation $(R - b_2, G \cup b_2)$ of M/b_1 is equivalent to (R, G). Since (R, G) is not sequential and $|R - b_2| = 4$, we deduce that $R - b_2$ is a quad of M/b_1, that is, $\{a, p, d_1, d_2\}$ is a quad in M/b_1. Hence $\{a, p, d_1, d_2\}$ is a cocircuit in M. Moreover,

$r(\{a, p, d_1, d_2, b_1, b_2\}) = 4$. Since $\{b_1, b_2, c_1, c_2\}$ is a circuit of M, we have $r(\{a, p, d_1, d_2, b_1, b_2, c_1, c_2\}) \leq 5$. As M is 4-connected and $r(M) \geq 5$, it follows that $r(M) = 5 = r(\{a, p, d_1, d_2, b_1, b_2, c_1, c_2\})$. Since $r(D) = r(M) - 2$, we deduce that $r(D) = 3$. Thus (5.2.1) holds.

Next we show that (ii) holds. This requires some case analysis. First observe that, by Lemma 2.19 and duality, (ii) holds if $r(M) \leq 4$ or $r^*(M) \leq 4$. Thus, we may assume $r(M), r^*(M) \geq 5$. Hence $r(M) = 5$ and $r(D) = 3$. Moreover, either $|E(M)| = 10$ and $|D| = 3$; or $|E(M)| = 11$ and $|D| = 4$.

If $\mathrm{cl}_M(\{a, p, d_1, d_2\})$ contains $\{c_1, c_2\}$, then $\mathrm{fcl}_{M/b_1}(R) = E(M/b_1)$, so (R, G) is a sequential 3-separation of M/b_1; a contradiction. Thus, by possibly interchanging the labels on c_1 and c_2, we may assume that $c_1 \notin \mathrm{cl}_M(\{a, p, d_1, d_2\})$. As M has $\{a, p, d_1, d_2, b_1\}$ as a circuit, $r(\{a, p, d_1, d_2\}) = 4$, so $r(\{a, p, d_1, d_2, c_1\}) = 5$. Now suppose that M/c_1 is not sequentially 4-connected, and has (J_1, K_1) as a non-sequential 3-separation. Then $c_1 \in \mathrm{cl}_M(J_1) \cap \mathrm{cl}_M(K_1)$. As $r(M/c_1) = 4$, both J_1 and K_1 have rank 3 in M/c_1. Thus neither J_1 nor K_1 contains $\{a, p, d_1, d_2\}$. Also, as $\{a, p, d_1, d_2\}$ is a cocircuit of M/c_1, we deduce that each of J_1 and K_1 contains two elements of $\{a, p, d_1, d_2\}$. Since $\{c_2, b_1, b_2\}$ is a circuit of M/c_1, we may assume that $\{c_2, b_1, b_2\} \subseteq J_1$. As $c_1 \in \mathrm{cl}(K_1)$ and $\{b_1, b_2, c_1, c_2, p\}$ is a cocircuit of M, it follows by orthogonality that $p \in K_1$. Thus $p \notin \mathrm{cl}(J_1)$ otherwise we could move p into J_1 to get a contradiction. The circuit $\{a, p, b_1, b_2\}$ of M now implies that $a \in K_1$.

If at least one of d_1 and d_2 is in K_1, then, since $\{a, p, d_1, d_2\}$ is a cocircuit of M, we may assume that both are. Then the circuits $\{a, p, d_1, d_2, b_1\}, \{a, p, b_1, b_2\}$, and $\{b_1, b_2, c_1, c_2\}$ of M imply that $\mathrm{fcl}_{M/c_1}(K_1) \supseteq E(M/c_1)$; a contradiction. Thus we may assume that $\{d_1, d_2\} \subseteq J_1$. As $r_{M/c_1}(J_1) = 3$, we have $r_M(J_1 \cup c_1) = 4$, so $r_{M/b_1}((J_1 - b_1) \cup c_1) = 3$.

Now $r_{M/b_1}(\{a, p, d_1, d_2, b_2\}) = 3$ and $r_{M/b_1}(\{d_1, d_2, c_1, c_2, b_2\}) = 3$. If $r_{M/b_1}(\{d_1, d_2, b_2\}) = 3$, then $r_{M/b_1}(\{a, p, d_1, d_2, c_1, c_2, b_2\}) = 3$, so $\mathrm{cl}_{M/b_1}(R_1) \supseteq \{b_2, c_1, c_2\}$. Hence (R, G) is a sequential 3-separation of M/b_1; a contradiction. We deduce that $r_{M/b_1}(\{d_1, d_2, b_2\}) = 2$, so M has $\{d_1, d_2, b_1, b_2\}$ as a circuit.

Next consider c_2, supposing first that $c_2 \in \mathrm{cl}_M(\{a, p, d_1, d_2\})$. As $\{a, p, d_1, d_2, b_1\}, \{d_1, d_2, b_1, b_2\}$, and $\{b_1, b_2, c_2, c_1\}$ are circuits of M, we deduce that $|\mathrm{cl}_M(\{a, p, d_1, d_2\})| \geq 8$. But M has rank 5 and at most eleven elements. Hence M has a cocircuit with at most three elements. We deduce that $c_2 \notin \mathrm{cl}_M(\{a, p, d_1, d_2\})$. If we assume that M/c_2 is not sequentially 4-connected and has (J_2, K_2) as a non-sequential 3-separation, then the argument given for c_1 and (J_1, K_1) gives that we may assume that $\{c_1, b_1, b_2, d_1, d_2\} \subseteq J_2$ and $\{a, p\} \subseteq K_2$.

Now take i in $\{1, 2\}$. If J_i meets $D - \{d_1, d_2\}$, then, since D is either a 3-element independent set or a 4-element circuit, we may assume that $J_i \supseteq D$. Hence $K_i \subseteq \{a, p, x\}$; a contradiction. We deduce that $J_i \cap D = \{d_1, d_2\}$.

Suppose that x is in K_1 or K_2, say K_1. Then $K_2 \subseteq K_1$. But $c_i \in \mathrm{cl}_M(K_i)$, so $\mathrm{cl}_M(K_1) \supseteq \{a, p, x, c_1, c_2\} \cup (D - \{d_1, d_2\})$. But $r_{M/c_1}(K_1) = 3$, so $r_M(K_1 \cup c_1) = 4$. Hence $\{b_1, b_2, d_1, d_2\}$ contains and therefore is a cocircuit of M. However, this set is also a circuit of M; a contradiction.

We may now assume that $x \in J_1 \cap J_2$. Then K_i is a quad of M/c_i for each i. Thus $\{a, p, d_3, d_4\}$ is a cocircuit of M where $\{d_3, d_4\} = D - \{d_1, d_2\}$. Recall that $\{a, p, d_1, d_2\}$ is also a cocircuit of M. Then M has a cocircuit C^* contained in $\{p, d_1, d_2, d_3, d_4\}$. The circuit $\{a, p, b_1, b_2\}$ of M implies that $C^* \subseteq \{d_1, d_2, d_3, d_4\}$. But $r(\{d_1, d_2, d_3, d_4\}) = 3$, so $\{d_1, d_2, d_3, d_4\}$ is 3-separating in M; a contradiction. We conclude that (ii) holds.

Now assume that none of $M/b_2, M\backslash b_1$, or $M\backslash b_2$ is sequentially 4-connected and that $\{c_1, c_2\}$ is a clonal pair of M. Then $r(M) \geq 5$. Next we observe that $M\backslash b_1$ and $M\backslash b_2$ are $(4, 4)$-connected. This is certainly true if $|E(M)| = 10$, so suppose that $|E(M)| = 11$. Then the 3-separation (R, G) of M/b_1 has $|R| = |G| = 5$, so M/b_1 is not $(4, 4)$-connected. Hence, by Lemma 2.3, $M\backslash b_1$ is $(4, 4)$-connected. By symmetry, so is $M\backslash b_2$.

Since $M\backslash b_1$ is not sequentially 4-connected, it has a quad D_{b_1}.

5.2.2. $D_{b_1} \cap \{c_1, c_2\} = \emptyset$.

Suppose D_{b_1} meets $\{c_1, c_2\}$. Then, by Lemma 2.20, $\{c_1, c_2\} \subseteq D_{b_1}$. If $\{a, p, b_2\} \subseteq E(M\backslash b_1) - D_{b_1}$, then $b_1 \in \mathrm{cl}(E(M\backslash b_1) - D_{b_1})$, a contradiction. Hence D_{b_1} meets $\{a, p, b_2\}$. By orthogonality with the cocircuit $\{a, p, b_2, b_1, x\}$ of M, we deduce that $|D_{b_1} \cap \{a, p, b_2, x\}| \geq 2$. But $\{c_1, c_2\} \subseteq D_{b_1}$, so $|D_{b_1} \cap \{a, p, b_2, x\}| = 2$. Now the circuit $\{b_1, b_2, c_1, c_2\}$ implies that $b_2 \notin D_{b_1}$. Moreover, $x \notin D_{b_1}$ otherwise, as $D_{b_1} - x \subseteq \{c_1, c_2, b_1, b_2, p, a\} = E(M\backslash x) - D$, we have $x \in \mathrm{cl}(E(M\backslash x) - D)$. But $\lambda_{M\backslash x}(E(M\backslash x) - D) = 2$, so $\lambda_M(E(M) - D) = 2$ contradicting the fact that M is 4-connected. Thus $\{x, b_2\}$ avoids D_{b_1}, so $D_{b_1} = \{c_1, c_2, a, p\}$. The cocircuit $D_{b_1} \cup b_1$ contradicts the fact that $a \in \mathrm{cl}(D \cup x)$, so (5.2.2) holds.

By symmetry:

5.2.3. $D_{b_2} \cap \{c_1, c_2\} = \emptyset$.

We now establish (iii)(a) by showing the following.

5.2.4. *For some i in $\{1,2\}$, the set D has a subset $\{e_1,e_2\}$ so that $D_{b_i} = \{b_j,p\} \cup \{e_1,e_2\}$ where $\{i,j\} = \{1,2\}$.*

Assume that the assertion fails for $i = 1$. If $b_2 \notin D_{b_1}$, then $\{c_1,c_2,b_2\} \subseteq E(M\backslash b_1) - D_{b_1}$, so b_1 is in the closure of the last set; a contradiction. Hence $b_2 \in D_{b_1}$. The cocircuit $\{b_1,b_2,c_1,c_2,p\}$ of M implies that $b_2 \notin \mathrm{cl}(D \cup \{a,x\})$. Thus $D_{b_1} - b_2 \nsubseteq D \cup \{a,x\}$, so $p \in D_{b_1}$. Hence $a \notin D_{b_1}$ otherwise $b_1 \in \mathrm{cl}(D_{b_1})$. Since the assertion fails for $i = 1$, we deduce that $x \in D_{b_1}$. Thus, for some element d' of D, we have $D_{b_1} = \{b_2,p,x,d'\}$.

By symmetry, if the assertion fails for $i = 2$, then $D_{b_2} = \{b_1,p,x,d''\}$ for some element d'' of D. Now $d' \neq d''$ otherwise $b_1 \in \mathrm{cl}(\{p,x,d'\}) \subseteq \mathrm{cl}(D_{b_1})$; a contradiction. The circuits D_{b_1}, D_{b_2}, and $\{a,p,b_1,b_2\}$ imply that $\mathrm{cl}(\{b_1,b_2,p,x\}) \supseteq \{d',d'',a\}$. As $r(M) \geq 5$, we deduce that $\{c_1,c_2\} \cup (D - \{d',d''\})$ contains a cocircuit of M. As M is 4-connected and $|E(M)| \leq 11$, we deduce that $|D| = 4$. By (5.2.1), $r(D) = 3$, so D is a circuit of M. But D meets the cocircuit $D_{b_1} \cup b_1$ of M in the single element d', contradicting orthogonality. We conclude that (5.2.4) holds.

By (5.2.4), after a possible relabelling, we may assume that $D_{b_1} = \{b_2,p,e_1,e_2\}$ where $\{e_1,e_2\} \subseteq D$. Then $M\backslash x$ is $(4,4)$-connected with a quad $\{a,b_1,p,b_2\}$, and $\{p,b_2,e_1,e_2\}$ is a quad of $M\backslash b_1$. Part (iii)(b) holds if M/p is sequentially 4-connected so we may assume that it is not. Thus, by (ii) applied with (x,a,b_1,p,b_2,e_1,e_2) replacing (x,a,p,b_1,b_2,c_1,c_2), we deduce that $M/e_1, M/e_2, M\backslash e_1$, or $M\backslash e_2$ is sequentially 4-connected. □

Corollary 5.3. *Let M be a 4-connected matroid with at most 11 elements including x,a,p,b_1,b_2,c_1,c_2 where $\{c_1,c_2\}$ is a clonal pair. If $M\backslash x$ is $(4,4)$-connected having $\{a,p,b_1,b_2\}$ as a quad, and $M\backslash p$ has $\{b_1,b_2,c_1,c_2\}$ as a quad, then $E(M) - \{c_1,c_2\}$ contains an element y such that $M\backslash y$ or M/y is sequentially 4-connected.*

Proof. The last lemma showed that there is such an element y in $\{b_1,b_2,p,e_1,e_2\}$. □

Lemma 5.4. *Let M be a 4-connected matroid with a clonal pair $\{\alpha,\beta\}$ and suppose that x is an element of $E(M) - \{\alpha,\beta\}$ such that $M\backslash x$ is $(4,4)$-connected having a quad P that avoids $\{\alpha,\beta\}$. Then at least one of the following holds:*

(i) *M/x is sequentially 4-connected;*
(ii) *P contains an element z such that $M\backslash z$ is sequentially 4-connected; or*
(iii) *$|E(M)| \leq 12$ and there is an element y of $E(M) - \{\alpha,\beta\}$ such that $M\backslash y$ or M/y is sequentially 4-connected.*

Proof. Let $P = \{p, a, b_1, b_2\}$, where p is chosen so that, if possible, $M\backslash p$ is $(4,4)$-connected. Suppose that the lemma fails for M. Neither $M\backslash p$ nor M/x is sequentially 4-connected. Thus, by Lemma 2.19, M has rank and corank at least 5, so $|E(M)| \geq 10$.

Now $M\backslash p$ has a non-sequential 3-separation $(X_1 \cup x, X_2)$ where $x \notin X_1$.

5.4.1. (X_1, X_2) *is a* 3*-separation of* $M\backslash p, x$ *and* $r(X_1 \cup x) = r(X_1)$.

Suppose not. Then $x \notin \mathrm{cl}(X_1)$ and (X_1, X_2) is a 2-separation of $M\backslash p, x$. Hence $X_1 \cup x$ is not a quad of $M\backslash p$. Thus $|X_1| \geq 4$. But both $(X_1 \cup p, X_2)$ and $(X_1, X_2 \cup p)$ are 3-separations of $M\backslash x$. Thus we get a contradiction to the fact that $M\backslash x$ is $(4,4)$-connected unless $|X_1| = |X_2| = 4$. Consider the exceptional case. Then $|E(M)| = 10$ and $r(M) = 5$. As $r(X_1) + r(X_2) - r(M\backslash p, x) = 1$, we deduce that $r(X_1) = r(X_2) = 3$. Since M has no triangles, it follows that X_1 and X_2 are circuits of $M\backslash p, x$. As $\{a, b_1, b_2\}$ is a triad of the last matroid, it follows by orthogonality that $\{a, b_1, b_2\}$ is a subset of X_1 or X_2. But $\{p, a, b_1, b_2\}$ is a circuit, so p is in $\mathrm{cl}(X_1)$ or $\mathrm{cl}(X_2)$. Hence $(X_1 \cup p, X_2)$ or $(X_1, X_2 \cup p)$ is a 2-separation of $M\backslash x$; a contradiction. Thus (5.4.1) holds.

Now $\{a, x, b_1, b_2\}$ is a cocircuit of $M\backslash p$. If either $X_1 \cup x$ or X_2 contains at least three elements of this set, then $M\backslash p$ has a 3-separation (Y_1, Y_2) with $\{a, x, b_1, b_2\}$ contained in Y_1 or Y_2. But $p \in \mathrm{cl}(\{a, b_1, b_2\})$, so (Y_1, Y_2) induces a 3-separation of M; a contradiction. Thus we may assume that $a \in X_1$ and $\{b_1, b_2\} \subseteq X_2$. Let $C = X_2 - \{b_1, b_2\}$ and $D = X_1 - a$. The cocircuit $P \cup x$ of M implies that $\mathrm{cl}_M(C)$ avoids $P \cup x$. If $\mathrm{cl}_M(C)$ meets D in D', say, then we replace (C, D, X_1, X_2) by $(C \cup D', D - D', X_1 - D, X_2 - D)$. Thus we may assume that C is closed.

We may also assume that:

5.4.2. *Either* $\{\alpha, \beta\} \subseteq C$ *or* $\{\alpha, \beta\} \cap C = \emptyset$.

If not, we may suppose that $\alpha \in C$ and $\beta \in D$. If $\alpha \in \mathrm{cl}(X_2 - \alpha)$, then $\beta \in \mathrm{cl}(X_2 - \alpha)$ and we can move β from D into C. Thus we may assume that $\alpha \notin \mathrm{cl}(X_2 - \alpha)$. Then we can move α from C to D to get a 3-separation of $M\backslash p$ equivalent to $(X_1 \cup x, X_2)$. Hence (5.4.2) holds.

5.4.3. $\lambda_{M\backslash p,x}(D) = 2 = \lambda_{M\backslash x}(D)$ *and* $|D| \leq 4$.

By uncrossing, as both X_2 and $\{a, b_1, b_2\}$ are 3-separating in $M\backslash p, x$, so too is D, the complement of their union. But D avoids $\{a, b_1, b_2\}$, and $\{a, p, b_1, b_2\}$ is a circuit, so D is also 3-separating in $M\backslash x$. As the last matroid is $(4,4)$-connected, $|D| \leq 4$.

5.4.4. $a \in {\mathrm{cl}^*}_{M\backslash p,x}(X_2)$ *and* $a \notin \mathrm{cl}(D)$.

The first assertion follows since $\{a, b_1, b_2\}$ is a triad of $M\backslash p, x$ and $\{b_1, b_2\} \subseteq X_2$. The second assertion follows by orthogonality.

5.4.5. $x \in \text{cl}(X_1) - \text{cl}(D)$ *and* $a \in \text{cl}(D \cup x)$.

Since $\lambda_{M\backslash x}(D) = 2$ and M is 4-connected, $x \notin \text{cl}(D)$. By (5.4.1), $x \in \text{cl}(X_1) = \text{cl}(D \cup a)$. Hence $a \in \text{cl}(D \cup x)$.

5.4.6. $\lambda_M(C) = \lambda_{M\backslash p}(C) = \lambda_{M\backslash p,x}(C) = 3$.

As $x \in \text{cl}(X_1)$ and $p \in \text{cl}(\{a, b_1, b_2\})$, we deduce that $\lambda_{M\backslash p,x}(C) = \lambda_{M\backslash p}(C) = \lambda_M(C)$. As $|X_2| \geq 4$, we have $|C| \geq 2$, so $\lambda_{M\backslash p,x}(C) \geq 2$. Now $\lambda_{M\backslash p,x}(X_1) = 2 = \lambda_{M\backslash p,x}(\{a, b_1, b_2\})$. Thus, by uncrossing, $\lambda_{M\backslash p,x}(X_1 \cup \{a, b_1, b_2\}) \leq 3$. But $\lambda_{M\backslash p,x}(X_1 \cup \{a, b_1, b_2\}) = \lambda_{M\backslash p,x}(C)$, so $\lambda_{M\backslash p,x}(C) \in \{2, 3\}$. Assume that $\lambda_{M\backslash p,x}(C) = 2$. Then $\lambda_M(C) = 2$, so $|C| = 2$. Let $C = \{c_1, c_2\}$. Then $\{c_1, c_2, b_1, b_2\}$ is a quad of $M\backslash p$. But $|D| \leq 4$, so $|E(M)| \leq 11$. By Lemma 5.2(ii), one of $M/b_1, M/c_1, M/c_2, M\backslash c_1$, or $M\backslash c_2$ is sequentially 4-connected. Thus (iii) holds unless $\{c_1, c_2\}$ meets $\{\alpha, \beta\}$. In the exceptional case, by (5.4.2), $\{\alpha, \beta\} = \{c_1, c_2\}$ and (ii) holds by Lemma 5.2(iii). Since M is a counterexample to the lemma, we deduce that $\lambda_{M\backslash p,x}(C) \neq 2$. Hence (5.4.6) holds.

5.4.7. $|D| \leq 3$.

Suppose not. Then, by (5.4.3), $|D| = 4$ and $\lambda_{M\backslash x}(D) = 2$. Thus $\lambda_M(D \cup x) = 3$. As $a \in \text{cl}_M(D \cup x)$, it follows that $(D \cup x, X_2 \cup p)$ is a 3-separation of M/a. Hence the last matroid is not $(4,4)$-connected. Thus, by Lemma 2.3, $M\backslash a$ is $(4,4)$-connected. The choice of p implies that $M\backslash p$ is $(4,4)$-connected. But, by (5.4.6), $|C| \geq 3$, so $|X_2| \geq 5$ and the 3-separation $(X_1 \cup x, X_2)$ shows that $M\backslash p$ is not $(4,4)$-connected. Hence (5.4.7) holds.

5.4.8. $b_1 \in \text{cl}(C \cup b_2)$.

As P is a quad of $M\backslash x$, neither b_1 nor b_2 is in $\text{cl}(C)$. If $b_1 \notin \text{cl}(C \cup b_2)$, then $(X_1 \cup x \cup \{b_1, b_2\}, C)$ is an equivalent 3-separation of $M\backslash p$ to $(X_1 \cup x, X_2)$, so $\lambda_{M\backslash p}(C) = 2$, contradicting (5.4.6).

5.4.9. $r(X_1 \cup \{b_1, b_2\}) = r(X_1) + 2$.

Assume not. Then $r(X_1 \cup \{b_1, b_2\}) \leq r(X_1) + 1$. As $C = X_2 - \{b_1, b_2\}$, we have $r(X_2 - \{b_1, b_2\}) = r(X_2) - 1$. Thus $\lambda_{M\backslash p}(C) \leq \lambda_{M\backslash p}(X_2) = 2$; a contradiction.

Consider any 3-separation (Q, Q') of $M\backslash a$.

5.4.10. *Both* Q *and* Q' *meet both* $D \cup x$ *and* $\{p, b_1, b_2\}$.

This follows immediately from the facts that $a \in \mathrm{cl}(D \cup x) \cap \mathrm{cl}(\{p, b_1, b_2\})$ and M is 4-connected.

5.4.11. *If $C \subseteq Q'$, then $b_1, b_2 \in \mathrm{cl}(Q')$.*

Suppose that $b_1 \notin \mathrm{cl}(Q')$. Then, by (5.4.8), $b_2 \notin \mathrm{cl}(Q')$. Thus $\{b_1, b_2\} \subseteq Q$ so, by (5.4.10), $p \in Q'$. Moreover, $(Q \cup p, Q' - p)$ is not a 3-separation of $M\backslash a$ since $Q \cup p \supseteq \{p, b_1, b_2\}$. Thus $p \in \mathrm{cl}(Q' - p)$ and $p \notin \mathrm{cl}(Q)$. As $\{p, b_1, b_2, x\}$ is a cocircuit of $M\backslash a$, it follows by orthogonality that $x \in Q'$. Since $r(X_1 \cup \{b_1, b_2\}) = r(X_1) + 2$, we have $r(Q - \{b_1, b_2\}) = r(Q) - 2$. Hence $Q - \{b_1, b_2\}$ is 3-separating in $M\backslash a$. As $Q' \cup \{b_1, b_2\}$ contains $\{p, b_1, b_2\}$, and $\{a, p, b_1, b_2\}$ is a circuit, we deduce that $Q - \{b_1, b_2\}$ is 3-separating in M. Thus $|Q - \{b_1, b_2\}| \leq 2$. Hence Q is independent, so Q is a cosegment of $M\backslash a$. Choose d in $Q - \{b_1, b_2\}$. Then $\{d, b_1, b_2\}$ is a triad of $M\backslash a$, so $\{a, d, b_1, b_2\}$ is a cocircuit of M. Thus $d \in \mathrm{cl}^*{}_M(P)$, so $d \in \mathrm{cl}^*{}_{M\backslash x}(P)$. Hence $(P \cup d, E(M\backslash x) - (P \cup d))$ is a 3-separation of the $(4,4)$-connected matroid $M\backslash x$; a contradiction. Thus b_1 is in $\mathrm{cl}(Q')$ and, by symmetry, so is b_2.

5.4.12. *If $C \subseteq Q'$, then $Q - \{b_1, b_2\}$ is a triad of $M\backslash a$ containing p and two elements of D.*

By (5.4.11), $Q' \cup \{b_1, b_2\}$ is 3-separating in $M\backslash a$. As $M\backslash a$ has no triangles, Q' does not span $M\backslash a$. Hence $Q - \{b_1, b_2\}$ contains a cocircuit of $M\backslash a$. Thus $(Q' \cup \{b_1, b_2\}, Q - \{b_1, b_2\})$ is a 3-separation of $M\backslash a$. Hence $p \in Q - \{b_1, b_2\}$. As $Q - \{p, b_1, b_2\} \subseteq X_1 \cup x$ and $p \notin \mathrm{cl}(X_1 \cup x)$, we have $p \notin \mathrm{cl}(Q - \{p, b_1, b_2\})$. Thus $Q - \{p, b_1, b_2\}$ is 3-separating in $M\backslash a$. But $a \in \mathrm{cl}(Q' \cup \{p, b_1, b_2\})$, so $Q - \{p, b_1, b_2\}$ is 3-separating in M. Hence $|Q - \{p, b_1, b_2\}| \leq 2$. Thus $|Q - \{b_1, b_2\}| = 3$. Hence $Q - \{b_1, b_2\}$ is a triad of $M\backslash a$ containing p. It remains to show that $x \notin Q - \{b_1, b_2\}$ which will imply that $Q - \{p, b_1, b_2\} \subseteq D$. Suppose $x \in Q - \{b_1, b_2\}$. Then $Q - \{b_1, b_2\}$ meets D in a single element, say d. Now $(Q - \{b_1, b_2, x\}) \cup a$ is a triad $\{a, p, d\}$ of $M\backslash x$. Thus $(P \cup d, E(M\backslash x) - (P \cup d))$ is a 3-separation of $M\backslash x$, contradicting the fact that this matroid is $(4,4)$-connected. Hence $Q - \{p, b_1, b_2\} \subseteq D$.

5.4.13. *If $C \subseteq Q'$, then Q is a triad of $M\backslash a$ containing p and two elements of D.*

Assume that the assertion fails. Then Q meets $\{b_1, b_2\}$ and avoids x. Without loss of generality, assume $b_1 \in Q$. Then, by (5.4.10), $b_2 \in Q'$. Now $|Q| = 4$ and $\lambda_M(Q \cup a) = 3$. Thus $\lambda_{M\backslash x}(Q \cup a) \leq 3$. Also $\lambda_{M\backslash x}(P) = 2$ and $|P \cap (Q \cup a)| = 3$, so $\lambda_{M\backslash x}(P \cap (Q \cup a)) = 3$. The submodularity of λ implies that $\lambda_{M\backslash x}(P \cup Q \cup a) \leq 2$, so $\lambda_{M\backslash x}(Q' - (P \cup x)) \leq 2$. As $M\backslash x$ is $(4,4)$-connected, it follows that $|Q' - (P \cup x)| \leq 4$. By (5.4.6), $\lambda_{M\backslash x}(C) = 3$, so $C \neq Q' - (P \cup x)$ and $|C| \geq 3$. Thus $C \subsetneqq Q' - (P \cup x)$, so $|C| = 3$. Since $|E(M)| \geq 10$, it follows using (5.4.5) that $|D| = 3$ and $|Q' \cap D| = 1$.

Let d be the element of $Q' \cap D$. As $C \cup d = Q' - (P \cup x)$, we deduce that $\lambda_{M\backslash x}(C \cup d) = 2$. But $\lambda_{M\backslash x}(C) = 3$. Hence $d \in \mathrm{cl}(C)$ contradicting the fact that C is closed.

5.4.14. *If (R, R') is a 3-separation of $M\backslash a$ with x in R, and $|R|, |R'| \geq 4$, then $|R \cap \{p, b_1, b_2\}| = 1$ and $|R \cap C| \leq 2$. Moreover, if $|R \cap D| \neq 0$, then $|R' \cap C| \leq 2$.*

By (5.4.10), both R and R' meet $\{p, b_1, b_2\}$. Suppose that $R' \cap \{p, b_1, b_2\}$ contains a single element, t. Since $(P \cup x) - a$ is a cocircuit of $M\backslash a$ and three elements of it are in R, it follows that $(R \cup t, R' - t)$ is a 3-separation of $M\backslash a$. But $\{p, b_1, b_2\} \subseteq R \cup t$, contradicting (5.4.10). Hence $|R' \cap \{p, b_1, b_2\}| \geq 2$, so $|R \cap \{p, b_1, b_2\}| = 1$.

Now $\lambda_{M\backslash a}(R) = 2$ and $\lambda_{M\backslash a}(C) \leq 3$. Thus, by submodularity, $\lambda_{M\backslash a}(R \cap C) \leq 2$ or $\lambda_{M\backslash a}(R \cup C) \leq 2$. We have $|R' - C| \geq 2$. If equality holds here, then $D \cup x \subseteq R$, so, by (5.4.5), $a \in \mathrm{cl}(R)$; a contradiction. Thus $|R' - C| > 2$. Hence if $\lambda_{M\backslash a}(R \cup C) \leq 2$, then $(R \cup C, R' - C)$ is a 3-separation of $M\backslash a$. Therefore, by (5.4.13), $R' - C$ is a triad of $M\backslash a$ containing p and two elements of D; a contradiction as $|R' \cap \{p, b_1, b_2\}| \geq 2$. Thus $\lambda_{M\backslash a}(R \cup C) \not\leq 2$, so $\lambda_{M\backslash a}(R \cap C) \leq 2$. But $E(M\backslash a) - (R \cap C)$ contains $\{p, b_1, b_2\}$, and $\{a, p, b_1, b_2\}$ is a circuit of M, so $\lambda_M(R \cap C) \leq 2$. Hence $|R \cap C| \leq 2$.

Suppose we have chosen R so that $|R \cap D| \neq 0$. Then $|R - C| \geq 3$. As $\lambda_{M\backslash a}(C) \leq 3$ and $\lambda_{M\backslash a}(R') = 2$, either $\lambda_{M\backslash a}(R' \cap C) \leq 2$ or $\lambda_{M\backslash a}(R' \cup C) \leq 2$. By (5.4.13), the latter does not arise, so $\lambda_{M\backslash a}(R' \cap C) \leq 2$. As $E(M\backslash a) - (R' \cap C) \supseteq \{p, b_1, b_2\}$, we deduce that $\lambda_M(R' \cap C) \leq 2$, so $|R' \cap C| \leq 2$.

5.4.15. $|C| \leq 4$ *and* $|E(M)| \leq 12$.

Suppose that $|C| \geq 5$. Let (R, R') be an arbitrary 3-separation of $M\backslash a$ with x in R and $|R|, |R'| \geq 4$. By (5.4.14), $|R \cap C| \leq 2$, so $|R' \cap C| \geq 3$ and $|R \cap D| = 0$. Also $|R \cap \{p, b_1, b_2\}| = 1$, so $|R| \leq 4$. Hence $|R| = 4$ and $M\backslash a$ is $(4, 4)$-connected.

The choice of p implies that $M\backslash p$ is also $(4, 4)$-connected. Since $(X_1 \cup x, X_2)$ is a non-sequential 3-separation of $M\backslash p$ and $|X_2| = |C| + 2 \geq 7$, it follows that $|X_1 \cup x| = 4$, so $|D| = 2$ and $X_1 \cup x$ is a quad of $M\backslash p$. Also $|E(M)| \geq 12$. Since the lemma fails for M, it follows that $M\backslash a$ is not sequentially 4-connected. As $M\backslash a$ is $(4, 4)$-connected, it has a quad, R. From the previous paragraph, $x \in R$ otherwise $x \in R'$ and $|E(M)| = 9$. Let t be the unique element of $R \cap \{p, b_1, b_2\}$. If $t \neq p$, then $R - x \subseteq X_2$ so $x \in \mathrm{cl}(X_2)$. But $X_1 \cup x$ is a cocircuit of $M\backslash p$, so we have a contradiction to orthogonality. Thus $t = p$. We may now apply Lemma 5.1 with $(t_1, t_2, t_3) =$

(x, a, p). By that lemma, M/x is sequentially 4-connected; a contradiction. We conclude that $|C| \leq 4$. Hence $|E(M)| \leq 12$.

We now know that M has at most 12 elements and that the lemma fails for it. Thus we may assume $M\backslash a$ has a non-sequential 3-separation (R, R') with x in R.

5.4.16. $|R \cap D| \leq 1$.

Assume $|R \cap D| \geq 2$. Then $|R' \cap D| \leq 1$. We have $\lambda_{M\backslash a}(R) = 2$ and $\lambda_{M\backslash a}(D \cup x) \leq 3$. Hence $\lambda_{M\backslash a}((D \cap R) \cup x) \leq 2$ or $\lambda_{M\backslash a}(D \cup R) \leq 2$. But $a \in \mathrm{cl}(D \cup x) \subseteq \mathrm{cl}(D \cup R)$ and $|E(M\backslash a) - (D \cup R)| = |R' - D| \geq 3$. Hence $\lambda_{M\backslash a}(D \cup R) \nleq 2$, so $\lambda_{M\backslash a}((D \cap R) \cup x) \leq 2$. As $\{p, b_1, b_2\}$ avoids $(D \cap R) \cup x$, and $\{a, p, b_1, b_2\}$ is a circuit, we deduce that $\lambda_M((D \cap R) \cup x) \leq 2$, so $|(D \cap R) \cup x)| \leq 2$; a contradiction.

5.4.17. $1 \leq |R' \cap D| \leq 2$.

Suppose $|R' \cap D| \geq 3$. Hence $D \subsetneqq R'$. Now $\lambda_{M\backslash a}(R') = 2$ and $\lambda_{M\backslash a}(D \cup x) \leq 3$, so $\lambda_{M\backslash a}(R' \cap (D \cup x)) \leq 2$ or $\lambda_{M\backslash a}(R' \cup D \cup x) \leq 2$. As $a \in \mathrm{cl}(D \cup x)$, the latter implies that $\lambda_M(R' \cup D \cup x \cup a) \leq 2$, so $|R - (D \cup x)| \leq 2$. But $R \cap D = \emptyset$, so $|R| \leq 3$; a contradiction. Thus we may assume that $\lambda_{M\backslash a}(R' \cap (D \cup x)) \leq 2$, that is, $\lambda_{M\backslash a}(D) \leq 2$. As $a \in \mathrm{cl}(E(M\backslash a) - D)$, we deduce that $\lambda_M(D) \leq 2$; a contradiction since $|D| \geq 3$. Thus $|R' \cap D| \leq 2$. Finally, $R' \cap D \neq \emptyset$ otherwise $D \cup x \subseteq R$, so $a \in \mathrm{cl}(R)$ and (R, R') is a 3-separation of the 4-connected matroid M.

5.4.18. *If* $|D| = 2$*, then* $|E(M)| \geq 11$ *and* $M\backslash a$ *is* $(4, 4)$*-connected.*

Observe that $D \cup \{x, a\}$ is a circuit of M. As M is a counterexample to the lemma, M/a has a non-sequential 3-separation (S, S'') where we may assume that S contains the triangle $D \cup x$ of M/a. We may also suppose that the triangle $\{p, b_1, b_2\}$ of M/a is contained in S or S'. As M is 4-connected, $a \in \mathrm{cl}_M(S')$. Orthogonality using the cocircuit $\{p, a, b_1, b_2, x\}$ of M implies that $\{p, b_1, b_2\} \subseteq S'$. Thus both S and S' contain triangles, so neither is a quad. Hence $|S|, |S'| \geq 5$ and M/a is not $(4, 4)$-connected. Thus, by Lemma 2.3, $M\backslash a$ is $(4, 4)$-connected.

5.4.19. $|R' \cap D| = 2$.

Suppose, to the contrary, that $|R' \cap D| = 1$. Then, by (5.4.16), $|D| \leq 2$. As $D \cup \{a, x\} = X_1 \cup x$ and $|X_1 \cup x| \geq 4$, we deduce that $|D| = 2$, that $D \cup \{a, x\}$ is a quad of $M\backslash p$, and that $|D \cap R| = 1$. By (5.4.14), the last equation implies that $|R|, |R'| \leq 5$, so $|E(M)| \leq 11$. But, by (5.4.18), $|E(M)| \geq 11$ and $M\backslash a$ is $(4, 4)$-connected. This is a contradiction since we must have $|E(M)| = 11$, so $|R| = 5 = |R'|$ and (R, R') is a 3-separation of $M\backslash a$. We conclude that $|R' \cap D| = 2$.

5.4.20. $|D| = 3$.

Assume, to the contrary, that $|D| = 2$. Then $|R \cap D| = 0$. Also, by (5.4.18), $M\backslash a$ is $(4,4)$-connected. The choice of p implies that $M\backslash p$ is $(4,4)$-connected. Now P is a quad of $M\backslash x$ and $D \cup \{a, x\}$ is a quad of $M\backslash p$. Moreover, by (5.4.14), $|R| = 4$, so R is a quad of $M\backslash a$. The cocircuit $D \cup \{a, x, p\}$ of M implies, by orthogonality, that $p \in R$. We may now apply Lemma 5.1 with $(t_1, t_2, t_3) = (x, a, p)$ to get that M/x is sequentially 4-connected, contradicting the fact that M is a counterexample to the lemma.

5.4.21. $r(M) > r^*(M)$.

Assume that $r(M) \leq r^*(M)$. By (5.4.20), $|D| = 3$. As $\{a, p, b_1, b_2\}$ is a cocircuit of $M\backslash x$, we deduce that $r(D \cup a) = r(D) + 1 = 4$. By (5.4.9), $r(D \cup \{a, b_1, b_2\}) = r(D \cup a) + 2$. Hence $r(M) \geq 6$. As $|E(M)| \leq 12$, we deduce that $r^*(M) = 6 = r(M)$ and $|E(M)| = 12$. Thus $|C| = 4$. Hence, by (5.4.14), $|R \cap C| = 2 = |R' \cap C|$. Since $\lambda_M(C) = 3$, we deduce that C is a circuit or a cocircuit of M. But $r(D \cup \{a, b_1, b_2\}) = r(M)$, so C is not a cocircuit. Thus C is a circuit.

As $|(R - C) \cap D| = |R \cap D| \leq 1$, (5.4.13) implies that $\lambda_{M\backslash a}(R - C) \not\leq 2$. Since C is a circuit, $r(R' \cup C) \leq r(R) + 1$. Thus $r(R - C) = r(R)$. But $|R - C| \leq 3$, so $r(R) = 3 = |R - C|$. Therefore $R - P$ is a circuit of M containing x, so $x \in \mathrm{cl}(E(M\backslash x) - P)$ and $(P, E(M) - P)$ is a 3-separation of M; a contradiction.

5.4.22. *For each z in P, the matroid $M\backslash z$ is not $(4,4)$-connected.*

By (5.4.20) and (5.4.6), $|D| = 3$ and $|C| \geq 3$. Thus $|X_1 \cup x|, |X_2| \geq 5$, so $M\backslash p$ is not $(4,4)$-connected. The choice of M now implies that none of $M\backslash a, M\backslash b_1$, or $M\backslash b_2$ is $(4,4)$-connected.

By (5.4.22) and Lemma 2.3, M/z is $(4,4)$-connected for all z in P. But M/z is not sequentially 4-connected, so it has a quad D_z. Moreover, D_z is fully closed in M/z otherwise M/z is not $(4,4)$-connected.

5.4.23. *For each z in P, the quad D_z contains $\{\alpha, \beta, x\}$ and avoids P.*

By Lemma 2.20, either $\{\alpha, \beta\} \subseteq D_z$ or $\{\alpha, \beta\} \cap D_z = \emptyset$. Assume that the latter holds and consider $M^*\backslash z$. It is $(4,4)$-connected having D_z as a quad. By (5.4.21), $r(M^*) < r^*(M^*)$. Thus M^* is not a counterexample to the lemma. Hence neither is M; a contradiction. Thus $\{\alpha, \beta\} \subseteq D_z$.

Since $P - z$ is a triangle of M/z, and D_z is a fully closed quad, we have $(P - z) \cap D_z = \emptyset$. Thus $D_z \subseteq C \cup D \cup x$. Now $z \in \mathrm{cl}_M(D_z)$ and $P \cup x$ is a cocircuit of M containing z. Hence $x \in D_z$.

5.4.24. $D_p = D_{b_1} = D_{b_2}$.

Since each of D_p, D_{b_1}, and D_{b_2} is a cocircuit of M and all three contain $\{\alpha, \beta, x\}$, if two of these sets, say D_{z_1} and D_{z_2}, are distinct, then $D_{z_2} \subseteq \mathrm{fcl}_{M/z_1}(D_{z_1})$, contradicting the fact that D_{z_1} is fully closed in M/z_1.

By the last observation, $\mathrm{cl}(D_p)$ contains $\{x, p, b_1, b_2\}$ and hence also contains a. In addition, it contains at least three elements of $D \cup C$. Thus $\mathrm{cl}(D_p)$ avoids at most four elements of $E(M)$, so $4 = r(D_p) \geq r(M) - 1$. Hence $r(M) \leq 5$. But $r(M) > r^*(M) \geq 5$ and this contradiction completes the proof of Lemma 5.4. □

Next is the main result of this section.

Theorem 5.5. *Let M be a 4-connected matroid having a clonal pair $\{\alpha, \beta\}$. Then M has an element x not in $\{\alpha, \beta\}$ such that $M \backslash x$ or M/x is sequentially 4-connected.*

Proof. Let M be a counterexample to the theorem. First we show:

5.5.1. *If $x \in E(M) - \{\alpha, \beta\}$ and $M \backslash x$ is $(4,4)$-connected, then $M \backslash x$ has a unique quad D_x, this quad contains $\{\alpha, \beta\}$, and $|E(M)| \geq 10$.*

By Lemma 5.4, $M \backslash x$ has no quad avoiding $\{\alpha, \beta\}$. Now, since $M \backslash x$ is not sequentially 4-connected, it has a 3-separation (X, Y) such that neither X nor Y is sequential. But $M \backslash x$ is $(4,4)$-connected, so $|X| \leq 4$ or $|Y| \leq 4$. We may assume the former. As X is non-sequential, X is a quad. Therefore, as noted above, $\{\alpha, \beta\}$ meets X. Thus, by Lemma 2.20, X contains $\{\alpha, \beta\}$. Hence $|Y| \geq 5$ and $|E(M)| \geq 10$. Furthermore, X is the unique quad of $M \backslash x$ containing $\{\alpha, \beta\}$ since, by uncrossing, the union of two such quads is 3-separating. Hence (5.5.1) holds.

Now choose e in $E(M) - \{\alpha, \beta\}$. Then, by Lemma 2.3 and duality, we may assume that $M \backslash e$ is $(4,4)$-connected. Let $D_e = \{\alpha, \beta, f, g\}$.

5.5.2. *Both $M \backslash f$ and $M \backslash g$ are $(4,4)$-connected.*

Assume that $M \backslash f$ is not $(4,4)$-connected. Then $|E(M)| \geq 11$ and, by Lemma 2.3, M/f is $(4,4)$-connected. By (5.5.1), M/f has a quad P containing $\{\alpha, \beta\}$. But M/f has $\{\alpha, \beta, g\}$ as a circuit. Hence $P \cup g$ is 3-separating in M/f, contradicting the fact that this matroid is $(4,4)$-connected. We conclude that $M \backslash f$ is $(4,4)$-connected. By symmetry, so is $M \backslash g$.

5.5.3. $D_f \cap D_g = \{\alpha, \beta\}$.

Assume this assertion fails. Suppose first that $D_f \neq D_g$. Then $|D_f \cap D_g| = 3$. As D_g is a circuit, we deduce that $D_g \subseteq \mathrm{cl}_{M\backslash f}(D_f)$. Hence $D_f \cup D_g$ is 3-separating in $M\backslash f$, contradicting the fact that it is $(4,4)$-connected. We may now assume that $D_f = D_g$.

Certainly $(D_f, E(M) - (D_f \cup f))$ is exactly 3-separating in $M\backslash f$. If $(D_f, E(M) - (D_f \cup f \cup g))$ is not exactly 3-separating in $M\backslash f, g$, then g is a coloop of $M|(E(M) - (D_f \cup f))$ so $g \in \mathrm{cl}^*{}_{M\backslash f}(D_f)$. Hence $D_f \cup g$ is 3-separating in $M\backslash f$; a contradiction. Thus $\lambda_{M\backslash f,g}(D_f) = 2$. Since $\lambda_{M\backslash f}(D_f) = 2 = \lambda_{M\backslash g}(D_f)$, it follows that $\{f,g\} \subseteq \mathrm{cl}(E(M) - (D_f \cup \{f,g\}))$. Hence $\lambda_M(D_f) = 2$; a contradiction. Thus (5.5.3) holds.

Let $D_f = \{\alpha, \beta, f_1, f_2\}$ and $D_g = \{\alpha, \beta, g_1, g_2\}$. By (5.5.3), the elements $\alpha, \beta, f_1, f_2, g_1$, and g_2 are distinct.

5.5.4. *$M\backslash e, f$ is 3-connected.*

Suppose not. Let (X, Y) be a 2-separation of $M\backslash e, f$. As D_e is a circuit, we may assume that $|D_e \cap X| = 2$ and $|D_e \cap Y| = 1$. But $D_e - f$ is a triad of $M\backslash e, f$. Hence we obtain the contradiction that $(X \cup D_e, Y - D_e)$ is a 2-separation of $M\backslash e$ unless $|Y - D_e| = 1$, that is, unless $|Y| = 2$. In the exceptional case, Y is a cocircuit of $M\backslash e, f$, so $Y \cup f$ is a triad T^* of $M\backslash e$. Thus $|T^* \cap D_e| = 2$, so $D_e \cup T^*$ is 3-separating in $M\backslash e$ contradicting the fact that this matroid is $(4,4)$-connected. Hence (5.5.4) holds.

5.5.5. *The set $\{f_1, f_2, g_1, g_2\}$ is a quad of $M\backslash e$.*

We know that the circuits of M include $\{\alpha, \beta, f, g\}, \{\alpha, \beta, f_1, f_2\}$, and $\{\alpha, \beta, g_1, g_2\}$. Also, since M has no quads, the cocircuits of M include $\{\alpha, \beta, f, g, e\}, \{\alpha, \beta, f_1, f_2, f\}$, and $\{\alpha, \beta, g_1, g_2, g\}$. Now $(\{\alpha, \beta, f_1, f_2\} \cup \{\alpha, \beta, g_1, g_2\}) - \alpha$ contains a circuit C. By orthogonality with the cocircuit $\{\alpha, \beta, f, g, e\}$, we deduce that $\beta \notin C$, so $C \subseteq \{f_1, f_2, g_1, g_2\}$. Then, as M is 4-connected, $C = \{f_1, f_2, g_1, g_2\}$, that is, $\{f_1, f_2, g_1, g_2\}$ is a circuit of M.

Now M has a cocircuit C^* contained in $(\{\alpha, \beta, f, g, e\} \cup \{\alpha, \beta, f_1, f_2, f\}) - \alpha$ and containing e. By orthogonality with the circuit $\{\alpha, \beta, g_1, g_2\}$, we deduce that $\beta \notin C^*$. Thus $C^* \subseteq \{e, f, g, f_1, f_2\}$. Orthogonality with the circuits $\{\alpha, \beta, f, g\}$ and $\{\alpha, \beta, f_1, f_2\}$ implies that C^* contains an even number of elements of each of $\{f, g\}$ and $\{f_1, f_2\}$. If C^* avoids $\{f, g\}$ or $\{f_1, f_2\}$, then $|C^*| \leq 3$; a contradiction. Thus $C^* = \{e, f, g, f_1, f_2\}$. Hence $\{f, g, f_1, f_2\}$ is a cocircuit of $M\backslash e$. By symmetry, $\{f, g, g_1, g_2\}$ is a cocircuit of $M\backslash e$. By elimination, $M\backslash e$ has a cocircuit D^* contained in $\{g, f_1, f_2, g_1, g_2\}$. By orthogonality with $\{\alpha, \beta, f, g\}$, we deduce that $D^* = \{f_1, f_2, g_1, g_2\}$. Hence (5.5.5) holds. As (5.5.5) contradicts (5.5.1), Theorem 5.5 must hold. □

6. Proof of Theorem 1.2 when $|A| \geq 11$

Lemma 4.13 leads us to consider 4-connected matroids with a clonal pair. The goal of this section is to prove Theorem 1.2 when $|A| \geq 11$. This proof is given at the end of this section following a sequence of preliminary results. The proof of Theorem 1.2 for $|A| \leq 10$ is given in the next section.

Lemma 6.1. *Let M be a 4-connected matroid with a clonal pair $\{\alpha, \beta\}$. Assume that $|E(M)| \geq 13$. Then there is an element e of $E(M) - \{\alpha, \beta\}$ such that, for some M_1 in $\{M, M^*\}$,*

- (i) *$M_1 \backslash e$ is $(4, 4, S)$-connected; or*
- (ii) *$M_1 \backslash e$ is sequentially 4-connected and if Z_1 is a sequential 3-separating set of $M_1 \backslash e$ with $|Z_1| \geq 5$, then there is a sequential ordering of Z_1 that begins $(\alpha, \beta, z_3, z_4, z_5)$ where $M_1 \backslash e$ has $\{\alpha, \beta, z_3\}$ as a triad and $\{\alpha, \beta, z_3, z_4\}$ as a circuit, and $z_5 \in \mathrm{cl}^*{}_{M_1 \backslash e}(\{\alpha, \beta, z_3, z_4\}) - \mathrm{cl}^*{}_{M_1 \backslash e}(\{\alpha, \beta\})$.*

Proof. Assume that (i) does not hold. By Theorem 5.5, there is an element e of $E(M) - \{\alpha, \beta\}$ such that, up to duality, $M \backslash e$ is sequentially 4-connected. Then $M \backslash e$ is not $(4, 4)$-connected. Thus, by Lemma 2.3, M/e is $(4, 4)$-connected. Hence M/e is not sequentially 4-connected. Let Z_1 be a 3-separating set in $M \backslash e$ with at least 5 elements and having a sequential ordering that begins $(z_1, z_2, z_3, z_4, z_5)$. Let $Z = \{z_1, z_2, z_3, z_4, z_5\}$. Then, by Lemma 2.16, we may assume that either $(z_1, z_2) = (\alpha, \beta)$, or $|\{\alpha, \beta\} \cap \{z_1, z_2, z_3, z_4\}| \leq 1$.

Now $\{z_1, z_2, z_3\}$ is a triad of $M \backslash e$. Clearly $z_4 \in \mathrm{cl}^{(*)}_{M \backslash e}(\{z_1, z_2, z_3\})$. We show next that

6.1.1. $|\{\alpha, \beta\} \cap \{z_1, z_2, z_3\}| \neq 1$.

Assume that $|\{\alpha, \beta\} \cap \{z_1, z_2, z_3\}| = 1$. Then, from above, $|\{\alpha, \beta\} \cap \{z_1, z_2, z_3, z_4\}| = 1$. By symmetry, we may assume that $z_1 = \alpha$. Thus $\{\alpha, z_2, z_3\}$ is a triad of $M \backslash e$. Hence $\{\beta, z_2, z_3\}$ is also a triad of $M \backslash e$. Suppose $z_4 \in \mathrm{cl}_{M \backslash e}(\{\alpha, z_2, z_3\})$. Then $\{\alpha, z_2, z_3, z_4\})$ and $\{\beta, z_2, z_3, z_4\})$ are circuits of $M \backslash e$. Thus $M \backslash e$ has $\{\alpha, \beta, z_2, z_3\}$ as a circuit, so

$$r_{M \backslash e}(\{\alpha, \beta, z_2, z_3\}) + r^*_{M \backslash e}(\{\alpha, \beta, z_2, z_3\}) - |\{\alpha, \beta, z_2, z_3\}| \leq 3 + 2 - 4 = 1,$$

that is, $\lambda_{M \backslash e}(\{\alpha, \beta, z_2, z_3\}) \leq 1$; a contradiction. Hence $z_4 \in \mathrm{cl}^*{}_{M \backslash e}(\{\alpha, z_2, z_3\})$. Thus $M^*|\{\alpha, \beta, z_2, z_3, z_4, e\} \cong U_{3,6}$. Hence, by Theorem 2.23, as $M^*|\{\beta, z_2, z_3, z_4, e\} \cong U_{3,5}$, there is an element f of $\{z_2, z_3, z_4, e\}$ such that $M^* \backslash f$ is internally 4-connected. Hence M/f is $(4, 4, S)$-connected; a contradiction. We conclude that (6.1.1) holds.

6.1.2. $(z_1, z_2) = (\alpha, \beta)$.

Assume this does not hold. Then, by (6.1.1), $\{\alpha, \beta\} \cap \{z_1, z_2, z_3\} = \emptyset$. If $z_4 \in \mathrm{cl}^*_{M\backslash e}(\{z_1, z_2, z_3\})$, then $M^*|\{z_1, z_2, z_3, z_4, e\} \cong U_{3,5}$ and so, by Theorem 2.23, there is an element f of $E(M) - \{\alpha, \beta\}$ such that M/f is $(4, 4, S)$-connected; a contradiction. We may now assume that $z_4 \in \mathrm{cl}_{M\backslash e}(\{z_1, z_2, z_3\})$. Then, by [12, Theorem 5.1], for some x in $\{z_1, z_2, z_3\}$, the matroid M/x is $(4, 4, S)$-connected. This contradiction establishes that (6.1.2) holds.

Now consider the sequential ordering $(\alpha, \beta, z_3, z_4, z_5)$ of Z_1. Certainly $\{\alpha, \beta, z_3\}$ is a triad of $M\backslash e$.

6.1.3. $z_4 \notin \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta, z_3\})$.

Assume that $z_4 \in \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta, z_3\})$. Then $\{\alpha, \beta, z_3, z_4\}$ has rank 2 in M^*/e. Hence $\{\alpha, \beta, z_3, z_4, e\}$ is a rank-3 set P in M^*. Suppose $z_5 \in \mathrm{cl}_{M\backslash e}(\{\alpha, \beta, z_3, z_4\})$. Then $z_5 \in \mathrm{cl}^*_{M^*/e}(\{\alpha, \beta, z_3, z_4\})$, so $z_5 \in \mathrm{cl}^*_{M^*}(\{\alpha, \beta, z_3, z_4, e\})$. Hence $z_5 \in \mathrm{cl}^*_{M^*}(P) - P$. Thus, by Lemma 2.18, M^*/z_5 is $(4, 4, S)$-connected, so $M\backslash z_5$ is $(4, 4, S)$-connected and (i) holds; a contradiction. We may now assume that $z_5 \in \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta, z_3, z_4\})$. Then $\{\alpha, \beta, z_3, z_4, z_5\}$ has rank 2 in M^*/e, so $M^*|\{\alpha, \beta, z_3, z_4, z_5, e\} \cong U_{3,6}$. By Theorem 2.23, for some f in $\{z_3, z_4, z_5, e\}$, the matroid M/f is $(4, 4, S)$-connected; a contradiction. We conclude that (6.1.3) holds.

By (6.1.3), $z_4 \in \mathrm{cl}_{M\backslash e}(\{\alpha, \beta, z_3\})$ and, since M has no triangles, $\{\alpha, \beta, z_3, z_4\}$ is a circuit of M.

6.1.4. $z_5 \in \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta, z_3, z_4\}) - \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta\})$.

Assume that $z_5 \in \mathrm{cl}_{M\backslash e}(\{\alpha, \beta, z_3, z_4\})$. Then $M|\{\alpha, \beta, z_3, z_4, z_5\} \cong U_{3,5}$. As $\{\alpha, \beta, z_3, e\}$ is a cocircuit of M, we have $e \in \mathrm{cl}^*_M(\{\alpha, \beta, z_3, z_4, z_5\})$. Thus, by Lemma 2.18, M/e is $(4, 4, S)$-connected; a contradiction. We deduce that $z_5 \notin \mathrm{cl}_{M\backslash e}(\{\alpha, \beta, z_3, z_4\})$. Hence $z_5 \in \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta, z_3, z_4\})$. If $z_5 \in \mathrm{cl}^*_{M\backslash e}(\{\alpha, \beta\})$, then $(\alpha, \beta, z_3, z_5, z_4)$ is a sequential ordering of Z. Thus we can interchange the labels on z_4 and z_5 and thereby obtain a contradiction to (6.1.3). We deduce that (6.1.4) holds and, hence, so does the lemma. □

As well as being used to establish Theorem 1.2 when $|A| \geq 11$, the next lemma is frequently used in the proof of Theorem 1.2 for $|A| \leq 10$.

Lemma 6.2. *Let M be a 3-connected matroid having a 3-separation (A, B). Assume that there is no triangle or triad of M that contains two or more elements of A. Let N be the clonal replacement of B by $\{\alpha, \beta\}$ and assume that N is 4-connected having at least seven elements. Let e be an element of A. Then $M\backslash e$ is 3-connected and $r_N(A - e) = r_N(A)$. Furthermore:*

(i) *If e exposes a 3-separation in $M\backslash e$ and $N\backslash e$ is sequentially 4-connected, then there is a flower $\Phi = (\{\alpha,\beta\}, A_1, A_2)$ in $N\backslash e$, where $A_1 \not\subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha,\beta\})$ and $A_2 \not\subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha,\beta\})$.*

(ii) *If e exposes a 3-separation in $M^*\backslash e$ and $r(N) = 4$, then there is a flower $(\{\alpha,\beta\}, A_1, A_2)$ in N/e for some A_1 and A_2, where $r_{N/e}(A_1) = 2$, $r_{N/e}(A_2) = 2$, $A_1 \not\subseteq \mathrm{fcl}_{N/e}(\{\alpha,\beta\})$, $A_2 \not\subseteq \mathrm{fcl}_{N/e}(\{\alpha,\beta\})$, and*
$$\sqcap_{N/e}(\{\alpha,\beta\}, A_1) = \sqcap_{N/e}(A_1, A_2) = \sqcap_{N/e}(A_2, \{\alpha,\beta\}) = 1.$$

(iii) *If e exposes a 3-separation in $M^*\backslash e$, and $r(N) \geq 5$ and $|E(N)| \geq 10$, then one of the following holds.*
   (a) *Some element x of $E(N)-\{\alpha,\beta\}$ does not expose a 3-separation in $M_1\backslash x$ for some $M_1 \in \{M, M^*\}$.*
   (b) *There is a 3-separation (U,V) in N/e, where $r_{N/e}(U) \geq 3$, $r_{N/e}(V) \geq 3$, and either $\{\alpha,\beta\} \subseteq U$ or $\{\alpha,\beta\} \subseteq V$.*
   (c) *$|E(N)| = 10$ and there is a copaddle $(\{\alpha,\beta\}, A_1, A_2)$ in N/e for some A_1 and A_2, where $r_{N/e}(A_1) = 2$, $r_{N/e}(A_2) = 2$, $|A_1| = 3$, and $|A_2| = 4$.*

Proof. First observe that, as N is 4-connected, $r_N(A-e) = r_N(A) = r(N)$, otherwise $\{e,\alpha,\beta\}$ is a triad of N. We show next that $M\backslash e$ is 3-connected. Assume it has a 2-separation (X,Y). Then, without loss of generality, $|X| \geq |Y \cup e| \geq 3$. Thus $(X, Y \cup e)$ is a 3-separation of M and $r(Y \cup e) = r(Y) + 1$. Moreover, $r(X \cup e) = r(X) + 1$. This is immediate if $|Y| \geq 3$; if $|Y| = 2$, then $Y \cup e$ is a triad and again it holds.

Now assume that $|(Y \cup e) \cap A| = 1$. Then $A - e \subseteq X$. Now $N|A = M|A$, so $r_M(A-e) = r_M(A)$. Thus $r_M(X) = r_M(X \cup e)$; a contradiction. Hence $|(Y \cup e) \cap A| \geq 2$.

Suppose $B \cap X = \emptyset$. Then $X \subseteq A$, so $Y \cup e \supseteq B$. By the construction of N, we have $r_N(\{\alpha,\beta\} \cup (A \cap (Y \cup e))) + r_N(X) - r(N) = r_M(B \cup (A \cap (Y \cup e))) + r_M(X) - r(M) = \lambda_M(X) = 2$; a contradiction to the fact that N is 4-connected.

Next let $|B \cap X| = 1$. Then, by uncrossing, $\lambda_M(Y \cup e \cup B) = 2$. Replacing $(X, Y \cup e)$ by $(X - B, Y \cup e \cup B)$ and using the previous paragraph, we get that $\lambda_N(X - B) = 2$. This is a contradiction since $|X| \geq |Y \cup e|$, so $|X| \geq \lceil \frac{|E(M)|}{2} \rceil \geq 4$ and $|X - B| \geq 3$.

We may now assume that $|B \cap X| \geq 2$. As $|A \cap (Y \cup e)| \geq 2$, an application of uncrossing implies that $\lambda_M(B \cup X) = 2$. Then, replacing $(X, Y \cup e)$ by $(X \cup B, (Y \cup e) - B)$, we get the contradiction that $(\{\alpha,\beta\} \cup (X \cap A), A \cap (Y \cup e))$ is a 3-separation of N unless $|A \cap (Y \cup e)| = 2$. Consider the exceptional case. We have $|A \cap X| = |A| - 2 \geq 3$. If $|B \cap (Y \cup e)| \geq 2$, then, by uncrossing,

$\lambda_M(B \cup Y \cup e) = 2$. Replacing $(X, Y \cup e)$ by $(X - B, B \cup Y \cup e)$ and arguing similarly to the above, we get that N has a 3-separation; a contradiction. Now suppose $|B \cap (Y \cup e)| = 1$. Then $Y \cup e$ is a triad of M containing two elements of A; a contradiction. We conclude that $M\backslash e$ is 3-connected.

Let $M_1 \in \{M, M^*\}$. By Lemma 4.11, we may assume that the clonal replacement of B by $\{\alpha, \beta\}$ in M^* is N^*. If $M_1 = M$, set $N_1 = N$, while if $M_1 = M^*$, set $N_1 = N^*$. Now suppose that $M_1\backslash e$ has a 3-separation that is exposed by e. Choose such a 3-separation (R, G) to minimize

$$\min\{|(A - e) \cap R|, |(A - e) \cap G|, |B \cap R|, |B \cap G|\}.$$

Suppose first that this minimum is 0. If R or G, say R, contains $A - e$, then, by Lemma 4.6, $e \in \mathrm{cl}_{M_1}(R)$, so $(R \cup e, G)$ is a 3-separation of M_1; a contradiction. Hence $|R \cap (A - e)|$ and $|G \cap (A - e)|$ are positive. Suppose R or G, say R, contains B. Then $G \subseteq A - e$ and so, by Lemma 4.9, $\lambda_{N_1\backslash e}(G) = \lambda_{M_1\backslash e}(G) = 2$. Hence $(G, (R \cap A) \cup \{\alpha, \beta\})$ is a 3-separating partition of $N_1\backslash e$. Now $|R \cap A| > 1$, otherwise $(G, R) \cong (A - e, B)$ and (G, R) is not exposed. Thus $(G, (R \cap A) \cup \{\alpha, \beta\})$ is a 3-separation of $N_1\backslash e$.

Assume that $(G, (R \cap A) \cup \{\alpha, \beta\})$ is a sequential 3-separation of $N_1\backslash e$. Then either G or $(R \cap A) \cup \{\alpha, \beta\}$ is sequential in $N_1\backslash e$. In the first case, by Corollary 4.10, G is sequential in $M_1\backslash e$, contradicting Lemma 2.15. Thus G is not sequential in $N_1\backslash e$, and so $(R \cap A) \cup \{\alpha, \beta\}$ is sequential in $N_1\backslash e$. Choose a sequential ordering $(z_1, z_2, \ldots, z_k)$ of $(R \cap A) \cup \{\alpha, \beta\}$ with the least j such that $\{z_1, z_2, \ldots, z_j\} \supseteq \{\alpha, \beta\}$. We may assume that $\{\alpha, \beta\} = \{z_{j-1}, z_j\}$. Suppose first that $j \leq 3$. The choice of j then implies that $j = 2$. Thus, by Lemma 4.9, for all i in $\{3, 4, \ldots, k\}$, we have

$$2 = \lambda_{N_1\backslash e}(G \cup \{z_i, z_{i+1}, \ldots, z_k\}) = \lambda_{M_1\backslash e}(G \cup \{z_i, z_{i+1}, \ldots, z_k\}).$$

Thus $(G, R) \cong (A - e, B)$; a contradiction. Hence we may assume that $j \geq 4$, in which case, $R \cap A$ is not a subset of $\mathrm{fcl}(\{\alpha, \beta\})$. If $M_1 = M$, then $(\{z_1, z_2, \ldots, z_{j-2}\}, \{\alpha, \beta\}, \{z_{j+1}, z_{j+2}, \ldots, z_k\} \cup G)$ is a flower in $N\backslash e$ and (i) holds.

Next assume that $M_1 = M^*$ and $r(N) \geq 4$. As G is not sequential in $N^*\backslash e$, it is not sequential in N/e, so $r_{N/e}(G) \geq 3$. Thus if $r(N) = 4$, then $r_{N/e}((R \cap A) \cup \{\alpha, \beta\}) = 2$, so $\{\alpha, \beta\} = \{z_1, z_2\}$; a contradiction. Hence we may assume that $r(N) \geq 5$. As $R \cap A$ is not a subset of $\mathrm{fcl}(\{\alpha, \beta\})$, we have $r_{N/e}((R \cap A) \cup \{\alpha, \beta\}) \geq 3$ and so (iii)(b) holds.

Now assume that $(G, (R \cap A) \cup \{\alpha, \beta\})$ is not sequential in $N_1\backslash e$. Then $N_1\backslash e$ is not sequentially 4-connected, so we may assume that $M_1 = M^*$. Furthermore, $r_{N/e}(G) \geq 3$ and $r_{N/e}((R \cap A) \cup \{\alpha, \beta\}) \geq 3$, so $r(N) \geq 5$ and (iii)(b) holds. Hence we may suppose that $\min\{|(A - e) \cap R|, |(A - e) \cap G|, |B \cap R|, |B \cap G|\}$ is positive.

Assume next that $\min\{|(A-e)\cap R|, |(A-e)\cap G|, |B\cap R|, |B\cap G|\} = 1$. Suppose $|B\cap R| = 1$. Then $|(A-e)\cap R|, |B\cap G| \geq 2$, so $\lambda_{M\backslash e}((A-e)\cap R) = 2$ and $((A-e)\cap R, B\cup G) \cong (R, G)$. But $(A-e)\cap R$ avoids B, contradicting the choice of (R, G). Hence $|B\cap R| > 1$. By symmetry, $|B\cap G| > 1$, and then $|(A-e)\cap R|, |(A-e)\cap G| > 1$.

We may now assume that

$$\min\{|(A-e)\cap R|, |(A-e)\cap G|, |B\cap R|, |B\cap G|\} \geq 2.$$

Let $A_1 = (A-e)\cap R$ and $A_2 = (A-e)\cap G$. Then, by uncrossing, each of A_1 and A_2 is 3-separating in $M_1\backslash e$ and hence, by Lemma 4.9, in $N_1\backslash e$. Thus $\Phi = (\{\alpha, \beta\}, A_1, A_2)$ is a flower in $N_1\backslash e$.

Suppose that A_1 or A_2 is a subset of $\mathrm{fcl}_{N_1\backslash e}(\{\alpha, \beta\})$. If $A_1 \cup A_2 \subseteq \mathrm{fcl}_{N_1\backslash e}(\{\alpha, \beta\})$, then there is a sequential ordering $(\alpha, \beta, y_1, y_2, \ldots, y_k)$ of $E(N_1\backslash e)$. By symmetry and relabelling, we may assume that $\{y_{k-1}, y_k\} \subseteq A_2$. Then, by uncrossing, there is a sequential ordering $(\alpha, \beta, z_1, z_2, \ldots, z_l)$ of $A_1 \cup \{\alpha, \beta\}$. Such a sequential ordering also exists if $A_1 \subseteq \mathrm{fcl}_{N_1\backslash e}(\{\alpha, \beta\})$ but $A_2 \not\subseteq \mathrm{fcl}_{N_1\backslash e}(\{\alpha, \beta\})$. Using this sequential ordering, we have, by Lemma 4.9, that, for all $i \in \{1, 2, \ldots, l\}$,

$$2 = \lambda_{N_1\backslash e}(A_2 \cup \{z_i, z_{i+1}, \ldots, z_l\}) = \lambda_{M_1\backslash e}(A_2 \cup \{z_i, z_{i+1}, \ldots, z_l\}).$$

Since $|A_2| \geq 2$, it follows by uncrossing that $G \cup (A_2 \cup \{z_i, z_{i+1}, \ldots, z_l\}) = G \cup \{z_i, z_{i+1}, \ldots, z_l\}$ is 3-separating in $M_1\backslash e$. Thus $(G, R) \cong (A-e, B)$; a contradiction. Hence neither A_1 nor A_2 is a subset of $\mathrm{fcl}_{N_1\backslash e}(\{\alpha, \beta\})$. We conclude that if $M_1 = M$, then (i) holds. This finishes the proof of (i).

We may now assume that $M_1 = M^*$ and $r(N) \geq 4$. Without loss of generality, we may also assume that $|A_1| \leq |A_2|$. Since $A_1 \not\subseteq \mathrm{fcl}_{N/e}(\{\alpha, \beta\})$, we have $r_{N/e}(\{\alpha, \beta\} \cup A_1) \geq 3$. If $r_{N/e}(A_2) \geq 3$, then $r(N) \geq 5$ and (iii)(b) holds. Therefore we may assume that $r_{N/e}(A_2) = 2$. If $r(N) = 4$, then a symmetrical argument shows that $r_{N/e}(A_1) = 2$. Furthermore, if Φ is a paddle or copaddle in N/e, then $r(N/e) \in \{2, 4\}$. But $r(N/e) = 3$. Thus $\sqcap_{N/e}(\{\alpha, \beta\}, A_1) = \sqcap_{N/e}(A_1, A_2) = \sqcap_{N/e}(A_2, \{\alpha, \beta\}) = 1$, and (ii) holds.

Now assume that $r(N) \geq 5$ and $|E(N)| \geq 10$. Suppose $|E(N)| \geq 11$. Then, as $r_N(A_2 \cup e) = 3$ and $A_2 \cup e$ avoids α and β, it follows by Theorem 2.23 that there is an element y of $A_2 \cup e$ such that $N\backslash y$ is internally and hence sequentially 4-connected. If y does not expose a 3-separation of $M\backslash y$, then (iii)(a) holds. If y does expose a 3-separation of $M\backslash y$, then, by applying (i) with $y = e$, we get that $N\backslash y$ has a flower $(\{\alpha, \beta\}, Y_1, Y_2)$ with $|Y_1| \geq |Y_2|$. As $|E(N\backslash y)| \geq 10$, we have $|Y_1| \geq 4$. Then the 3-separation $(Y_1, \{\alpha, \beta\} \cup Y_2)$ contradicts the fact that $N\backslash y$ is internally 4-connected.

We may now suppose that $|E(N)| = 10$. Then $|A_1| \in \{2, 3\}$ as $|A_1| \leq |A_2|$. Since N has no triads, $r_{N/e}(A_1) = 2$. Thus, as $r(N) \geq 5$, the flower Φ is a copaddle in N/e. If $|A_1| = 2$, then $r^*_{N/e}(\{\alpha,\beta\} \cup A_1) = 2$ and so $A_1 \subseteq \mathrm{fcl}_{N_1\backslash e}(\{\alpha,\beta\})$; a contradiction. Thus $|A_1| = 3$ and so (iii)(c) holds. □

Corollary 6.3. *Let M be a 3-connected matroid having a 3-separation (A, B). Assume that there is no triangle or triad of M that contains two or more elements of A. Let N be the clonal replacement of B by $\{\alpha,\beta\}$ and assume that N is 4-connected. Let e be an element of A such that either*

(i) *$N\backslash e$ is internally 4-connected and $|E(N)| \geq 10$; or*
(ii) *$N\backslash e$ is $(4, 4, S)$-connected and $|E(N)| \geq 13$; or*
(iii) *$N\backslash e$ is sequentially 4-connected, $|E(N)| \geq 13$, and every 5-element sequential 3-separating set Z of $N\backslash e$ contains $\{\alpha,\beta\}$ and has a sequential ordering $(\alpha,\beta,z_3,z_4,z_5)$ with $\{\alpha,\beta,z_3\}$ as a triad and $\{\alpha,\beta,z_3,z_4\}$ as a circuit, and $z_5 \in \mathrm{cl}^*_{N\backslash e}(\{\alpha,\beta,z_3,z_4\}) - \mathrm{cl}^*_{N\backslash e}(\{\alpha,\beta\})$.*

Then e does not expose any 3-separations in $M\backslash e$.

Proof. In each of (i)–(iii), $N\backslash e$ is sequentially 4-connected and $|E(N)| \geq 10$. Thus, by the last lemma, $M\backslash e$ is 3-connected and $r_N(A - e) = r_N(A)$. Assume that e exposes a 3-separation of $M\backslash e$. Then, by the last lemma again, $N\backslash e$ has a flower $(\{\alpha,\beta)\}, A_1, A_2)$ for some A_1 and A_2 where neither A_1 nor A_2 is contained in $\mathrm{fcl}_{N\backslash e}(\{\alpha,\beta\})$. We may assume that $|A_2| \geq |A_1|$. Then $|A_2| \geq 4$. Thus $(A_1 \cup \{\alpha,\beta\}, A_2)$ is a 3-separation of $N\backslash e$. Hence if $N\backslash e$ is internally 4-connected, we obtain a contradiction. We deduce that the corollary holds when (i) occurs. Now assume (ii) or (iii) holds. Then $|E(N)| \geq 13$. As $|A_2| \geq |A_1|$, we deduce that $|A_2| \geq 5$. We show next that

6.3.1. $|A_1| \geq 3$.

Assume the contrary. Then $|A_1| = 2$ and $|A_2| \geq 8$. If A_2 is sequential, then, for some element z of A_2, we have $(A_1 \cup \{\alpha,\beta\} \cup z, A_2 - z)$ as a 3-separation of $N\backslash e$ with $|A_1 \cup \{\alpha,\beta\} \cup z|, |A_2 - z| \geq 5$ and $A_2 - z$ sequential avoiding $\{\alpha,\beta\}$. This contradicts the hypothesis governing $N\backslash e$. Thus A_2 is non-sequential. Hence $A_1 \cup \{\alpha,\beta\}$ is sequential. By Lemma 2.16, $A_1 \cup \{\alpha,\beta\}$ has a sequential ordering of the form (α,β,x,y) so $A_1 \subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha,\beta\})$; a contradiction. Thus (6.3.1) holds.

As $|A_1| \geq 3$, we have $|A_2|, |A_1 \cup \{\alpha,\beta\}| \geq 5$. Since A_2 avoids $\{\alpha,\beta\}$, the choice of $N\backslash e$ means that A_2 is non-sequential. Thus $A_1 \cup \{\alpha,\beta\}$ is sequential in $N\backslash e$ having a sequential ordering of the form $(\alpha,\beta,z_3,z_4,\dots,z_n)$ for some $n \geq 5$. Again we obtain the contradiction that $A_1 \subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha,\beta\})$. □

Theorem 6.4. *Let (A, B) be a non-sequential 3-separation in a 3-connected matroid M. Suppose that B is fully closed, A meets no triangle or triad of*

M, and if (X,Y) is a non-sequential 3-separation of M, then either $A \subseteq \text{fcl}(X)$ or $A \subseteq \text{fcl}(Y)$. If $|A| \geq 11$, then A contains an element whose deletion from M or M^* is 3-connected but does not expose any 3-separations.

Proof. By Lemma 4.13, the clonal replacement, N, of B by $\{\alpha, \beta\}$ is 4-connected. Since $|A| \geq 11$, we have $|E(N)| \geq 13$. Thus, by Lemma 6.1, N has an element e not in $\{\alpha, \beta\}$ such that, for some M_1 in $\{M, M^*\}$, the matroid $M_1\backslash e$ satisfies one of the connectivity conditions 6.1(i) or (ii). Because M has no triangles or triads having at least two elements in A, it follows by Corollary 6.3 that e does not expose any 3-separations in $M_1\backslash e$. □

7. Proof of Theorem 1.2 when $|A| \leq 10$.

The proof of Theorem 1.2 for $|A| \leq 10$ is given at the end of this section, and is an amalgamation of three lemmas. The third of these lemmas requires one additional preliminary which we state and prove first.

Lemma 7.1. *Let M be a 3-connected matroid having a 3-separation (A, B). Assume that there is no triangle or triad of M that contains two or more elements of A. Let N be the clonal replacement of B by $\{\alpha, \beta\}$ and assume that N is 4-connected. If $|E(N)| \geq 11$ and X is a 5-element rank-3 subset of $E(N)$ that avoids at least one element in $\{\alpha, \beta\}$, then there is an element x of $X - \{\alpha, \beta\}$ such that x does not expose any 3-separation in $M\backslash x$. In particular, if $e \in E(N) - \{\alpha, \beta\}$ and Y is a 4-element cosegment of $N\backslash e$ that avoids at least one element in $\{\alpha, \beta\}$, then there is an element y in $(Y - \{\alpha, \beta\}) \cup e$ such that y does not expose any 3-separation in $M^*\backslash y$.*

Proof. By Theorem 2.23, there is an element x in $X - \{\alpha, \beta\}$ such that $N\backslash x$ is internally 4-connected. It follows by Corollary 6.3(i) that x does not expose any 3-separation in $M\backslash x$. □

Lemma 7.2. *Let $(S, E(M) - S)$ be a non-sequential 3-separation in a 3-connected matroid M. Suppose no triangle or triad of M contains more than one element of S. If $r(S) \leq 3$, then S contains an element e such that $M^*\backslash e$ is 3-connected and e does not expose any 3-separations of M^*.*

Proof. Clearly $r_M(S) = 3$. Moreover, $\text{cl}(E(M) - S) \neq E(M)$. Take e in $S - \text{cl}(E(M) - S)$. Then M has no triangle containing e. Let (X, Y) be a non-minimal 2-separation or an exposed 3-separation of M/e. Then, without loss of generality, we may assume that $|X \cap (S - e)| \geq 2$. Hence X spans $S - e$ in M/e, so we may assume that X contains $S - e$. Thus $Y \subseteq \text{cl}(E(M) - S)$, so $r_M(Y \cup e) = r_M(Y) + 1$. Hence $(X \cup e, Y)$ is a 2- or 3-separation of M. This contradiction establishes the lemma. □

Lemma 7.3. *Let M be a 3-connected matroid having a 3-separation (A, B). Suppose that there is no triangle or triad of M that contains two or more elements of A. Let N be the clonal replacement of B by $\{\alpha, \beta\}$ and assume that N is 4-connected. If $|E(N)| \geq 8$, and either $r(N) = 4$ or $r^*(N) = 4$, then there is an element in $E(N) - \{\alpha, \beta\}$ whose deletion from M or M^* does not expose any 3-separations.*

Proof. By Lemma 4.11, we may assume that $r(A) = 4$. Suppose that every element f of $E(N) - \{\alpha, \beta\}$ exposes a 3-separation in each of $M\backslash f$ and $M^*\backslash f$. Since N is 4-connected, $N\backslash\{\alpha, \beta\}$ is connected. Assume first that $N\backslash\{\alpha, \beta\}$ is 3-connected. If there is no element $y \in E(N) - \{\alpha, \beta\}$ such that $N\backslash\{\alpha, \beta\}/y$ is 3-connected, then, by [7, Theorem 2.5], $N\backslash\{\alpha, \beta\}$ has a triangle; a contradiction. Therefore there is such an element y. By Lemma 6.2(ii), there is a flower $(\{\alpha, \beta\}, P_1, P_2)$ in N/y where $\sqcap_{N/y}(P_1, P_2) = 1$. Hence (P_1, P_2) is a 2-separation in $N\backslash\{\alpha, \beta\}/y$, contradicting the choice of y. Thus $N\backslash\{\alpha, \beta\}$ is not 3-connected.

We may now assume that $N\backslash\{\alpha, \beta\}$ is not 3-connected. Suppose first $|E(N)| \geq 9$. Then $N\backslash\{\alpha, \beta\}$ has a 2-separation (X, Y). Since $r(N) = 4$, we may assume that $r(X) = 2$ and $r(Y) = 3$. Since N has no triangles, X is a series pair in $N\backslash\{\alpha, \beta\}$. Let $X = \{y, z\}$. By Lemma 6.2(ii), there is a flower $(\{\alpha, \beta\}, P_1, P_2)$ in N/y, where $r_{N/y}(P_1) = 2 = r_{N/y}(P_2)$, and

$$\sqcap_{N/y}(\{\alpha, \beta\}, P_1) = \sqcap_{N/y}(P_1, P_2) = \sqcap_{N/y}(P_2, \{\alpha, \beta\}) = 1.$$

As $z \notin \mathrm{cl}_N((P_1 \cup P_2) - z)$ and N has no triangles, $|(P_1 \cup P_2) - z| \leq 4$ and so $|E(N)| \leq 8$; a contradiction. Thus if $r(N) = 4$ and $|E(N)| \geq 9$, then the lemma holds.

Now suppose that $|E(N)| = 8$. Since $N\backslash\{\alpha, \beta\}$ has rank 4 and 6 elements, its dual $N^*/\{\alpha, \beta\}$ has rank 2 and 6 elements. Therefore, as $N\backslash\{\alpha, \beta\}$ is connected, but not 3-connected, and it contains no triangles, it is not difficult to check that $N^*/\{\alpha, \beta\}$ has at least one non-trivial parallel class and any such parallel class has exactly two elements. If $N^*/\{\alpha, \beta\}$ has exactly one non-trivial parallel class $\{z, z'\}$, then $N^*/\{\alpha, \beta\}\backslash z$ is isomorphic to $U_{2,5}$ and so $N\backslash\{\alpha, \beta\}/z$ is isomorphic to $U_{3,5}$. But, by Lemma 6.2(ii), $E(N) - \{\alpha, \beta, z\}$ is the union of two segments in $N\backslash\{\alpha, \beta\}/z$; a contradiction. Since $r^*(N) = 4$, it now follows by Lemma 4.11 that, up to isomorphism, $N/\{\alpha, \beta\}$ is either (a) the 6-element rank-2 matroid with exactly three non-trivial parallel classes, $\{x, x'\}$, $\{y, y'\}$, and $\{z, z'\}$, or (b) the 6-element rank-2 matroid with exactly two non-trivial parallel classes, $\{y, y'\}$ and $\{z, z'\}$, where $E(N) - \{\alpha, \beta\} = \{x, x', y, y', z, z'\}$. In the analysis of (a) and (b), we freely use the consequence of the following observation. If N contains a 5-element rank-3 subset, then N is not 4-connected and so, for all $a \in E(N)$, the matroid N/a contains no 4-element segment.

First assume that (a) holds. Then, as N has no triangles, $\{\alpha,\beta,x,x'\}$, $\{\alpha,\beta,y,y'\}$, and $\{\alpha,\beta,z,z'\}$ are circuits in N. Furthermore, as N is 4-connected, this implies that none of $\{y,y',z,z'\}$, $\{x,x',z,z'\}$, and $\{x,x',y,y'\}$ are circuits in N. Consider N/z. By Lemma 6.2(ii), there is a flower $(\{\alpha,\beta\},P_1,P_2)$ in N/z, where $r_{N/z}(P_1)=2=r_{N/z}(P_2)$ and neither P_1 nor P_2 is contained in $\mathrm{fcl}_{N/z}(\{\alpha,\beta\})$. If $|\mathrm{cl}_{N/z}(P_1)-z'|=3$, then $z'\in P_2$ and so, as $z'\in \mathrm{cl}_{N/z}(\{\alpha,\beta\})$, it follows that $P_2\subseteq \mathrm{fcl}_{N/z}(\{\alpha,\beta\})$; a contradiction. Thus $|\mathrm{cl}_{N/z}(P_1)-z'|\neq 3$ and, similarly, $|\mathrm{cl}_{N/z}(P_2)-z'|\neq 3$. Hence, without loss of generality, we may assume that $|P_1|=3$ with $z'\in P_1$, and $|P_2|=2$ with $\mathrm{cl}_{N/z}(P_2)\cap(P_1-z')$ empty. If $P_1=\{x,x',z'\}$, then, as N has no triangles, $\{x,x',z,z'\}$ is a circuit in N; a contradiction. Thus $|P_1\cap\{x,x'\}|\leq 1$ and, similarly, $|P_1\cap\{y,y'\}|\leq 1$. So, without loss of generality, we may assume that $P_1=\{x,y,z'\}$. Then $\{x,y,z,z'\}$ is a circuit in N. Now consider N/x. In N/x, both $\{x',\alpha,\beta\}$ and $\{y,z,z'\}$ are triangles. Therefore, $(\alpha,\beta,x',y',y,z,z')$ is a sequential ordering of $E(N)-x$ in N/x. But then N/x has no flower $(\{\alpha,\beta\},P'_1,P'_2)$, where $P'_1\not\subseteq \mathrm{fcl}_{N/x}(\{\alpha,\beta\})$ and $P'_2\not\subseteq \mathrm{fcl}_{N/x}(\{\alpha,\beta\})$, contradicting Lemma 6.2(ii). Thus (a) does not hold.

Now assume that (b) holds. Since N has no triangles, $\{\alpha,\beta,y,y'\}$ and $\{\alpha,\beta,z,z'\}$ are circuits in N. Therefore, as N is 4-connected, neither $\{x,x',z,z'\}$ nor $\{x,x',y,y'\}$ is a circuit in N. Consider N/x. By Lemma 6.2(ii), there is a flower $(\{\alpha,\beta\},P_1,P_2)$ in N/x, where $r_{N/x}(P_1)=2$, $r_{N/x}(P_2)=2$, $P_1\not\subseteq \mathrm{fcl}_{N/x}(\{\alpha,\beta\})$, and $P_2\not\subseteq \mathrm{fcl}_{N/x}(\{\alpha,\beta\})$. Therefore, as either $|P_1|=3$ or $|P_2|=3$, N/x has a triangle $T\subseteq\{x',y,y',z,z'\}$. Since neither $\{x,x',z,z'\}$ nor $\{x,x',y,y'\}$ is a circuit in N, this triangle is neither $\{x',z,z'\}$ nor $\{x',y,y'\}$. Furthermore, if $\{x,y,y',z\}$ is a circuit in N, then $\{x,y,y'\}$ is a triangle in N/z. But $\{\alpha,\beta,z'\}$ is also a triangle in N/z and so $(\alpha,\beta,z',x',x,y,y')$ is a sequential ordering of $E(N)-z$ in N/z. Thus there is no flower $(\{\alpha,\beta\},P'_1,P'_2)$ in N/z, where $P'_1\not\subseteq \mathrm{fcl}_{N/z}(\{\alpha,\beta\})$ and $P'_2\not\subseteq \mathrm{fcl}_{N/z}(\{\alpha,\beta\})$, contradicting Lemma 6.2(ii). Hence $\{x,y,y',z\}$ is not a circuit in N and so $T\neq\{y,y',z\}$. Similarly, $T\notin\{\{y,y',z'\},\{y,z,z'\},\{y',z,z'\}\}$. Therefore $x'\in T$ and, without loss of generality, we may assume that $T=\{x',y,z\}$ and so $\{x,x',y,z\}$ is a circuit in N. But then $(\alpha,\beta,z',y',x,x',y)$ is a sequential ordering of $E(N)-z$ in N/z and so there is no flower $(\{\alpha,\beta\},P'_1,P'_2)$, where $P'_1\not\subseteq \mathrm{fcl}_{N/z}(\{\alpha,\beta\})$ and $P'_2\not\subseteq \mathrm{fcl}_{N/z}(\{\alpha,\beta\})$. This last contradiction to Lemma 6.2(ii) implies that (b) does not hold. This completes the proof of the lemma. □

Lemma 7.4. *Let M be a 3-connected matroid having a 3-separation (A,B). Suppose that there is no triangle or triad of M that contains two or more elements of A. Let N be the clonal replacement of B by $\{\alpha,\beta\}$ and assume that N is 4-connected. Let e be an element of $E(N)-\{\alpha,\beta\}$ such that $N\backslash e$ is sequentially 4-connected. If*

(I) $r(N)=5$ *and* $|E(N)|\in\{10,11,12\}$, *or*

(II) $r(N) = 6$ *and* $|E(N)| \in \{11, 12\}$, *or*
(III) $r(N) = 7$ *and* $|E(N)| = 12$,

then there is an element in $E(N) - \{\alpha, \beta\}$ *whose deletion from* M *or* M^* *does not expose any* 3*-separations.*

Proof. Suppose that $|E(N)| \geq 10$ and every element f of $E(N) - \{\alpha, \beta\}$ exposes a 3-separation in each of $M\backslash f$ and $M^*\backslash f$. By Lemma 6.2, there is a flower $\Phi = (\{\alpha, \beta\}, P_1, P_2)$ in $N\backslash e$ with the property that neither P_1 nor P_2 is a subset of $\mathrm{fcl}_{N\backslash e}(\{\alpha, \beta\})$. The proof of the lemma is partitioned into three parts depending on which of (I), (II), and (III) holds. Furthermore, each part is partitioned into three cases depending on whether Φ is (i) a paddle, (ii) a copaddle, or (iii) $\sqcap(\{\alpha, \beta\}, P_1) = \sqcap(P_1, P_2) = \sqcap(P_2, \{\alpha, \beta\}) = 1$.

(I) $r(N) = 5$ and $|E(N)| \in \{10, 11, 12\}$.

(i) Φ is a paddle. Since Φ is a paddle,

$$5 = r(N\backslash e) = r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 4.$$

Therefore, $r(P_1) + r(P_2) = 7$. Since neither P_1 nor P_2 is contained in $\mathrm{fcl}_{N\backslash e}(\{\alpha, \beta\})$, it follows that $r(P_1) \geq 3$ and $r(P_2) \geq 3$. Thus we may assume that $r(P_1) = 3$ and $r(P_2) = 4$. Since $e \notin \mathrm{cl}(P_1 \cup \{\alpha, \beta\})$, the set $\mathrm{cl}(P_1 \cup \{\alpha, \beta, e\})$ has rank 4, so its complement is a cocircuit of N. In $N\backslash e$, this complement contains a cocircuit C^*. Since N has no triangles and $N\backslash e$ is 3-connected, it follows by Lemma 2.8 that C^* contains an element $y \in P_2$ such that $y \notin \mathrm{cl}(P_1 \cup \{\alpha, \beta, e\})$ and $N\backslash e/y$ is 3-connected.

Consider N/y. By Lemma 6.2(iii), either (a) N/y has a 3-separation (R, G), where $r_{N/y}(R)$, $r_{N/y}(G) \geq 3$, and, without loss of generality, $\{\alpha, \beta\} \subseteq R$; or (b) $|E(N)| = 10$ and there is a copaddle $(\{\alpha, \beta\}, A_1, A_2)$ in N/y, where $r_{N/y}(A_1) = 2 = r_{N/y}(A_2)$, and $|A_1| = 3$, and $|A_2| = 4$.

Since $y \notin \mathrm{cl}(P_1 \cup \{\alpha, \beta\})$, we have $r_{N/y}(P_1 \cup \{\alpha, \beta\}) = 3$ and $P_1 \cup \{\alpha, \beta\}$ contains no triangles in N/y. If (b) holds, then $r_{N/y}(\{\alpha, \beta\} \cup A_1) = 4 = r_{N/y}(\{\alpha, \beta\} \cup A_2)$, and so, as either $|P_1 \cap A_1| \geq 2$ or $|P_1 \cap A_2| \geq 2$, we have $r_{N/y}(P_1 \cup \{\alpha, \beta\}) = 4$; a contradiction. Thus we may assume that (a) holds. As $r(N/y) = 4$, it follows that $r_{N/y}(R) = r_{N/y}(G) = 3$. Since $r_{N/y}(P_1 \cup \{\alpha, \beta\}) = 3 = r_{N/y}(P_2 - y)$, it follows that $(P_1 \cup \{\alpha, \beta\}, P_2 - y)$ is a 3-separation in $N\backslash e/y$. Moreover, as $\{\alpha, \beta\} \subseteq \mathrm{cl}(P_2)$, we have that $\{\alpha, \beta\} \subseteq \mathrm{cl}_{N\backslash e/y}(P_2 - y)$. If $|R \cap P_1| \geq 1$, then, by replacing (R, G) by a closure-equivalent 3-separation, we may assume that $P_1 \cup \{\alpha, \beta\} \subseteq R$. If $|R \cap P_1| = 0$, then, by replacing (R, G) by a closure-equivalent 3-separation, we may assume that $P_1 \cup \{\alpha, \beta\} \subseteq G$. We deduce that, by interchanging R and G if needed, we may assume that $P_1 \cup \{\alpha, \beta\} \subseteq R$. As $N/y\backslash e$ is 3-connected, it follows by Lemma 2.21 that $e \in \mathrm{cl}_{N/y}(R-e)$ or $e \in \mathrm{cl}_{N/y}(G-e)$.

Now $y \notin \mathrm{cl}(P_1 \cup \{\alpha, \beta, e\})$ and $e \notin \mathrm{cl}(P_1 \cup \{\alpha, \beta\})$ so, by Lemma 2.2, $e \notin \mathrm{cl}_{N/y}(P_1 \cup \{\alpha, \beta\})$. But $\mathrm{cl}_{N/y}(P_1 \cup \{\alpha, \beta\}) = \mathrm{cl}_{N/y}(R - e)$ so $e \notin \mathrm{cl}_{N/y}(R - e)$. Hence $e \in \mathrm{cl}_{N/y}(G - e)$ so $e \in \mathrm{cl}_{N/y}(P_2 - y)$ and therefore $e \in \mathrm{cl}_N(P_2)$; a contradiction. It now follows that Φ is not a paddle.

(ii) $\boldsymbol{\Phi}$ is a copaddle. If $|P_1| = 2$, then $P_1 \subseteq \mathrm{fcl}(\{\alpha, \beta\})$; a contradiction. Therefore, as N contains no triangles, it follows by symmetry that $r(P_1) \geq 3$ and $r(P_2) \geq 3$, so $r(P_1) + r(P_2) \geq 6$. But, as Φ is a copaddle,

$$\begin{aligned} 5 = r(N\backslash e) &= r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 2 \\ &= r(P_1) + r(P_2). \end{aligned}$$

This contradiction implies that Φ is not a copaddle.

(iii) $\boldsymbol{\sqcap(\{\alpha, \beta\}, P_1) = \sqcap(P_1, P_2) = \sqcap(P_2, \{\alpha, \beta\}) = 1}$. Since $P_1 \not\subseteq \mathrm{fcl}(\{\alpha, \beta\})$ and $P_2 \not\subseteq \mathrm{fcl}(\{\alpha, \beta\})$, it follows by Lemma 3.1 that P_1 and P_2 are both sequential. Furthermore, as

$$5 = r(N\backslash e) = r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 3,$$

we have $r(P_1) + r(P_2) = 6$. Without loss of generality, we may assume that $r(P_2) \in \{2, 3\}$. The analysis of (iii) is partitioned into two subcases depending on the rank of P_2.

In the analysis of the two subcases, we constantly consider matroids obtained from N by contracting an element. The next result helps us avoid considering of the possibility arising via Lemma 6.2(iii)(c) when $|E(N)| = 10$.

7.4.1. *Suppose that $|E(N)| = 10$. Let a be an element of $E(N) - \{\alpha, \beta\}$ such that $N\backslash e/a$ contains a triad avoiding α and β. Then there is no copaddle of the form $(\{\alpha, \beta\}, A_1, A_2)$ in N/a, where $r_{N/a}(A_1) = 2$, $r_{N/a}(A_2) = 2$, $|A_1| = 3$, and $|A_2| = 4$.*

If there were such a copaddle, then, as $N\backslash e/a$ contains a triad T, the complement of T in $N\backslash e/a$ has rank 3. But a simple check shows that either $|A_1 - (T \cup e)| \geq 2$ or $|A_2 - (T \cup e)| \geq 2$, and so, as $\sqcap_{N/a}(\{\alpha, \beta\}, A_1) = 0 = \sqcap_{N/a}(\{\alpha, \beta\}, A_2)$, the complement of T in $N\backslash e/a$ has rank 4; a contradiction. Thus (7.4.1) holds.

(iii)(a) $\boldsymbol{r(P_2) = 2}$. As N has no triangles, it follows that $|P_2| = 2$ and $r(P_1) = 4$. The next result is used frequently in this subcase.

7.4.2. *Let $a \in P_1$ such that either*

(i) *$a \notin \mathrm{cl}(\{\alpha, \beta, e\} \cup P_2)$ or*
(ii) *$a \notin \mathrm{cl}(\{\alpha, \beta\} \cup P_2)$ and $P_1 - a$ contains a triad in $N\backslash e/a$.*

Suppose that $N/a\backslash e$ is 3-connected. If N/a contains a 3-separation (R, G), where $r_{N/a}(R) = 3 = r_{N/a}(G)$, and $\{\alpha, \beta\} \subseteq R$, then $P_2 \subseteq G$.

Suppose that N/a has such a 3-separation (R, G) and assume that it is chosen to maximize $|P_2 \cap R|$. If $|P_2 \cap R| = 0$, then (7.4.2) holds, so we may assume that $|P_2 \cap R| \geq 1$. Then, as N has no triangles and $a \notin \mathrm{cl}(P_2 \cup \{\alpha, \beta\})$, it follows that $r_{N/a}(\{\alpha, \beta\} \cup (R \cap P_2)) = 3$. The choice of (R, G) now implies that $\{\alpha, \beta\} \cup P_2 \subseteq R$. As $3 = r_{N/a}(\{\alpha, \beta\} \cup P_2) \leq r_{N/a}(R - e) \leq r_{N/a}(R) = 3$, it follows that $(\{\alpha, \beta\} \cup P_2, P_1 - a)$ and $(R - e, G \cup e)$ are closure-equivalent 3-separations of $N/a\backslash e$. By Lemma 2.21, $e \in \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)$ or $e \in \mathrm{cl}_{N/a}(P_1 - a)$. The latter does not occur as $e \notin \mathrm{cl}(P_1)$. Thus $e \in \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)$ so $e \in \mathrm{cl}(\{\alpha, \beta\} \cup P_2 \cup a)$. But $e \notin \mathrm{cl}(\{\alpha, \beta\} \cup P_2)$. Hence $a \in \mathrm{cl}(\{\alpha, \beta\} \cup P_2 \cup e)$. As this contradicts (7.4.2)(i), it follows that (7.4.2)(ii) holds. Then, since $P_1 - a$ contains a triad of $N\backslash e/a$, Lemma 2.22 implies that $e \in \mathrm{cl}_{N/a}(P_1 - a)$, which we already eliminated. Thus (7.4.2) holds.

Let $(z_1, z_2, \ldots, z_k)$ be a sequential ordering of P_1 in $N\backslash e$, where $k \geq 5$ as $|E(N)| \geq 10$. Since N has no triangles, $\{z_1, z_2, z_3\}$ is a triad.

7.4.3. $z_4 \in \mathrm{cl}(\{z_1, z_2, z_3\})$.

Assume the contrary. Then $z_4 \in \mathrm{cl}^*(\{z_1, z_2, z_3\})$ and so $z_5 \in \mathrm{cl}(\{z_1, z_2, z_3, z_4\})$ since $r(P_1) = 4$. If $|E(N)| \geq 11$, then, as $\{z_1, z_2, z_3, z_4\}$ is a 4-element cosegment in $N\backslash e$ avoiding α and β, the lemma holds by Lemma 7.1. Thus $|E(N)| = 10$ and so $k = 5$. Since $e \notin \mathrm{cl}(P_2 \cup \{\alpha, \beta\})$, the set $\mathrm{cl}(P_2 \cup \{\alpha, \beta, e\})$ has rank 4, so its complement is a cocircuit of N. In $N\backslash e$, this complement contains a cocircuit C^*. Since N has no triangles, it follows by Lemma 2.8 that C^* contains an element a such that $a \notin \mathrm{cl}(P_2 \cup \{\alpha, \beta, e\})$ and $N\backslash e/a$ is 3-connected. Then $z_5 \neq a$ since $z_5 \in \mathrm{cl}(P_2 \cup \{\alpha, \beta, e\})$.

Consider N/a. Since $\{z_1, z_2, z_3, z_4\} - a$ is a triad in $N\backslash e/a$, it follows by Lemma 6.2(iii) and (7.4.1) that N/a contains a 3-separation (R, G), where $r_{N/a}(R) = 3 = r_{N/a}(G)$, and $\{\alpha, \beta\} \subseteq R$. By (7.4.2), we may assume that $P_2 \subseteq G$. Since $r_N((P_1 - z_5) \cup \{\alpha, \beta\}) = 5$, we have $r_{N/a}((P_1 - \{z_5, a\}) \cup \{\alpha, \beta\}) = 4$. As $r_{N/a}(R) = 3$, it follows that $|G \cap (P_1 - \{z_5, a\})| \geq 1$. Similarly, $|R \cap (P_1 - \{z_5, a\})| \geq 1$.

Now consider z_5, which is in $\mathrm{cl}(\{z_1, z_2, z_3, z_4\})$. By closure-equivalence, $z_5 \in \mathrm{cl}_N(\{\alpha, \beta\} \cup P_2)$. As N has no triangles, it follows by the choice of a that $r_{N/a}(\{\alpha, \beta, z_5\}) = 3 = r_{N/a}(P_2 \cup z_5)$. Hence as $z_5 \in Z$ for some Z in $\{R, G\}$, we get $r_{N/a}(Z) \geq 4$; a contradiction. Thus (7.4.3) holds.

Now suppose that $z_5 \in \mathrm{cl}(\{z_1, z_2, z_3, z_4\})$. Since $r(P_1) = 4$, we have $k \geq 6$ and so $|E(N)| \geq 11$. Thus, by Lemma 7.1, there is an element x in $\{z_1, z_2, z_3, z_4, z_5\}$ such that x does not expose any 3-separation in $M\backslash x$. Thus $z_5 \notin \mathrm{cl}(\{z_1, z_2, z_3, z_4\})$, so $z_5 \in \mathrm{cl}^*(\{z_1, z_2, z_3, z_4\})$. If z_6 or z_7 exists, then $z_6, z_7 \in \mathrm{cl}(\{z_1, z_2, z_3, z_4, z_5\})$ as $r(P_1) = 4$.

The next result is used twice in the rest of the analysis of this subcase.

7.4.4. *Suppose that $|E(N)| = 10$. Let a be an element of $\{z_1, z_2, z_3\}$. Then there is no copaddle of the form $(\{\alpha, \beta\}, A_1, A_2)$ in N/a, where $r_{N/a}(A_1) = 2 = r_{N/a}(A_2)$, $|A_1| = 3$, and $|A_2| = 4$.*

Assume such a copaddle exists. Since $\{z_1, z_2, z_3, z_4\} - a$ is a triangle in N/a, this set is contained in A_1 or A_2. Now $e, z_5 \notin \mathrm{cl}_{N/a}(\{z_1, z_2, z_3, z_4\} - a)$. Furthermore, if $\mathrm{cl}_{N/a}(\{z_1, z_2, z_3, z_4\} - a)$ contains an element p of P_2, then $p \in \mathrm{cl}_{N\backslash e}(P_1)$, so $\mathrm{fcl}_{N\backslash e}(\{\alpha, \beta\}) \supseteq P_2$; a contradiction. Hence $P_2 \cup \{z_5, e\}$ avoids $\mathrm{cl}_{N/a}(\{z_1, z_2, z_3, z_4\} - a)$ and so is contained in A_2. But $r_{N/a}(P_2 \cup z_5) = 3$; a contradiction. Thus (7.4.4) holds.

7.4.5. *There is an element a of $\{z_1, z_2, z_3\}$ such that $a \notin \mathrm{cl}(\{\alpha, \beta, e\} \cup P_2)$.*

As $z_5 \in \mathrm{cl}^*(\{z_1, z_2, z_3, z_4\})$, it follows by Lemma 2.10 that $\sqcap(\{\alpha, \beta\} \cup P_2, \{z_1, z_2, z_3\}) \leq 1$ and so $\sqcap(\{\alpha, \beta, e\} \cup P_2, \{z_1, z_2, z_3\}) \leq 2$. Thus such an element a certainly exists.

For the element a just found, by Lemma 2.13, $N\backslash e/a$ is 3-connected. By Lemma 6.2(iii) and (7.4.4), N/a has a 3-separation (R, G), where $r_{N/a}(R) = 3 = r_{N/a}(G)$.

Suppose that z_6 exists. Then $z_6 \in \mathrm{cl}_{N\backslash e}(\{\alpha, \beta\} \cup P_2)$. Since N has no triangles and a is in a triad in $N\backslash e$ avoiding $\{\alpha, \beta, z_6\} \cup P_2$, it follows that $z_6 \notin \mathrm{cl}_{N/a}(\{\alpha, \beta\})$ and $z_6 \notin \mathrm{cl}_{N/a}(P_2)$. By closure-equivalence and (7.4.2), we may assume that $\{\alpha, \beta\} \subseteq R$ and $P_2 \subseteq G$. Now the rank of $(P_1 - a) - \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)$ in N/a is 3. If $[(P_1 - a) - \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)] \subseteq R$, then $r_{N/a}(R) \geq 4$; a contradiction. So $|[(P_1 - a) - \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)] \cap G| \geq 1$. Similarly, $|[(P_1 - a) - \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)] \cap R| \geq 1$. But then neither $z_6 \in G$ nor $z_6 \in R$; otherwise $r_{N/a}(G) \geq 4$ and $r_{N/a}(R) \geq 4$, respectively. Thus z_6 does not exist, in which case, $|P_1| = 5$ and so $|E(N)| = 10$.

7.4.6. *For some Q_1 and Q_2 such that $\{Q_1, Q_2\} = \{\{\alpha, \beta\}, P_2\}$,*

$$\sqcap(Q_1, P_1 - z_5) = 1 \text{ and } r_N(Q_1 \cup (P_1 - z_5)) = 4.$$

By Lemma 6.2(iii) and (7.4.2), we may assume that $Q_1 \subseteq R$ and $Q_2 \subseteq G$. Either $|(\{z_1, z_2, z_3, z_4\} - a) \cap R| \geq 2$ or $|(\{z_1, z_2, z_3, z_4\} - a) \cap G| \geq 2$, so we may assume that $|(\{z_1, z_2, z_3, z_4\} - a) \cap R| \geq 2$. Therefore, as $r_{N/a}(R) = 3$, it follows that $r_{N/a}(Q_1 \cup (\{z_1, z_2, z_3, z_4\} - a)) = 3$ and so $r_N(Q_1 \cup (P_1 - z_5)) = 4$. Thus $\sqcap(Q_1, P_1 - z_5) = 1$. Hence (7.4.6) holds.

Since $r(Q_1 \cup (P_1 - z_5)) = 4$ and $N\backslash e$ is 3-connected, $Q_2 \cup z_5$ is a triad in $N\backslash e$. We now consider N/z_5. Since $(Q_1 \cup Q_2, P_1 - z_5)$ is a 2-separation in $N\backslash\{e, z_5\}$ and N has no triangles, it follows by Lemma 2.5 that $N\backslash e/z_5$

is 3-connected. Now, since $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e/z_5$, we may assume by Lemma 6.2(iii), (7.4.1), and (7.4.2) that N/z_5 has a 3-separation (X, Y), where $r_{N/z_5}(X) = 3 = r_{N/z_5}(Y)$; $Q_2 \subseteq X$; and $Q_1 \subseteq Y$. If $\{z_1, z_2, z_3\} \subseteq X$, then $r_{N/z_5}(X) \geq 4$; a contradiction. So $|\{z_1, z_2, z_3\} \cap Y| \geq 1$. Similarly, $|\{z_1, z_2, z_3\} \cap X| \geq 1$. Since $Q_2 \cup z_5$ is a triad in $N\backslash e$ and N has no triangles, $z_4 \notin \mathrm{cl}_{N/z_5}(Q_1)$. Therefore $z_4 \notin Y$, otherwise $r_{N/z_5}(Y) \geq 4$. Thus $z_4 \in X$ and so $z_4 \in \mathrm{cl}_{N/z_5}(Q_2)$ otherwise $r_{N/z_5}(Q_2 \cup z_4) = 3$ and we obtain the contradiction that $r_{N/z_5}(X) > 3$ since X also meets the cocircuit $\{z_1, z_2, z_3, e\}$ of N/z_5. Noting that $\sqcap_N(Q_2, P_1 - z_5) \in \{0, 1\}$, we break the rest of the analysis into two parts depending on the value of $\sqcap_N(Q_2, P_1 - z_5)$.

First assume that $\sqcap_N(Q_2, P_1 - z_5) = 1$. Then $r_N(Q_2 \cup (P_1 - z_5)) = 4$. Since $N\backslash e$ is 3-connected, $Q_1 \cup z_5$ is a triad in $N\backslash e$. Therefore, as N has no triangles, $z_4 \notin \mathrm{cl}_{N/z_5}(Q_2)$; a contradiction. Thus $\sqcap_N(Q_2, P_1 - z_5) \neq 1$.

Now assume that $\sqcap_N(Q_2, P_1 - z_5) = 0$. Then, as $r_{N/z_5}(X) = 3$ and N has no triangles, we have $|\{z_1, z_2, z_3\} \cap X| = 1$ and $|\{z_1, z_2, z_3\} \cap Y| = 2$. Letting $\{u, u'\} = \{z_1, z_2, z_3\} \cap Y$, we have $r_N(Q_1 \cup \{u, u'\}) = 3$ since $r_{N/z_5}(Y) = 3$ and $Q_2 \cup \{z_5, e\}$ is a cocircuit of N. Let $w = \{z_1, z_2, z_3\} - \{u, u'\}$. If $u \in \mathrm{cl}_N(Q_1 \cup Q_2 \cup e)$, then, as $r_N(Q_1 \cup \{u, u'\}) = 3$, it follows that $u' \in \mathrm{cl}_N(Q_1 \cup Q_2 \cup e)$. But then $\{w, z_4, z_5\}$ is a triad in N; a contradiction. So $u \notin \mathrm{cl}_N(Q_1 \cup Q_2 \cup e)$ and, similarly, $u' \notin \mathrm{cl}_N(Q_1 \cup Q_2 \cup e)$.

For $\{v, v'\} = \{u, u'\}$, consider N/v. By Lemma 2.13, $N\backslash e/v$ is 3-connected. By Lemma 6.2(iii) and (7.4.4), N/v has a 3-separation (U, V), where $r_{N/v}(U) = 3 = r_{N/v}(V)$. Since $v \notin \mathrm{cl}_N(Q_1 \cup Q_2 \cup e)$, it follows by (7.4.2) that we may assume $Q_2 \subseteq U$ and $Q_1 \subseteq V$. Say $|U \cap \{v', w, z_4\}| \geq 2$. Then $r((U \cap \{v', w, z_4\}) \cup v) \geq 3$. Since $\sqcap(Q_2, P_1 - z_5) = 0$, it follows that $r(U \cup v) \geq 5$, so $r_{N/v}(U) \geq 4$; a contradiction. Therefore $|V \cap \{v', w, z_4\}| \geq 2$. But $\{v', w, z_4\}$ is a triangle of M/v, so we may assume that $\{v', w, z_4\} \subseteq V$. If $z_5 \in V$, then $r_{N/v}(V) \geq 4$; a contradiction. Thus $z_5 \in U$ and $Q_2 \cup z_5 = U - e$. Since $N\backslash e/v$ is 3-connected and since $Q_2 \cup z_5$ is a triad in $N\backslash e$ and therefore in $N\backslash e/v$, it follows, by Lemma 2.22, that $e \in \mathrm{cl}_{N/v}(U - e)$. Thus $e \in \mathrm{cl}_N(Q_2 \cup \{v, z_5\})$. As v was arbitrarily chosen in $\{u, u'\}$, we have that $e \in \mathrm{cl}_N(Q_2 \cup \{u, z_5\})$ and $e \in \mathrm{cl}_N(Q_2 \cup \{u', z_5\})$. If $e \in \mathrm{cl}_N(Q_2 \cup z_5)$, then $Q_2 \cup \{e, z_5\}$ is 3-separating in N; a contradiction. Thus $e \notin \mathrm{cl}_N(Q_2 \cup z_5)$ and so $u \in \mathrm{cl}_N(Q_2 \cup \{e, z_5\})$ and $u' \in \mathrm{cl}_N(Q_2 \cup \{e, z_5\})$. Therefore $r_N(Q_2 \cup (P_1 - w) \cup e) = 4$ as $z_4 \in \mathrm{cl}_{N/z_5}(Q_2)$. But then $Q_1 \cup w$ is a triad in N; a contradiction. This completes the analysis of (iii)(a).

(iii)(b) $r(P_2) = 3$. Since $r(P_1) = 3$, we may assume without loss of generality that $|P_1| \geq |P_2|$. As $|E(N)| \in \{10, 11, 12\}$, this implies that $|P_1| \geq 4$. Let $(z_1, z_2, \ldots, z_k)$ be a sequential ordering of P_1 in $N\backslash e$. Since N has no triangles and $r(P_1) = 3$, it follows that $\{z_1, z_2, z_3\}$ is a triad of $N\backslash e$ and $z_4 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3\})$. If $k \geq 5$, then, as $r(P_1) = 3$, we have

$z_5 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3, z_4\})$. But then $|E(N)| \geq 11$ and so, by Lemma 7.1, there is an element x in $\{z_1, z_2, z_3, z_4, z_5\}$ such that x does not expose any 3-separation in $M\backslash x$; a contradiction. Thus $k = 4$. Similarly, if $(y_1, y_2, \ldots, y_l)$ is a sequential ordering of P_2 in $N\backslash e$, then $\{y_1, y_2, y_3\}$ is a triad, $3 \leq l \leq 4$, and $y_4 \in \mathrm{cl}_{N\backslash e}(\{y_1, y_2, y_3\})$ when $l = 4$.

7.4.7. *There is an element a of $\{z_1, z_2, z_3\}$ such that $a \notin \mathrm{cl}(P_2 \cup e)$.*

Since $\sqcap(P_1, P_2) = 1$, it follows that $\sqcap(P_1, P_2 \cup e) \leq 2$. Thus such an element a certainly exists.

For this element a, by Lemma 2.13, $N\backslash e/a$ is 3-connected. By Lemma 6.2, either N/a has a 3-separation (R, G), where $r_{N/a}(R), r_{N/a}(G) \geq 3$, and, without loss of generality, $\{\alpha, \beta\} \subseteq R$; or $|E(N)| = 10$ and there is a copaddle $(\{\alpha, \beta\}, A_1, A_2)$ in N/a, where $r_{N/a}(A_1) = 2 = r_{N/a}(A_1)$, $|A_1| = 3$, and $|A_2| = 4$. By (7.4.1), since $\{y_1, y_2, y_3\}$ is a triad of $N\backslash e/a$ avoiding $\{\alpha, \beta\}$, the second possibility does not occur. Thus $r(N/a) = 4$, so $r_{N/a}(R) = 3 = r_{N/a}(G)$.

By our choice of a, if $X \subseteq P_2 \cup \{\alpha, \beta\}$, then $r_{N/a}(X) = r(X)$. If $\{y_1, y_2, y_3\} \subseteq R$, then $r_{N/a}(R) \geq 4$; a contradiction. If $\{y_1, y_2, y_3\} \subseteq G$, then, as $\{y_1, y_2, y_3\}$ is a triad in $N\backslash e/a$, it follows by Lemma 2.22 that $e \in \mathrm{cl}_{N/a}(G - e)$. Since $P_2 \subseteq \mathrm{cl}_{N/a}(G)$, it follows by (7.4.7) that $r_{N/a}(G) \geq 4$; a contradiction. Thus $|R \cap \{y_1, y_2, y_3\}| \geq 1$ and $|G \cap \{y_1, y_2, y_3\}| \geq 1$. If $|R \cap (P_1 - a)| \geq 2$, then, as $r_{N/a}(\{\alpha, \beta\} \cup (P_1 - a)) = 3$ and $\{y_1, y_2, y_3\}$ is a triad in $N\backslash e/a$, we have $r_{N/a}(R) \geq 4$. Thus $|G \cap (P_1 - a)| \geq 2$ and, by closure-equivalence, we may assume that $P_1 - a \subseteq G$.

7.4.8. *The element y_4 does not exist.*

If y_4 exists, then, as $P_2 \cup \{\alpha, \beta\}$ contains no triangles in N/a, it follows that $y_4 \in G$; otherwise, $r_{N/a}(R) \geq 4$. If $y_4 \notin \mathrm{cl}(P_1)$, then $y_4 \notin \mathrm{cl}_{N/a}(P_1 - a)$. As $\{y_1, y_2, y_3\}$ is a triad of $N\backslash e/a$, it follows that $r_{N/a}(G) \geq 4$; a contradiction. Thus $y_4 \in \mathrm{cl}(P_1)$. But y_4 and z_4 are distinct, and so $P_1 \cup y_4$ is a 5-element rank-3 set in N, contradicting Lemma 7.1. Hence (7.4.8) holds.

Assume that $|R \cap \{y_1, y_2, y_3\}| = 2$ and consider $N\backslash e$. By our choice of a, as $r_{N/a}(\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})) = 3$, we have $r_{N\backslash e}(\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})) = 3$. Since $r_N(\mathrm{cl}(P_1) \cup (G \cap \{y_1, y_2, y_3\})) = 4$, it follows that $\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})$ is 3-separating in $N\backslash e$. In particular,

$$(\{\alpha, \beta\}, \mathrm{cl}(P_1) \cup (G \cap \{y_1, y_2, y_3\}), R \cap \{y_1, y_2, y_3\})$$

is a flower in $N\backslash e$. Also $\sqcap_N(\{\alpha, \beta\}, R \cap \{y_1, y_2, y_3\}) = 1$. Thus $(\{\alpha, \beta\}, \mathrm{cl}(P_1) \cup (G \cap \{y_1, y_2, y_3\}), R \cap \{y_1, y_2, y_3\})$ is a flower in $N\backslash e$ of the form analyzed in (iii)(a).

We may now assume that $|G\cap\{y_1,y_2,y_3\}|=2$. Let $R\cap\{y_1,y_2,y_3\}=\{v\}$ and $G\cap\{y_1,y_2,y_3\}=\{u,u'\}$. Since $r_{N\backslash e/a}(G-e)=3$, it follows that $\{\alpha,\beta,v\}$ is a triad in $N\backslash e/a$, and therefore a triad in $N\backslash e$. Furthermore, as

$$3=r_{N/a}((P_1-a)\cup\{u,u'\})=r_N(P_1\cup\{u,u'\})-1,$$

$r_N(P_1\cup\{u,u'\})=4$ and so $\sqcap_N(P_1,\{u,u'\})=1$. Since $|\mathrm{cl}_N(\{\alpha,\beta,v\})\cap\{u,u'\}|\leq 1$, we may assume that $u\notin\mathrm{cl}(\{\alpha,\beta,v\})$.

Consider N/u and note that $u'\in\mathrm{cl}_{N/u}(P_1)$. Since $\{z_1,z_2,z_3\}$ is a triad in $N\backslash e/u$, it follows by Lemma 6.2(iii) and (7.4.1) that there is a 3-separation (U,V) in N/u where $r_{N/u}(U)=3=r_{N/u}(V)$, and $\{\alpha,\beta\}\subseteq U$.

We show next that

7.4.9. *$\{\alpha,\beta,v\}$ is a not a triangle in N/u, the element $e\notin\mathrm{cl}_{N/u}(\{\alpha,\beta,v,u'\})$, and $e\notin\mathrm{cl}_{N/u}(P_1\cup u')$.*

Since $u\notin\mathrm{cl}_N(\{\alpha,\beta,v\})$ and N has no triangles, $\{\alpha,\beta,v\}$ is not a triangle in N/u. If $e\in\mathrm{cl}_{N/u}(\{\alpha,\beta,v,u'\})$, then $e\in\mathrm{cl}_N(\{\alpha,\beta,v,u,u'\})$. This implies that $e\in\mathrm{cl}_N(\{\alpha,\beta\}\cup P_2)$ and so, as $\{\alpha,\beta\}\cup P_2$ is 3-separating in $N\backslash e$, it is 3-separating in N; a contradiction. Thus $e\notin\mathrm{cl}_{N/u}(\{\alpha,\beta,v,u'\})$. Lastly, if $e\in\mathrm{cl}_{N/u}(P_1\cup u')$, then $e\in\mathrm{cl}_N(P_1\cup\{u,u'\})$. But then $\{\alpha,\beta,v\}$ is a triad in N; a contradiction. Thus $e\notin\mathrm{cl}_{N/u}(P_1\cup u')$ and (7.4.9) holds.

If $v\in U$, then, as $r_{N/u}(U)=3$, we have $U\subseteq\mathrm{cl}_{N/u}(\{\alpha,\beta,v\})$. Therefore $\{z_1,z_2,z_3\}\subseteq V$. By (7.4.9), $e\notin\mathrm{cl}_{N/u}(\{\alpha,\beta,v\})$, so $e\notin U$. Thus $e\in V$. But then, by (7.4.9), $e\notin\mathrm{cl}_{N/u}(P_1)$, so $r_{N/u}(V)\geq 4$; a contradiction. Hence $v\in V$. If $\{z_1,z_2,z_3\}\subseteq U$, then $r_{N/u}(U)\geq 4$; a contradiction. Also, if $\{z_1,z_2,z_3\}\subseteq V$, then, as $\{\alpha,\beta,v\}$ is a triad in $N\backslash e$, we have that $v\notin\mathrm{cl}_{N/u}(P_1)$, so $r_{N/u}(V)\geq 4$; a contradiction. It now follows that $|U\cap\{z_1,z_2,z_3\}|\geq 1$ and $|V\cap\{z_1,z_2,z_3\}|\geq 1$. Since $\{v,u,u'\}$ is a triad in $N\backslash e$, we have $r_{N/u}(\{\alpha,\beta,z_4\})=r_N(\{\alpha,\beta,z_4\})$. As N has no triangles, this implies that $r_{N/u}(\{\alpha,\beta,z_4\})=3$, so $z_4\notin U$; otherwise, $r_{N/u}(U)\geq 4$. Therefore $z_4\in V$. If $u'\in V$, then $r_{N/u}(V)\geq 4$ as $r_N((P_1\cap V)\cup\{u,u',v\})=5$. This contradiction implies that $u'\in U$.

Assume $|V\cap\{z_1,z_2,z_3\}|=2$. Then $r_{N/u}(V\cap P_1)=3$ as $r_N(V\cap P_1)=3$, and so $r_{N/u}(V)\geq 4$; a contradiction. Thus $|V\cap\{z_1,z_2,z_3\}|=1$ and so $|U\cap\{z_1,z_2,z_3\}|=2$. If $u'\notin\mathrm{cl}_{N/u}(\{\alpha,\beta\})$, then $r_{N/u}(U)\geq 4$; a contradiction. Therefore $u'\in\mathrm{cl}_{N/u}(\{\alpha,\beta\})$. Consider $N\backslash e$. Since N has no triangles, it follows that

$$\begin{aligned}\sqcap(\{\alpha,\beta\},\{u,u'\})&=r(\{\alpha,\beta\})+r(\{u,u'\})-r(\{\alpha,\beta,u,u'\})\\&=2+2-3=1.\end{aligned}$$

Furthermore, $r(P_1 \cup v) = 4$ and so $\{\alpha, \beta, u, u'\}$ is a 3-separation in $N\backslash e$. It now follows that $(\{\alpha, \beta\}, \{u, u'\}, P_1 \cup v)$ is a flower in $N\backslash e$ of the form analyzed in (iii)(a). This completes the analysis of (iii)(b) and therefore completes the analysis of (I).

(II) $r(N) = 6$ and $|E(N)| \in \{11, 12\}$.

(i) Φ is a paddle. Since Φ is a paddle,

$$6 = r(N\backslash e) = r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 4.$$

Thus $8 = r(P_1) + r(P_2) \leq |P_1| + |P_2| \leq 9$, so either P_1 or P_2 is independent. Also, as neither $P_1 \subseteq \text{fcl}(\{\alpha, \beta\})$ nor $P_2 \subseteq \text{fcl}(\{\alpha, \beta\})$, we have $r(P_1) \geq 3$ and $r(P_2) \geq 3$. Without loss of generality, there are two possibilities to consider: either $r(P_1) = 3$ and $r(P_2) = 5$; or $r(P_1) = r(P_2) = 4$.

If $r(P_1) = r(P_2) = 4$, then we may assume that P_1 is independent. Then, as $\alpha, \beta \in \text{cl}(P_2)$,

$$r^*_{N\backslash e}(P_1) = |P_1| - r(N\backslash e) + r(P_2 \cup \{\alpha, \beta\}) = 2.$$

Thus P_1 is a 4-element cosegment of $N\backslash e$ that avoids α and β. Hence, by Lemma 7.1, that there is an element y in $P_1 \cup e$ such that y does not expose any 3-separation in $M^*\backslash y$; a contradiction.

We may now assume that $r(P_1) = 3$ and $r(P_2) = 5$. Consider N/e. By Lemma 6.2(iii), N/e has a 3-separation (R, G), where $r_{N/e}(R), r_{N/e}(G) \geq 3$, and R or G contains $\{\alpha, \beta\}$. Since $e \notin \text{cl}(P_1 \cup \{\alpha, \beta\})$, we have $r_{N/e}(P_1 \cup \{\alpha, \beta\}) = 3$ and $P_1 \cup \{\alpha, \beta\}$ contains no triangles in N/e. Therefore, as $|P_1 \cup \{\alpha, \beta\}| \geq 5$, we may also assume by switching to a closure-equivalent 3-separation that $P_1 \cup \{\alpha, \beta\} \subseteq R$ and so $G \subseteq P_2$. Since N is 4-connected, $e \notin \text{cl}^*(R)$. Therefore, by Lemma 2.1, $e \in \text{cl}(G)$. Then $e \in \text{cl}(P_2)$, so P_2 is 3-separating in N; a contradiction. We conclude that Φ is not a paddle.

(ii) Φ is a copaddle. Since neither P_1 nor P_2 is a subset of $\text{fcl}(\{\alpha, \beta\})$, we have $|P_1|, |P_2| \geq 3$. Also, as N has no triangles, $r(P_1), r(P_2) \geq 3$. Thus, as

$$6 = r(N\backslash e) = r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 2,$$

$r(P_1) = r(P_2) = 3$. If $|P_1| \geq 5$, then, by Lemma 7.1, there is an element $x \in P_1$ such that x does not expose any 3-separation in $M\backslash x$. Thus, by symmetry, we may assume that $|P_1|, |P_2| \leq 4$. But $|E(N)| \in \{11, 12\}$ and so $|P_1| = |P_2| = 4$. Now, by Lemma 3.1, either P_1 or P_2 is sequential. Without loss of generality, we may assume that P_2 is sequential. Let (y_1, y_2, y_3, y_4) be a sequential ordering of P_2. Since N has no triangles, $\{y_1, y_2, y_3\}$ is a triad in $N\backslash e$. Now, as $\sqcap(P_1, P_2) = 0$, we have $\sqcap(P_1, P_2 \cup e) \leq 1$, and so there is an element $a \in P_1 - \text{cl}(\{\alpha, \beta\} \cup P_2)$ such that $a \notin \text{cl}(P_2 \cup e)$.

Consider N/a and note that, as a is either in a triad or a quad of $N\backslash e$, it follows by Lemma 2.13 that $N\backslash e/a$ is 3-connected. Furthermore, we have

$$\sqcap_{N/a}(\{\alpha,\beta\}, P_1 - a) = \sqcap_{N/a}(P_1 - a, P_2) = \sqcap_{N/a}(P_2, \{\alpha,\beta\}) = 0.$$

By Lemma 6.2(iii), we may assume that N/a has a 3-separation (R, G), where $r_{N/a}(R), r_{N/a}(G) \geq 3$, and $\{\alpha, \beta\} \subseteq R$. As $P_1 - a$ is a triangle of N/a, we may also assume that either $P_1 - a \subseteq R$ or $P_1 - a \subseteq G$. Suppose that $P_1 - a \subseteq R$. Then $r_{N/a}(R) = 4$, and so $R \cap \{y_1, y_2, y_3\}$ is empty; otherwise, $r_{N/a}(R) \geq 5$ and so $r_{N/a}(G) \leq 2$; a contradiction. Thus $\{y_1, y_2, y_3\} \subseteq G$ so $\{y_1, y_2, y_3\}$ spans G in N/a. By our choice of a, we have that $e \notin \mathrm{cl}_{N/a}(G-e)$. Therefore, as $N/a\backslash e$ is 3-connected, it follows by Lemma 2.21 that $e \in \mathrm{cl}_{N/a}(R - e)$. Hence $\{y_1, y_2, y_3\}$ is a triad in N; a contradiction. Thus $P_1 - a \subseteq G$. If $|R \cap P_2| = |G \cap P_2| = 2$, then $r_{N/a}(R), r_{N/a}(G) \geq 4$; a contradiction as (R, G) is a 3-separation in N/a. Therefore either $|R \cap P_2| \geq 3$ or $|G \cap P_2| \geq 3$. But, as N has no triangles and $a \notin \mathrm{cl}(P_2)$, any 3-element subset of P_2 is independent in N/a. Therefore either $r_{N/a}(R) \geq 5$ or $r_{N/a}(G) \geq 5$; a contradiction. Thus Φ is not a copaddle.

(iii) $\boldsymbol{\sqcap(\{\alpha,\beta\}, P_1) = \sqcap(P_1, P_2) = \sqcap(P_2, \{\alpha,\beta\}) = 1}$. By Lemma 3.1, both P_1 and P_2 are sequential. Furthermore, as

$$6 = r(N\backslash e) = r(\{\alpha,\beta\}) + r(P_1) + r(P_2) - 3,$$

$r(P_1) + r(P_2) = 7$, and so we may assume that $r(P_2) \in \{2, 3\}$.

Before partitioning (iii) into two subcases depending on the rank of P_2, consider P_i, where $i \in \{1, 2\}$. Let $|P_i| = k$, and suppose that $3 \leq k \leq 5$. Let $(z_1, z_2, \ldots, z_k)$ be a sequential ordering of P_i. Since N has no triangles, it follows that $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e$. If $z_4 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3\})$, then $\{z_1, z_2, z_3, z_4\}$ is a 4-element cosegment in $N\backslash e$ avoiding α and β, so the lemma holds by Lemma 7.1. Thus, if $k \geq 4$, then $z_4 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3\})$. If $z_5 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3, z_4\})$, then $\{z_1, z_2, z_3, z_4, z_5\}$ is a 5-element rank-3 subset of $E(N)$ avoiding α and β, and so the lemma holds by Lemma 7.1. Therefore, if $k = 5$, then $z_5 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3, z_4\})$.

(iii)(a) $\boldsymbol{r(P_2) = 2}$. Since N has no triangles, $|P_2| = 2$ and so $|P_1| \in \{6, 7\}$. Let $(z_1, z_2, \ldots, z_k)$ be a sequential ordering of P_1. Then, from above, $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e$, the element $z_4 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3\})$, and $z_5 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3, z_4\})$. Now $r_{N\backslash e}(P_1) = 5$. Thus if $k = 6$, then $z_6 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3, z_4, z_5\})$. Moreover, if $k = 7$, then either $z_6 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3, z_4, z_5\})$ and $z_7 \in \mathrm{cl}_{N\backslash e}(P_1 - z_7)$; or $z_6 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3, z_4, z_5\})$ and $z_7 \in \mathrm{cl}^*_{N\backslash e}(P_1 - z_7)$.

To maintain symmetry, let $\{Q_1, Q_2\} = \{\{\alpha, \beta\}, P_2\}$. First suppose that $k = 6$. Then $|E(N)| = 11$ and so, by (I) of the lemma, which we have already proved, it suffices to show that there is an element a of $E(N) - \{\alpha, \beta\}$ such that N/a is sequentially 4-connected. We assume no such element exists.

7.4.10. *Let $k \in \{1, 2\}$. If $Q_k \cup z_i$ is a triad in $N\backslash e$ for some $i \in \{5, 6\}$, then $Q_k \cup z_j$ is not a triad in $N\backslash e$, where $j \in \{5, 6\} - i$.*

To show this, suppose that $Q_k \cup z_i$ is a triad for some k and i. Let $p \in Q_k$. If $Q_k \cup z_j$ is a triad in $N\backslash e$ where $i \neq j$, then, by circuit elimination, $\{p, z_5, z_6\}$ is a triad in $N\backslash e$. But $Q_1 \cup Q_2$ is a circuit of $N\backslash e$ and $|(Q_1 \cup Q_2) \cap \{p, z_5, z_6\}| = 1$, contradicting orthogonality. Hence (7.4.10) holds.

7.4.11. *There is an element a of $\{z_1, z_2, z_3\}$ such that $a \notin \mathrm{cl}_N(Q_1 \cup Q_2 \cup \{z_i, e\})$, where i is chosen in $\{5, 6\}$ so that if a triad of the type described in (7.4.10) exists in $N\backslash e$, then $Q_1 \cup z_i$ or $Q_2 \cup z_i$ is a triad of $N\backslash e$.*

By Lemma 2.10(ii), $\sqcap(\{z_1, z_2, z_3\}, Q_1 \cup Q_2) = 0$, so $\sqcap(\{z_1, z_2, z_3\}, Q_1 \cup Q_2 \cup \{z_i, e\}) \leq 2$. Hence there is such an element a in $\{z_1, z_2, z_3\}$.

Consider N/a and note that, by Lemma 2.13, $N\backslash e/a$ is 3-connected. As N/a is not sequentially 4-connected, it has a non-sequential 3-separation (R, G). Since N/a has $\{z_1, z_2, z_3, z_4\} - a$ as a circuit, we may assume that either $\{z_1, z_2, z_3, z_4\} - a \subseteq R$ or $\{z_1, z_2, z_3, z_4\} - a \subseteq G$.

7.4.12. *Neither $Q_1 \cup Q_2 \subseteq R$ nor $Q_1 \cup Q_2 \subseteq G$.*

Assume that $Q_1 \cup Q_2 \subseteq R$. Then $G - e \subseteq P_1 - a$. If $\{z_1, z_2, z_3, z_4\} - a \subseteq R$, then $|G| \leq 3$; a contradiction as (R, G) is non-sequential. Therefore $\{z_1, z_2, z_3, z_4\} - a \subseteq G$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $e \in \mathrm{cl}(P_1)$, and so $(P_1, \{\alpha, \beta\} \cup P_2)$ is a 3-separation of N; a contradiction. Therefore $e \in \mathrm{cl}_{N/a}(R - e)$ so $(R \cup e, G - e)$ is a non-sequential 3-separation of N/a. Since $(z_1, z_2, z_3, z_4, z_5, z_6)$ is a sequential ordering of P_1 in $N\backslash e$, it follows that

$$(\{z_1, z_2, z_3, z_4\} - a, z_5, z_6, \{\alpha, \beta\} \cup P_2)$$

is a 3-sequence in $N/a\backslash e$. By [5, Lemma 5.8], $(\{z_1, z_2, z_3, z_4\} - a, z_6, z_5, \{\alpha, \beta\} \cup P_2)$ is also a 3-sequence of $N/a\backslash e$. Thus $G - e$ is sequential in $N/a\backslash e$ and therefore, as $e \in \mathrm{cl}_{N/a}(R - e)$, we deduce that $G - e$ is sequential in N/a; a contradiction. So $Q_1 \cup Q_2 \not\subseteq R$ and, by symmetry, $Q_1 \cup Q_2 \not\subseteq G$; that is, (7.4.12) holds.

By Lemma 2.12 and (7.4.12), we may now assume that $Q_1 \subseteq R$ and $Q_2 \subseteq G$. Furthermore, without loss of generality, we may also assume that $\{z_1, z_2, z_3, z_4\} - a \subseteq R$. Then, by Lemma 2.10, $r_{N/a}(R) \geq 4$. Since $|G| \geq 4$, we have $|G \cap \{z_5, z_6\}| \geq 1$. If $|G \cap \{z_5, z_6\}| = 2$, then $r_{N/a}(G) \geq 4$, contradicting the fact that (R, G) is a 3-separation of N/a. So $|G \cap \{z_5, z_6\}| =$

1, and $G-e$ is a triad in $N\backslash e/a$ and therefore a triad in $N\backslash e$. Let $\{s\} = R \cap \{z_5, z_6\}$ and $\{g\} = G \cap \{z_5, z_6\}$.

By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R-e)$ or $e \in \mathrm{cl}_{N/a}(G-e)$. If $e \in \mathrm{cl}_{N/a}(R-e)$, then, arguing as in the proof of (7.4.12), we get that $G-e$ is sequential; a contradiction. So $e \in \mathrm{cl}_{N/a}(G-e)$. If $Q_1 \cup s$ is a triad in $N\backslash e$, then, as $e \in \mathrm{cl}_{N/a}(G-e)$, we have $Q_1 \cup s$ is a triad in N; a contradiction. Therefore, $Q_1 \cup s$ is not a triad in $N\backslash e$. Since $Q_2 \cup g$ is a triad in $N\backslash e$, it follows by (7.4.10) that $Q_2 \cup s$ is not a triad in $N\backslash e$. Thus, by the choice of a in (7.4.11), $a \notin \mathrm{cl}_N(Q_1 \cup Q_2 \cup \{g, e\})$. Since $e \notin \mathrm{cl}_N(Q_2 \cup g)$, it follows that $e \notin \mathrm{cl}_{N/a}(Q_2 \cup g)$; a contradiction as $Q_2 \cup g = G - e$. It now follows that we may suppose that $k = 7$.

Assume that $z_6 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3, z_4, z_5\})$ and $z_7 \in \mathrm{cl}_{N\backslash e}(P_1 - z_7)$. Consider N/z_5 and note that, by Lemma 2.11, $N\backslash e/z_5$ is 3-connected. By Lemma 6.2(iii), N/z_5 has a 3-separation (R, G), where $r_{N/z_5}(R), r_{N/z_5}(G) \geq 3$, and R or G contains $\{\alpha, \beta\}$. Furthermore, as $r_{N/z_5}(Q_1 \cup Q_2 \cup z_7) = 3$, we may assume that $Q_1 \cup Q_2 \cup z_7 \subseteq R$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/z_5}(R-e)$ or $e \in \mathrm{cl}_{N/z_5}(G-e)$. If $e \in \mathrm{cl}_{N/z_5}(G-e)$, then $Q_1 \cup Q_2$ is 3-separating in N; a contradiction. Therefore $e \in \mathrm{cl}_{N/z_5}(R-e)$. As $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e/z_5$, it follows by Lemma 2.22 that $|\{z_1, z_2, z_3\} \cap R| \geq 1$ and so $r_{N/z_5}(R) \geq 4$. If $z_6 \in R$, then $r_{N/z_5}(R) \geq 5$; a contradiction as $r_{N/z_5}(G) \geq 3$ and (R, G) is a 3-separation of N/z_5. Thus $z_6 \in G$. Now $|G \cap \{z_1, z_2, z_3, z_4\}| \leq 2$, otherwise $r_{N/z_5}(G) \geq 4$; a contradiction as $r_{N/z_5}(R) \geq 4$. But this implies that $G-e$ is a triad in $N\backslash e/z_5$ and so, by Lemma 2.22, $e \in \mathrm{cl}_{N/z_5}(G-e)$; a contradiction.

Now assume that $z_6 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3, z_4, z_5\})$ and $z_7 \in \mathrm{cl}^*_{N\backslash e}(P_1 - z_7)$. If $z_6 \in \mathrm{cl}(Q_1 \cup Q_2)$, then, by interchanging the roles of z_6 and z_7 in the analysis of the previous paragraph, we deduce that z_5 does not expose any 3-separation of $M^*\backslash z_5$. Thus we may assume that $z_6 \notin \mathrm{cl}(Q_1 \cup Q_2)$. Furthermore, $z_6 \notin \mathrm{cl}(\{z_1, z_2, z_3, z_4\})$; otherwise N has a 5-element rank-3 set that avoids α and β, and so the lemma holds by Lemma 7.1.

The next assertion holds because $\sqcap(\{z_1, z_2, z_3\}, Q_1 \cup Q_2 \cup \{z_7, e\}) \leq 2$.

7.4.13. *There is an element a of $\{z_1, z_2, z_3\}$ such that $a \notin \mathrm{cl}_N(Q_1 \cup Q_2 \cup \{z_7, e\})$.*

For the element a just found, by Lemma 2.13, $N\backslash e/a$ is 3-connected. By Lemma 6.2(iii), there is a 3-separation (R, G) of N/a such that $r_{N/a}(R), r_{N/a}(G) \geq 3$, and R or G contains $\{\alpha, \beta\}$. Furthermore, we may assume that either $\{z_1, z_2, z_3, z_4\} - a \subseteq R$ or $\{z_1, z_2, z_3, z_4\} - a \subseteq G$.

7.4.14. *Neither $Q_1 \cup Q_2 \subseteq R$ nor $Q_1 \cup Q_2 \subseteq G$.*

Assume that $Q_1 \cup Q_2 \subseteq R$. By Lemma 2.21, $e \in \mathrm{cl}_{N/a}(R-e)$ or $e \in \mathrm{cl}_{N/a}(G-e)$. If $e \in \mathrm{cl}_{N/a}(G-e)$, then, as $G-e \subseteq P_1-a$, we have that $Q_1 \cup Q_2$ is 3-separating in N; a contradiction. Thus $e \in \mathrm{cl}_{N/a}(R-e)$. If $\{z_1, z_2, z_3, z_4\}-a \subseteq R$, then $\{z_5, z_6, z_7\} = G-e$, so $G-e$ is a triad in $N\backslash e/a$. But $e \in \mathrm{cl}_{N/a}(R-e)$. Thus G is a triad in N/a, and therefore a triad in N; a contradiction. So $\{z_1, z_2, z_3, z_4\} - a \subseteq G$. If $z_5 \in G$ or $z_6 \in G$, then $z_6 \in \mathrm{cl}_{N/a}(G)$ or $z_5 \in \mathrm{cl}_{N/a}(G)$, respectively, and so we may assume that $\{z_5, z_6\} \subseteq G$. In this instance, $R-e \subseteq Q_1 \cup Q_2 \cup z_7$ and so, by (7.4.13) and Lemma 2.2, $e \notin \mathrm{cl}_{N/a}(R-e)$. This contradiction implies that $z_5, z_6 \in R$. But then, as $z_6 \notin \mathrm{cl}(Q_1 \cup Q_2)$, we have $r_{N/a}(R) \geq 5$; a contradiction as $r_{N/a}(G) \geq 3$. Hence $Q_1 \cup Q_2 \not\subseteq R$ and so, by symmetry, (7.4.14) holds.

By Lemma 2.12 and (7.4.14), we may now assume that $Q_1 \subseteq R$ and $Q_2 \subseteq G$. Furthermore, we may also assume that $\{z_1, z_2, z_3, z_4\} - a \subseteq G$. Thus, by Lemma 2.10, $r_{N/a}(G) \geq 4$. If $z_5 \in G$ or $z_6 \in G$, then $z_6 \in \mathrm{cl}_{N/a}(G)$ or $z_5 \in \mathrm{cl}_{N/a}(G)$, respectively, and so we may assume that $\{z_5, z_6\} \subseteq G$. In this instance, $z_7 \in R$; otherwise $r_{N/a}(G) \geq 5$, contradicting the fact that $r_{N/a}(R) \geq 3$. Therefore $R - e \subseteq Q_1 \cup Q_2 \cup z_7$ and so, by (7.4.13), $e \notin \mathrm{cl}_{N/a}(R-e)$. By Lemma 2.21, this implies that $e \in \mathrm{cl}_{N/a}(G-e)$ which, in turn implies that $Q_1 \cup z_7$ is a triad in N/a and therefore a triad in N; a contradiction. Thus $z_5, z_6 \in R$. As $z_6 \notin \mathrm{cl}(Q_1 \cup Q_2)$ and $\{z_1, z_2, z_3, e\}$ is a cocircuit of N containing a, it follows that $z_5, z_6 \notin \mathrm{cl}_{N/a}(Q_1 \cup Q_2)$ and $z_5 \notin \mathrm{cl}_{N/a}(Q_1 \cup Q_2 \cup z_6)$. Thus $r_{N/a}(R) \geq 4$; a contradiction as $r_{N/a}(G) \geq 4$. This completes the subcase when $r(P_2) = 2$.

(iii)(b) $\boldsymbol{r(P_2) = 3}$. Let $(z_1, z_2, \ldots, z_k)$ be a sequential ordering of P_1. Since $r(P_2) = 3$, it follows that $|P_2| \geq 3$ and $r(P_1) = 4$. Therefore, by the set-up prior to (iii)(a), $k \in \{5, 6\}$, and $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e$; $z_4 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3\})$; and $z_5 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3, z_4\})$. Moreover, if $k = 6$, then $z_6 \in \mathrm{cl}_{N\backslash e}(P_1 - z_6)$. Now let $(y_1, y_2, \ldots, y_l)$ be a sequential ordering of P_2. By the set-up prior to (iii)(a), $l \in \{3, 4\}$, and $\{y_1, y_2, y_3\}$ is a triad in $N\backslash e$. Also $y_4 \in \mathrm{cl}_{N\backslash e}(\{y_1, y_2, y_3\})$ if $l = 4$. Without loss of generality, we may assume that P_1 is closed. Thus if y_4 exists and belongs to $\mathrm{cl}(P_1)$, then $k = 5$ and we relabel y_4 as z_6. Hence we may assume that $y_4 \notin \mathrm{cl}(P_1)$.

Noting that $\sqcap(\{\alpha, \beta\}, \{z_1, z_2, z_3, z_4\}) \in \{0, 1\}$, we partition (iii)(b) into cases depending on the value of $\sqcap(\{\alpha, \beta\}, \{z_1, z_2, z_3, z_4\})$. First assume that

$$\sqcap(\{\alpha, \beta\}, \{z_1, z_2, z_3, z_4\}) = 1.$$

Then $r(\{\alpha, \beta, z_1, z_2, z_3, z_4\}) = 4$. Consider N/z_5 and note that, by Lemma 2.11, $N\backslash e/z_5$ is 3-connected. Furthermore, observe that, as N has no triangles, $\{z_1, z_2, z_3, z_4\}$ contains no triangles in N/z_5 and, if y_4 exists, $\{y_1, y_2, y_3, y_4\}$ contains no triangles in N/z_5. By Lemma 6.2(iii), N/z_5 has a 3-separation (R, G), where $r_{N/z_5}(R), r_{N/z_5}(G) \geq 3$ and $\alpha, \beta \in R$.

7.4.15. $|\{y_1, y_2, y_3\} \cap R| \neq 3$.

If $|\{y_1, y_2, y_3\} \cap R| = 3$, then, by closure-equivalence, we may assume that $E(N) - \{e, z_1, z_2, z_3, z_5\} \subseteq R$. Thus $r_{N/z_5}(R) \geq 4$, so $\{z_1, z_2, z_3\} \subseteq G$, otherwise $r_{N/z_5}(R) \geq 5$; a contradiction. But then both $R - e$ and $G - e$ contain a triad in $N\backslash e/z_5$, contradicting Lemma 2.22. Hence (7.4.15) holds.

7.4.16. $|\{y_1, y_2, y_3\} \cap R| \neq 2$.

Suppose that $|\{y_1, y_2, y_3\} \cap R| = 2$. If $|\{z_1, z_2, z_3, z_4\} \cap R| \geq 3$, then, by closure-equivalence, we may assume that $P_1 - z_5 \subseteq R$, and so $r_{N/z_5}(R) \geq 5$; a contradiction. Thus $|\{z_1, z_2, z_3, z_4\} \cap G| \geq 2$. Therefore, as R and $\{\alpha, \beta\} \cup P_2$ are 3-separating sets in $N/z_5\backslash e$, it follows by uncrossing that $R \cap (\{\alpha, \beta\} \cup P_2)$ is a 3-separating set R' in $N/z_5\backslash e$. Let $G' = E(N) - (R' \cup z_5 \cup e)$. Then (R', G') is a 3-separation of $N/z_5\backslash e$. Since $r_{N/z_5}(R') \geq 3$ and $r_{N/z_5}(G') \geq 4$, it follows that $r_{N/z_5}(R') = 3$ and $r_{N/z_5}(G') = 4$.

If y_4 exists, then $y_4 \in G'$, otherwise $y_4 \in R'$ and $r_{N/z_5}(R') \geq 4$. But then, as $y_4 \notin \mathrm{cl}(P_1)$, we have that $y_4 \notin \mathrm{cl}_{N/z_5}(P_1)$ and so $r_{N/z_5}(G') \geq 5$; a contradiction. Thus we may assume that y_4 does not exist.

Since y_4 does not exist, $|R'| = 4$. Furthermore, as $z_5 \notin \mathrm{cl}_N(R')$, we have that R' is 3-separating in $N\backslash e$ and $\sqcap_N(R' \cap P_2, \{\alpha, \beta\}) = 1$. It now follows that $(\{\alpha, \beta\}, (G' \cup z_5) - e, R' - \{\alpha, \beta\})$ is a flower in $N\backslash e$ of the form analyzed in (iii)(a). Hence $|\{y_1, y_2, y_3\} \cap R| \neq 2$; that is, (7.4.16) holds.

7.4.17. $|\{y_1, y_2, y_3\} \cap R| \neq 1$.

Suppose that $|\{y_1, y_2, y_3\} \cap R| = 1$. If $|\{z_1, z_2, z_3, z_4\} \cap R| \geq 3$, then, by closure-equivalence, we may assume that $\{z_1, z_2, z_3, z_4\} \subseteq R$ and so $r_{N/z_5}(R) \geq 5$; a contradiction. So $|\{z_1, z_2, z_3, z_4\} \cap R| \leq 2$. If $|\{z_1, z_2, z_3, z_4\} \cap G| \geq 3$, then, by closure-equivalence, we may assume that $\{z_1, z_2, z_3, z_4\} \subseteq G$ and $z_6 \in G$ if z_6 exists. Assume that y_4 does not exist or if it exists, then $y_4 \in G$. If $R - e = \{\alpha, \beta\} \cup (\{y_1, y_2, y_3\} \cap R)$, then $R - e$ is a triad in $N\backslash e/z_5$. But $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e/z_5$ and $\{z_1, z_2, z_3\} \subseteq G$. This contradiction to Lemma 2.22 implies that y_4 exists and $y_4 \in R$. But $\{\alpha, \beta, y_4\}$ is not a triangle in N/z_5, and so $r_{N/z_5}(R) \geq 4$. Since $r_{N/z_5}(G) \geq 4$, we have another contradiction. Thus $|\{z_1, z_2, z_3, z_4\} \cap G| \leq 2$, so $|\{z_1, z_2, z_3, z_4\} \cap R| = 2 = |\{z_1, z_2, z_3, z_4\} \cap G|$, in which case, $r_{N/z_5}(R) = 4$ and $r_{N/z_5}(G) = 3$.

If $z_4 \in R$, then $\{\alpha, \beta, z_4\}$ is a triangle in N/z_5 and so $\{\alpha, \beta, z_4, z_5\}$ is a circuit in N. But this implies that $r_N(\{z_1, z_2, z_3, z_4, \alpha, \beta\}) = 5$; a contradiction as, by assumption, $r_N(\{z_1, z_2, z_3, z_4, \alpha, \beta\}) = 4$. Thus $z_4 \in G$. Since $r_{N/z_5}(G) = 3$, it follows that $(G \cap \{y_1, y_2, y_3\}) \cup z_4$ is a triangle in N/z_5. If

y_4 exists, then, as $\{\alpha, \beta, y_4\}$ is not a triangle in N/z_5, it follows that $y_4 \in G$, otherwise $y_4 \in R$ and $r_{N/z_5}(R) \geq 5$. But then $r_{N/z_5}(G) \geq 4$; a contradiction. So y_4 does not exist and, similarly, z_6 does not exist. It now follows that

$$(\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\}), (G \cap \{z_1, z_2, z_3, z_4\}) \cup \{y_1, y_2, y_3\})$$

is a 3-separation of $N\backslash e/z_5$. If $z_5 \in \mathrm{cl}_N(\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\}))$, then $\mathrm{cl}(\{\alpha, \beta\} \cup (P_1 - z_5))$ is a hyperplane in N. But $r_N(\{\alpha, \beta\} \cup (P_1 - z_5)) = 4$; a contradiction. Thus $z_5 \notin \mathrm{cl}_N(\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\}))$, so $r_N(\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\})) = 3$. In particular, $\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\})$ is 3-separating in $N\backslash e$. Since $\sqcap_N(\{\alpha, \beta\}, R \cap \{z_1, z_2, z_3\}) = 1$, it now follows that $(\{\alpha, \beta\}, R \cap \{z_1, z_2, z_3\}, (G \cap \{z_1, z_2, z_3, z_4\}) \cup \{y_1, y_2, y_3, z_5\})$ is a flower in $N\backslash e$ of the form analyzed in (iii)(a). Hence $|\{y_1, y_2, y_3\}| \neq 1$; that is, (7.4.17) holds.

7.4.18. $|\{y_1, y_2, y_3\} \cap R| \neq 0$.

Suppose that $|\{y_1, y_2, y_3\} \cap R| = 0$. If $\{z_1, z_2, z_3\} \subseteq G$, then $r_{N/z_5}(G) \geq 5$; a contradiction. Thus $|\{z_1, z_2, z_3\} \cap R| \geq 1$. If $\{z_1, z_2, z_3\} \subseteq R$, then each of $R - e$ and $G - e$ contain a triad in $N\backslash e/z_5$, contradicting Lemma 2.22. Therefore $|\{z_1, z_2, z_3\} \cap G| \geq 1$, and so $r_{N/z_5}(R) = 3$ and $r_{N/z_5}(G) = 4$.

If $z_4 \in R$, then, as $r_{N/z_5}(R) = 3$ and $\{z_1, z_2, z_3\}$ is a triad in N/z_5, we have $z_4 \in \mathrm{cl}_{N/z_5}(\{\alpha, \beta\})$. Since $\{\alpha, \beta, z_4\}$ is not a triangle in N, it follows that $\{\alpha, \beta, z_4, z_5\}$ is a circuit in N. But $r_N(\{\alpha, \beta, z_1, z_2, z_3, z_4\}) = 4$ and so $r_N(\{\alpha, \beta\} \cup P_1) = 4$; a contradiction. Thus $z_4 \notin R$, and so $z_4 \in G$. Since $r_{N/z_5}(G) = 4$, we have $z_4 \in \mathrm{cl}_{N/z_5}(\{y_1, y_2, y_3\})$.

Assume that z_6 exists. If $z_6 \in \mathrm{cl}_{N/z_5}(\{\alpha, \beta\})$, then $z_6 \in \mathrm{cl}_N(\{\alpha, \beta, z_5\})$. But $z_6 \notin \mathrm{cl}_N(\{\alpha, \beta\})$, so $z_5 \in \mathrm{cl}_N(\{\alpha, \beta, z_6\})$; a contradiction. Thus $z_6 \notin \mathrm{cl}_{N/z_5}(\{\alpha, \beta\})$. Therefore, if $z_6 \in R$, then $r_{N/z_5}(R) \geq 4$; a contradiction. So $z_6 \in G$. But then either $r_{N/z_5}(G) \geq 5$ or $r_{N/z_5}(\{z_4, z_6\}) = 2$; a contradiction. Therefore z_6 does not exist. On the other hand, if y_4 exists, then, as $\{y_1, y_2, y_3\} \subseteq G$, we may assume that $y_4 \in G$.

If $|\{z_1, z_2, z_3\} \cap G| = 2$, then $R - e$ is a triad in $N\backslash e/z_5$. But $\{y_1, y_2, y_3\}$ is also a triad in $N\backslash e/z_5$ and $\{y_1, y_2, y_3\} \subseteq G - e$, contradicting Lemma 2.22. Thus $|\{z_1, z_2, z_3\} \cap R| = 2$. Since $z_5 \notin \mathrm{cl}_N(\{\alpha, \beta\} \cup (P_1 - z_5))$, it follows that $r_N(\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\}) = 3$. Therefore, $\{\alpha, \beta\} \cup (R \cap \{z_1, z_2, z_3\})$ is 3-separating in $N\backslash e$. Since $\sqcap_N(\{\alpha, \beta\}, R \cap \{z_1, z_2, z_3\}) = 1$, it now follows that $(\{\alpha, \beta\}, R \cap \{z_1, z_2, z_3\}, G \cup z_5)$ is a flower in $N\backslash e$ of the form analyzed in (iii)(a). Hence (7.4.18) holds.

It follows from (7.4.15)–(7.4.18) that $\sqcap(\{\alpha, \beta\}, \{z_1, z_2, z_3, z_4\}) \neq 1$.

Now assume that $\sqcap(\{\alpha, \beta\}, \{z_1, z_2, z_3, z_4\}) = 0$. By Lemma 2.10(i), we have the following result.

7.4.19. *There is an element a of $\{z_1, z_2, z_3\}$ such that $a \notin \mathrm{cl}_N(\{\alpha, \beta, e\} \cup P_2)$.*

Consider N/a. By Lemma 2.13, $N\backslash e/a$ is 3-connected and so, by Lemma 6.2(iii), there is a 3-separation (R, G) of N/a such that $r_{N/a}(R), r_{N/a}(G) \geq 3$ and $\alpha, \beta \in R$. We may assume that either $\{z_1, z_2, z_3, z_4\} - a \subseteq R$ or $\{z_1, z_2, z_3, z_4\} - a \subseteq G$. Suppose that $\{z_1, z_2, z_3, z_4\} - a \subseteq R$. Then, as $\sqcap(\{\alpha, \beta\}, \{z_1, z_2, z_3, z_4\}) = 0$, we have $r_{N/a}(R) \geq 4$. Therefore $r_{N/a}(R) = 4$ and $r_{N/a}(G) = 3$. If $|\{y_1, y_2, y_3\} \cap R| \geq 1$, then $r_{N/a}(R) \geq 5$; a contradiction. Thus $\{y_1, y_2, y_3\} \subseteq G$. If $z_5 \in G$, then $r_{N/a}(G) \geq 4$; a contradiction. Therefore $z_5 \in R$, and so $G - e \subseteq \mathrm{cl}_{N/a}(P_2 \cup \{\alpha, \beta\})$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $e \in \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)$. But then $e \in \mathrm{cl}_N(\{\alpha, \beta, a\} \cup P_2)$, and so, as $e \notin \mathrm{cl}_N(\{\alpha, \beta\} \cup P_2)$, we have $a \in \mathrm{cl}_N(\{\alpha, \beta, e\} \cup P_2)$, contradicting (7.4.19). Thus $e \in \mathrm{cl}_{N/a}(R - e)$, and so $\{y_1, y_2, y_3\}$ is a triad in N/a and therefore a triad in N; a contradiction. Therefore $\{z_1, z_2, z_3, z_4\} - a \subseteq G$.

7.4.20. $|R \cap \{y_1, y_2, y_3\}| \neq 3$.

Suppose $|R \cap \{y_1, y_2, y_3\}| = 3$. Then $r_{N/a}(R) = 4$ and so $z_5 \in G$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(R - e)$, then $e \in \mathrm{cl}_{N/a}(\{\alpha, \beta\} \cup P_2)$, contradicting our choice of a. Thus $e \in \mathrm{cl}_{N/a}(G - e)$. But then $\{y_1, y_2, y_3\}$ is a triad in N/a and therefore a triad in N; a contradiction. Hence $|R \cap \{y_1, y_2, y_3\}| \neq 3$; that is, (7.4.20) holds.

7.4.21. $|R \cap \{y_1, y_2, y_3\}| \neq 2$.

Suppose $|R \cap \{y_1, y_2, y_3\}| = 2$. If y_4 exists, then $y_4 \in G$ otherwise, by closure-equivalence, we may assume that $\{y_1, y_2, y_3\} \subseteq R$; a contradiction. Therefore, as $P_2 \cup \{\alpha, \beta\}$ and $R - e$ are 3-separating sets in $N\backslash e/a$, it follows by uncrossing that $\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})$ is a 3-separating set in $N\backslash e/a$. Since the complement of $\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})$ has rank at least 4, it follows that $r_{N\backslash e/a}(\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})) = 3$, which in turn implies that $r_{N\backslash e}(\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})) = 3$. Thus $\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})$ is 3-separating in $N\backslash e$, and so $(\{\alpha, \beta\}, P_1 \cup G, R \cap \{y_1, y_2, y_3\})$ is a flower in $N\backslash e$. Moreover, $\sqcap_N(\{\alpha, \beta\}, R \cap \{y_1, y_2, y_3\}) = 1$, so it is a flower of the form analyzed in (iii)(a). Hence $|R \cap \{y_1, y_2, y_3\}| \neq 2$; that is, (7.4.21) holds.

7.4.22. $|R \cap \{y_1, y_2, y_3\}| \neq 0$.

Suppose $|R \cap \{y_1, y_2, y_3\}| = 0$. Then $\{y_1, y_2, y_3\} \subseteq G$, and so we may assume that $y_4 \in G$ if y_4 exists. Moreover, $r_{N/a}(G) = 4$ and $r_{N/a}(R) = 3$. If $z_5 \in G$, then $r_{N/a}(G) \geq 5$; a contradiction. So $z_5 \in R$. Also, if z_6 exists, then $z_6 \notin R$, otherwise $r_{N/a}(R) \geq 4$ as $z_6 \notin \mathrm{cl}_{N/a}(\{\alpha, \beta\})$. Thus if z_6 exists, then $z_6 \in G$. It now follows that $R-e$ is a triad in $N\backslash e/a$. But $\{y_1, y_2, y_3\}$ is also a triad in $N\backslash e/a$ and $\{y_1, y_2, y_3\} \subseteq G-e$, contradicting Lemma 2.22. Hence (7.4.22) holds.

It follows from (7.4.20)-(7.4.22) that we may assume $|R \cap \{y_1, y_2, y_3\}| = 1$. Suppose $\sqcap_{N/a}(G \cap \{y_1, y_2, y_3\}, P_1 - a) = 0$. Then $r_{N/a}(G) \geq 4$, and so $r_{N/a}(G) = 4$ and $r_{N/a}(R) = 3$. If $z_5 \in R$, then $r_{N/a}(R) \geq 4$; a contradiction. If $z_5 \in G$, then $r_{N/a}(G) \geq 5$; a contradiction. Therefore $\sqcap_{N/a}(G \cap \{y_1, y_2, y_3\}, P_1 - a) \geq 1$.

Assume that $z_5 \in G$. Then, by closure-equivalence, we may assume that if z_6 exists, it is in G. Suppose that either y_4 does not exist, or y_4 exists and $y_4 \in G$. Then $\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})$ is a triad in $N\backslash e/a$. Now, by Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R-e)$, or $e \in \mathrm{cl}_{N/a}(G-e)$. Our choice of a implies that $e \in \mathrm{cl}_{N/a}(G-e)$ and so $\{\alpha, \beta\} \cup (R \cap \{y_1, y_2, y_3\})$ is a triad in N/a and therefore a triad in N; a contradiction. Thus we may assume that y_4 exists and $y_4 \in R$. But then $r_{N/a}(R) \geq 4$; a contradiction as $r_{N/a}(G) \geq 4$. Hence $z_5 \in R$ and so $r_{N/a}(R) = 4$ and $r_{N/a}(G) = 3$.

7.4.23. *Neither z_6 nor y_4 exists.*

If z_6 exists, then $z_6 \in G$; otherwise $r_{N/a}(R) \geq 5$ as $\{\alpha, \beta, z_6\}$ is a not a triangle in N. But then $z_6 \in \mathrm{cl}_{N/a}(G \cap \{z_1, z_2, z_3, z_4\})$ as $r_{N/a}(G) = 3$. So $\{z_1, z_2, z_3, z_4, z_6\}$ is a 5-element rank-3 subset of $E(N)$ in N avoiding α and β, contradicting Lemma 7.1. Thus z_6 does not exist.

If y_4 exists, then $y_4 \in G$; otherwise $r_{N/a}(R) \geq 5$. But then, as $r_{N/a}(G) = 3$, it follows that $(G \cap \{y_1, y_2, y_3\}) \cup y_4$ is a triangle in N/a and so, by our choice of a, is a triangle in N; a contradiction. So y_4 does not exist, and (7.4.23) holds.

Since the element of $R \cap \{y_1, y_2, y_3\}$ is a coloop of R in $N\backslash e/a$, it follows that $\{\alpha, \beta, z_5\}$ is a triad in $N\backslash e/a$. Thus $\{\alpha, \beta, z_5\}$ is a triad in $N\backslash e$. Since N is 4-connected, this implies that

7.4.24. *$\{\alpha, \beta, z_5, e\}$ is a cocircuit in N.*

7.4.25. *In N, there is no 4-element rank-3 subset of $\{\alpha, \beta\} \cup \{y_1, y_2, y_3\}$ that includes α and β.*

Suppose that C is such a subset. Then C is a circuit in N and hence in $N\backslash e/a$. The element of $\{\alpha, \beta, y_1, y_2, y_3\} - C$ is a coloop of this set in $N\backslash e/a$. Thus C is 3-separating in $N\backslash e/a$. The choice of a implies that C is 3-separating in $N\backslash e$. Let $C - \{\alpha, \beta\} = \{c_1, c_2\}$ and $P_2 - \{c_1, c_2\} = d$. Suppose C is not a cocircuit in $N\backslash e$. Then $\mathrm{cl}_{N\backslash e}(P_1 \cup d) \cap C$ is non-empty. If $\alpha \in \mathrm{cl}_{N\backslash e}(P_1 \cup d)$, then $\beta \in \mathrm{cl}_{N\backslash e}(P_1 \cup d)$ and so, as $N\backslash e$ is 3-connected, it follows that $C \subseteq \mathrm{cl}_{N\backslash e}(P_1 \cup d)$; a contradiction. Thus, without loss of generality, we may assume that $c_2 \in \mathrm{cl}_{N\backslash e}(P_1 \cup d)$. Since $N\backslash e$ is 3-connected, $c_1 \notin \mathrm{cl}_{N\backslash e}(P_1 \cup d)$. It now follows that $(\alpha, \beta, c_1, c_2, d)$ is a sequential ordering of $P_2 \cup \{\alpha, \beta\}$ in $N\backslash e$. But then $P_2 \subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha, \beta\})$; a contradiction. Hence C is a cocircuit in $N\backslash e$ and hence in $N\backslash e/a$.

If $|C \cap (G \cap P_2)| = 1$, then $r_{N/a}(G) \geq 4$; a contradiction. So $C = \{\alpha, \beta\} \cup (G \cap P_2)$. But then, as $r_{N/a}(G) = 3$, it follows that $(G - e) \cup \{\alpha, \beta\}$ has rank 4 in $N\backslash e/a$ and so $((R \cap P_2) \cup z_5, (G - e) \cup \{\alpha, \beta\})$ is a 2-separation of the 3-connected matroid $N\backslash e/a$; a contradiction. Thus (7.4.25) holds.

It follows from (7.4.23) and (7.4.25) that $|E(N)| = 11$ and $r(\{\alpha, \beta\} \cup \{p, q\}) = 4$ for all distinct $p, q \in \{y_1, y_2, y_3\}$. Now consider N/z_5. We show next that N/z_5 is sequentially 4-connected. As $|E(N)| = 11$, it will follow by the dual of (I) of this lemma, which we have already proved, that $E(N) - \{\alpha, \beta\}$ contains an element x that does not expose any 3-separation in $M'\backslash x$ for some M' in $\{M, M^*\}$ thereby completing the proof of the lemma when (II) holds.

Assume that (U, V) is a non-sequential 3-separation of N/z_5. Then $|U|, |V| \geq 4$ and $r_{N/z_5}(U), r_{N/z_5}(V) \geq 3$. Without loss of generality, we may assume that $\alpha, \beta \in U$. Note that, by Lemma 2.11, $N\backslash e/z_5$ is 3-connected.

7.4.26. $|U \cap \{y_1, y_2, y_3\}| \neq 3$.

Suppose $|U \cap \{y_1, y_2, y_3\}| = 3$. Since $|V| \geq 4$, we may assume that $\{z_1, z_2, z_3, z_4\} \subseteq V$. But then each of $U - e$ and $V - e$ contains a triad in $N\backslash e/z_5$, contradicting Lemma 2.22. Thus (7.4.26) holds.

7.4.27. $|U \cap \{y_1, y_2, y_3\}| \neq 2$.

Suppose $|U \cap \{y_1, y_2, y_3\}| = 2$. Then, as $r_{N/z_5}(\{\alpha, \beta, y_1, y_2, y_3\}) = 4$ and $r_{N/z_5}(\{\alpha, \beta\} \cup \{p, q\}) = 4$ for all distinct $p, q \in \{y_1, y_2, y_3\}$, it follows that $(U \cup P_2, V - P_2)$ is a non-sequential 3-separation of N/z_5. But, by (7.4.26), there is no such 3-separation, and so (7.4.27) holds.

7.4.28. $|U \cap \{y_1, y_2, y_3\}| \neq 1$.

Suppose that $|U \cap \{y_1, y_2, y_3\}| = 1$. Let $\{f\} = U \cap \{y_1, y_2, y_3\}$ and $\{g, h\} = V \cap \{y_1, y_2, y_3\}$. Assume that $\sqcap_{N/z_5}(\{g, h\}, P_1 - z_5) = 0$. If $P_1 \subseteq U$, then $|V| \leq 3$; a contradiction. If $P_1 \subseteq V$, then, as $\sqcap_{N/z_5}(\{g, h\}, P_1 - z_5) = 0$, we have $r_{N/z_5}(V) \geq 5$ and so $r_{N/z_5}(U) \leq 2$; a contradiction. Thus we may assume that P_1 is not spanned by $P_1 \cap U$ or $P_1 \cap V$ in N/z_5 so $|P_1 \cap U| = 2 = |P_1 \cap V|$. Thus $r_{N/z_5}(U) \geq 4$. Moreover, as $\sqcap_{N/z_5}(\{g, h\}, P_1 - z_5) = 0$, we have $r_{N/z_5}(V) = 4$; a contradiction.

We may now assume that $\sqcap_{N/z_5}(\{g,h\}, P_1 - z_5) = 1$. Then $\{\alpha, \beta, f\}$ is a triad in $N\backslash e/z_5$. Since $\{\alpha, \beta, f\} \subseteq U$, it follows by Lemma 2.22 that $e \in \mathrm{cl}_{N/z_5}(U - e)$. Now $P_1 \not\subseteq U$, otherwise $|V| \leq 3$. If $P_1 \subseteq V$, then $V - e$ and $U - e$ both contain a triad in $N\backslash e/z_5$, contradicting Lemma 2.22. Thus, as in the last paragraph, we may assume that $|P_1 \cap U| = 2 = |P_1 \cap V|$. Then $r_{N/z_5}(U) \geq 4$ and so $r_{N/z_5}(U) = 4$ and $r_{N/z_5}(V) = 3$.

Consider z_4. If $z_4 \in V$, then, as $r_{N/z_5}(V) = 3$, we have that $\{g, h, z_4\}$ is a triangle in N/z_5. Since $e \in \mathrm{cl}_{N/z_5}(U - e)$, it now follows that V is sequential in N/z_5; a contradiction. Therefore $z_4 \in U$ and $\{\alpha, \beta, z_4\}$ is a triangle in N/z_5 as $r_{N/z_5}(U) = 4$. Since $(U \cup e, V - e)$ is a 3-separation in N/z_5 and N is 4-connected, $(U \cup \{e, z_5\}, V - e)$ is not a 3-separation in N. This implies that $z_5 \in \mathrm{cl}_N(V - e)$. But then N has a circuit D containing z_5 such that $D - z_5 \subseteq V - e$. But, by (7.4.24), $\{\alpha, \beta, z_5, e\}$ is a cocircuit in N. This contradiction to orthogonality implies that (7.4.28) holds.

7.4.29. $|U \cap \{y_1, y_2, y_3\}| \neq 0$.

Suppose that $|U \cap \{y_1, y_2, y_3\}| = 0$. If $\{z_1, z_2, z_3\} \subseteq V$, then we may assume that $z_4 \in V$, so $|U| \leq 3$; a contradiction. Therefore $|\{z_1, z_2, z_3\} \cap U| \geq 1$. If $\{z_1, z_2, z_3\} \subseteq U$, then both $U - e$ and $V - e$ contain a triad in $N\backslash e/z_5$, contradicting Lemma 2.22. Therefore $|\{z_1, z_2, z_3\} \cap V| \geq 1$ and so $r_{N/z_5}(U) = 3$ and $r_{N/z_5}(V) = 4$.

Suppose $z_4 \in V$. Then $z_4 \in \mathrm{cl}_{N/z_5}(\{y_1, y_2, y_3\})$; otherwise $r_{N/z_5}(V) \geq 5$. Therefore $z_4 \in \mathrm{cl}_N(\{y_1, y_2, y_3, z_5\})$. If $z_4 \notin \mathrm{cl}_N(\{y_1, y_2, y_3\})$, then $z_5 \in \mathrm{cl}_N(\{y_1, y_2, y_3, z_4\})$ and so N has a circuit consisting of z_5 and a subset of $\{y_1, y_2, y_3, z_4\}$. But, by (7.4.24), $\{\alpha, \beta, z_5, e\}$ is a cocircuit of N, contradicting orthogonality. Thus $z_4 \in \mathrm{cl}_N(\{y_1, y_2, y_3\})$ and so $r^*_{N\backslash e}(\{z_1, z_2, z_3, z_5\}) = 2$. Hence $r_{N^*}(\{z_1, z_2, z_3, z_5, e\}) = 3$. By Lemma 7.1, there is an element x in this subset such that x does not expose any 3-separation of $M^*\backslash x$; a contradiction. Thus we may assume that $z_4 \in U$, in which case, as $r_{N/z_5}(U) = 3$, we have that $\{\alpha, \beta, z_4\}$ is a triangle in N/z_5. Furthermore, $|U \cap \{z_1, z_2, z_3\}| = 1$; otherwise N contains a triangle as $r_{N/z_5}(U) = 3$.

Since $\{y_1, y_2, y_3\}$ is a triad of $N\backslash e/z_5$ and $\{y_1, y_2, y_3\} \subseteq V$, it follows by Lemma 2.22 that $e \in \mathrm{cl}_{N/z_5}(V - e)$ and $e \notin \mathrm{cl}_{N/z_5}(U - e)$. Thus $e \in V$ and (α, β, z_4, u) is a sequential ordering of U, where $\{u\} = U \cap \{z_1, z_2, z_3\}$; a contradiction. Hence (7.4.29) holds.

It now follows by (7.4.26)–(7.4.29) that there is no non-sequential 3-separation (U, V) of N/z_5, thereby completing analysis of (II).

(III) $r(N) = 7$ and $|E(N)| = 12$.

It follows from (I) that we may assume that N/f is not sequentially 4-connected for all $f \in E(N) - \{\alpha, \beta\}$.

(i) Φ is a paddle. Since Φ is a paddle,

$$\begin{aligned}7 = r(N\backslash e) &= r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 4\\ &\leq 2 + |P_1| + |P_2| - 4\\ &= 11 - 4 = 7.\end{aligned}$$

Therefore P_1 and P_2 are independent sets and so, as each is 3-separating in $N\backslash e$, we have $r^*_{N\backslash e}(P_1) = 2 = r^*_{N\backslash e}(P_2)$. Without loss of generality, we may assume that $|P_2| \geq |P_1|$. In particular, $|P_2| \in \{5, 6\}$ and so, by Lemma 7.1, there is an element y in $P_2 \cup e$ such that y does not expose any 3-separation in $M^*\backslash y$; a contradiction. Hence Φ is not a paddle.

(ii) Φ is a copaddle. Since neither P_1 nor P_2 is a subset of $\mathrm{fcl}(\{\alpha, \beta\})$, we have $|P_1|, |P_2| \geq 3$. Also, as N has no triangles, $r(P_1), r(P_2) \geq 3$. Thus, as

$$7 = r(N\backslash e) = r(\{\alpha, \beta\}) + r(P_1) + r(P_2) - 2,$$

we have $r(P_1)+r(P_2) = 7$, and so, without loss of generality, we may assume that $r(P_1) = 4$ and $r(P_2) = 3$. If $|P_2| = 3$, then $r^*_{N\backslash e}(P_2 \cup \{\alpha, \beta\}) = 2$, and so $P_2 \subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha, \beta\})$; a contradiction. Thus $|P_2| \geq 4$. If $|P_2| = 5$, then, by Lemma 7.1, there is an element $x \in P_2$ such that x does not expose any 3-separation in $M\backslash x$. Therefore $|P_2| = 4$, so $|P_1| = 5$. We partition (ii) into two subcases depending on whether or not P_1 contains a 4-element circuit.

First suppose that P_1 contains such a 4-circuit Q, and let z be the element in $P_1 - Q$. Since $r(Q) = 3$ and $r(P_1) = 4$, it follows that $z \notin \mathrm{cl}_{N\backslash e}(Q)$ and so, by Lemma 2.1, $z \in \mathrm{cl}^*_{N\backslash e}(P_2 \cup \{\alpha, \beta\})$. Therefore Q is 3-separating in $N\backslash e$. Moreover, as Φ is a copaddle, $z \in \mathrm{cl}^*_{N\backslash e}(\{\alpha, \beta\})$. Thus

7.4.30. *$\{\alpha, \beta, z\}$ is a triad in $N\backslash e$. In particular, $(\{\alpha, \beta, z\}, Q, P_2)$ and $(\{\alpha, \beta\}, Q, P_2 \cup z)$ are copaddles in $N\backslash e$.*

Next we show the following.

7.4.31. *There is an element $a \in Q - \mathrm{cl}_N(\{\alpha, \beta, z\} \cup P_2)$ such that $a \notin \mathrm{cl}_N(P_2 \cup \{z, e\})$ and $a \notin \mathrm{cl}_N(\{\alpha, \beta\} \cup \{z, e\})$.*

Since $N\backslash e$ has $(\{\alpha, \beta, z\}, Q, P_2)$ as a copaddle, $\sqcap(\{\alpha, \beta, z\}, Q) = 0$. Thus $\sqcap(\{\alpha, \beta, z, e\}, Q) \leq 1$. Similarly, as $(\{\alpha, \beta\}, Q, P_2 \cup z)$ is a copaddle, $\sqcap(P_2 \cup \{z, e\}), Q) \leq 1$. Moreover, $\mathrm{cl}_N(\{\alpha, \beta, z\} \cup P_2)$ contains at most one element of Q. Hence the desired element a exists.

Consider N/a. As a is in a cocircuit of $N\backslash e$ contained in Q, it follows by Lemma 2.13 that $N\backslash e/a$ is 3-connected. Furthermore,

$$\sqcap_{N/a}(\{\alpha, \beta, z\}, Q - a) = \sqcap_{N/a}(Q - a, P_2) = \sqcap_{N/a}(P_2, \{\alpha, \beta, z\}) = 0.$$

Since N/a is not sequentially 4-connected, it has a non-sequential 3-separation (R, G). By Lemma 2.12, we may assume that $\alpha, \beta \in R$. As $Q - a$ is a triangle of N/a, we may also assume that $Q - a \subseteq R$ or $Q - a \subseteq G$.

7.4.32. *$Q - a \not\subseteq R$.*

Suppose that $Q - a \subseteq R$. Then $G - e \subseteq P_2 \cup z$. Since $N \backslash e/a$ is 3-connected, it follows by Lemma 2.21 that either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $e \in \mathrm{cl}_N((G - e) \cup a)$. But $e \notin \mathrm{cl}_N(G - e)$; otherwise Q is 3-separating in N. Therefore $a \in \mathrm{cl}_N(G \cup e)$, so $a \in \mathrm{cl}_N(P_2 \cup \{e, z\})$, contradicting (7.4.31). Thus $e \in \mathrm{cl}_{N/a}(R - e)$. Since (R, G) is non-sequential, it now follows that $|G \cap P_2| \geq 3$ and so, as P_2 contains no triangles in N/a, we may assume that $P_2 \subseteq G$. But then, as $e \in \mathrm{cl}_{N/a}(R - e)$, we have $e \in \mathrm{cl}_N(P_1 \cup \{\alpha, \beta\})$, so P_2 is 3-separating in N; a contradiction. Thus (7.4.32) holds.

7.4.33. *$Q - a \not\subseteq G$.*

Suppose that $Q - a \subseteq G$. Suppose also that $|P_2 \cap R| \geq 3$. Then, by closure-equivalence, we may assume that $P_2 \subseteq R$. If $z \in R$, then, as $|G| \geq 4$, it follows that $G = (Q - a) \cup e$, in which case, G is sequential as $Q - a$ is a triangle in N/a; a contradiction. Therefore $z \in G$. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $e \in \mathrm{cl}_N(P_1)$ and so P_2 is 3-separating in N; a contradiction. Therefore, by Lemma 2.21, $e \in \mathrm{cl}_{N/a}(R - e)$ and $e \in G$. Thus $G = (P_1 - a) \cup e$ and, as $Q_1 - a$ is a triangle of N/a, and $P_1 - a$ is 3-separating in N/a, it follows that G is sequential; a contradiction. Thus $|P_2 \cap G| \geq 2$.

Suppose $|P_2 \cap G| \geq 3$. Then, by closure-equivalence, we may assume that $P_2 \subseteq G$, and so $z \in R$ as $|R| \geq 4$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(R - e)$, then $e \in \mathrm{cl}_N(\{\alpha, \beta\} \cup P_1)$, and so P_2 is 3-separating in N; a contradiction. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $\{\alpha, \beta, z\}$ is a triad in N/a and therefore in N; a contradiction. Thus we may assume that $|P_2 \cap R| = 2 = |P_2 \cap G|$, which implies that $r_{N/a}(R), r_{N/a}(G) \geq 4$. Since $r(N/a) = 6$, we deduce that $r_{N/a}(R) = 4 = r_{N/a}(G)$. By (7.4.30), $\{\alpha, \beta, z\}$ is a triad in $N \backslash e$, so it is a triad in $N \backslash e/a$. Therefore $r_{N/a}(G) \geq 5$ if $z \in G$; a contradiction. Thus $z \in R$. But then, as $\sqcap_{N/a}(P_2, \{\alpha, \beta, z\}) = 0$, we have $r_{N/a}(R) = 5$; a contradiction. Thus (7.4.33) holds.

It now follows that we may suppose P_1 contains no 4-element rank-3 subset. In particular, every 4-element subset of P_1 is independent. Since $r(\{\alpha, \beta\} \cup P_2) = 5$ and $e \notin \mathrm{cl}(\{\alpha, \beta\} \cup P_2)$, the set $\mathrm{cl}(\{\alpha, \beta, e\} \cup P_2)$ has rank 6, and so its complement is a cocircuit C^* of N contained in P_1. Since N has no triangles, it follows by Lemma 2.8 that C^* contains an element $a \in P_1$ such that $a \notin \mathrm{cl}(\{\alpha, \beta, e\} \cup P_2)$ and $N \backslash e/a$ is 3-connected.

Consider N/a. Since P_1 contains no 4-element rank-3 subset, $P_1 - a$ is a circuit in N/a. Moreover,

$$\sqcap_{N/a}(\{\alpha,\beta\}, P_1 - a) = \sqcap_{N/a}(P_1 - a, P_2) = \sqcap_{N/a}(P_2, \{\alpha,\beta\}) = 0.$$

Since N/a is not sequentially 4-connected, it has a non-sequential 3-separation (R, G). By Lemma 2.12, we may assume that $\alpha, \beta \in R$.

7.4.34. *$P_1 - a \not\subseteq R$.*

Suppose $P_1 - a \subseteq R$. Then, as (R, G) is non-sequential, $|P_2 \cap G| \geq 3$ and so we may assume that $P_2 \subseteq G$ as P_2 contains no triangles in N/a. Since $N\backslash e/a$ is 3-connected, it follows by Lemma 2.21 that either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $e \in \mathrm{cl}_{N/a}(P_2)$, and so $e \in \mathrm{cl}_N(P_2 \cup a)$. Since $e \notin \mathrm{cl}_N(P_2)$, it follows that $a \in \mathrm{cl}_N(P_2 \cup e)$, contradicting our choice of a. Thus $e \in \mathrm{cl}_{N/a}(R - e)$. But then $e \in \mathrm{cl}_N(\{\alpha,\beta\} \cup P_1)$ and so P_2 is 3-separating in N; a contradiction. Hence (7.4.34) holds.

7.4.35. *$P_1 - a \not\subseteq G$.*

Suppose $P_1 - a \subseteq G$ Then, as (R, G) is non-sequential, $|P_2 \cap R| \geq 1$. Assume that $P_2 \subseteq R$. Then $r_{N/a}(R) \geq 5$, and so $r_{N/a}(R) = 5$ and $r_{N/a}(G) = 3$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(R - e)$, then $e \in \mathrm{cl}_N(\{\alpha,\beta,a\} \cup P_2)$. Since $e \notin \mathrm{cl}_N(\{\alpha,\beta\} \cup P_2)$, it follows that $a \in \mathrm{cl}_N(\{\alpha,\beta,e\} \cup P_2)$, contradicting our choice of a. Thus $e \in \mathrm{cl}_{N/a}(G - e)$ and so $e \in \mathrm{cl}_N(P_1)$. But then $\{\alpha,\beta\} \cup P_2$ is 3-separating in N; a contradiction. It now follows that $|P_2 \cap G| \geq 1$, and so, as P_2 contains no triangles in N/a, we may assume that $|P_2 \cap R| = 2 = |P_2 \cap G|$. Since $\sqcap_{N/a}(P_2, \{\alpha,\beta\}) = 0 = \sqcap_{N/a}(P_1 - a, P_2)$, this implies that $r_{N/a}(R) \geq 4$ and $r_{N/a}(G) \geq 5$; a contradiction as $r(N/a) = 6$. Hence (7.4.35) holds.

It follows from (7.4.34) and (7.4.35) that we may assume $|(P_1 - a) \cap R| = 2 = |(P_1 - a) \cap G|$. If $P_2 \subseteq R$, then $|G| \leq 3$; a contradiction. So $|P_2 \cap G| \geq 1$. If $P_2 \subseteq G$, then $r_{N/a}(G) \geq 5$ and $r_{N/a}(R) \geq 4$; a contradiction. Therefore $|P_2 \cap R| \geq 1$, and so we may assume that $|P_2 \cap R| = 2 = |P_2 \cap G|$. Now $|\mathrm{cl}_{N/a}(P_2 \cup \{\alpha,\beta\}) \cap (P_1 - a)| \leq 1$, otherwise, as $N\backslash e/a$ is 3-connected, $E(N) - \{e, a\} \subseteq \mathrm{cl}_{N/a}(P_2 \cup \{\alpha,\beta\})$ contradicting the fact that $r_{N/a}(P_2 \cup \{\alpha,\beta\}) = 5$. Therefore $r_{N/a}(R) \geq 5$; a contradiction as $r_{N/a}(G) \geq 4$. We conclude that Φ is not a copaddle.

(iii) $\boldsymbol{\sqcap(\{\alpha,\beta\}, P_1) = \sqcap(P_1, P_2) = \sqcap(P_2, \{\alpha,\beta\}) = 1}$. Since $P_1 \not\subseteq \mathrm{fcl}(\{\alpha,\beta\})$ and $P_2 \not\subseteq \mathrm{fcl}(\{\alpha,\beta\})$, it follows by Lemma 3.1 that P_1 and P_2 are both sequential.

Now $7 = r(N\backslash e) = r(\{\alpha,\beta\}) + r(P_1) + r(P_2) - 3$. Therefore $r(P_1) + r(P_2) = 8$. Furthermore, as $|E(N)| = 12$, we have $|P_1| + |P_2| = 9$. It now follows that either P_1 is independent or P_2 is independent. Without loss of generality, we may assume that P_2 is independent. As P_2 is 3-separating in $N\backslash e$, it follows that $r^*_{N\backslash e}(P_2) = 2$. Therefore, if $|P_2| \geq 4$, then, by Lemma 7.1, there is an element $y \in P_2 \cup e$ such that y does not expose any 3-separation in $M^*\backslash y$. Thus we may assume that $|P_2| \in \{2, 3\}$.

Before partitioning (iii) into two subcases depending on the size of P_2, consider P_1. Now $|P_1| \in \{6, 7\}$. Let $(z_1, z_2, \ldots, z_k)$ be a sequential ordering of P_1. Since P_1 contains no triangles of $N\backslash e$, it follows that $\{z_1, z_2, z_3\}$ is a triad of $N\backslash e$. If $z_4 \in \mathrm{cl}^*_{N\backslash e}(\{z_1, z_2, z_3\})$, then $\{z_1, z_2, z_3, z_4\}$ is a 4-point cosegment in $N\backslash e$ avoiding α and β, and so the lemma holds by Lemma 7.1. Thus $z_4 \in \mathrm{cl}_{N\backslash e}(\{z_1, z_2, z_3\})$. Since $r_{N\backslash e}(P_1) = |P_1| - 1$, it now follows that $z_j \in \mathrm{cl}^*_{N\backslash e}(\{z_1, \ldots, z_{j-1}\})$ for all $j \geq 5$.

(iii)(a) $|P_2| = 2$. In this subcase, $|P_1| = 7$ and so $r(P_1) = 6$. Since $P_2 \not\subseteq \mathrm{fcl}_{N\backslash e}(\{\alpha, \beta\})$, it follows that both $\mathrm{cl}_{N\backslash e}(P_1) \cap \mathrm{cl}_{N\backslash e}(P_2)$ and $\mathrm{cl}_{N\backslash e}(\{\alpha, \beta\}) \cap \mathrm{cl}_{N\backslash e}(P_2)$ are empty. To maintain symmetry, let $\{Q_1, Q_2\} = \{\{\alpha, \beta\}, P_2\}$. Now, by Lemma 2.10, $r(\{z_1, z_2, z_3, z_4\} \cup Q_1 \cup Q_2) = 6$ and so $\{z_5, z_6, z_7\}$ is a triad in $N\backslash e$.

7.4.36. *Let $k \in \{1, 2\}$. If $Q_k \cup z_i$ is a triad in $N\backslash e$ for some $i \in \{5, 6, 7\}$, then $Q_k \cup z_j$ is not a triad in $N\backslash e$ for each $j \in \{5, 6, 7\} - i$.*

Suppose that $Q_k \cup z_i$ and $Q_k \cup z_j$ are triads in $N\backslash e$, where $i, j \in \{5, 6, 7\}$ and $i \neq j$. Then $Q_k \cup \{z_5, z_6, z_7\}$ is a cosegment in $N\backslash e$. In particular, there is a 4-element cosegment in $N\backslash e$ that avoids at least one element in $\{\alpha, \beta\}$. Therefore, by Lemma 7.1, there is an element y in $(Q_k \cup \{z_5, z_6, z_7, e\}) - \{\alpha, \beta\}$ such that y does not expose any 3-separation in $M^*\backslash y$; a contradiction. Hence (7.4.36) holds.

7.4.37. *There is an element a of $\{z_1, z_2, z_3\}$ such that $a \notin \mathrm{cl}_N(\{z_5, z_6, z_7, e\})$ and $a \notin \mathrm{cl}_N(P_2 \cup \{\alpha, \beta, z_i, e\})$, where i is chosen in $\{5, 6, 7\}$ so that, if possible, $Q_1 \cup z_i$ or $Q_2 \cup z_i$ is a triad in $N\backslash e$.*

As $\sqcap(\{z_1, z_2, z_3\}, \{z_5, z_6, z_7\}) = 0$, we have $\sqcap(\{z_1, z_2, z_3\}, \{z_5, z_6, z_7, e\}) \leq 1$. Furthermore, by [5, Lemma 5.8] and Lemma 2.10, $\sqcap(\{z_1, z_2, z_3\}, Q_1 \cup Q_2 \cup z_i) = 0$, so $\sqcap(\{z_1, z_2, z_3\}, Q_1 \cup Q_2 \cup \{z_i, e\}) \leq 1$. Thus there is such an element a in $\{z_1, z_2, z_3\}$ satisfying (7.4.37).

Consider N/a and note that, by Lemma 2.13, $N\backslash e/a$ is 3-connected. Since N/a is not sequentially 4-connected, it has a non-sequential 3-separation (R, G). By closure-equivalence, either $\{z_1, z_2, z_3, z_4\} - a \subseteq R$, or $\{z_1, z_2, z_3, z_4\} - a \subseteq G$.

7.4.38. *Neither $Q_1 \cup Q_2 \subseteq R$ nor $Q_1 \cup Q_2 \subseteq G$.*

Suppose that $Q_1 \cup Q_2 \subseteq R$. Suppose also that $\{z_1, z_2, z_3, z_4\} - a \subseteq R$. Then $G = \{z_5, z_6, z_7, e\}$ as $|G| \geq 4$. Now $a \notin \mathrm{cl}_N(\{z_5, z_6, z_7, e\})$ and $e \notin \mathrm{cl}_N(\{z_5, z_6, z_7\})$, so, by Lemma 2.2, $e \notin \mathrm{cl}_{N/a}(G - e)$. Therefore, by Lemma 2.21, $e \in \mathrm{cl}_{N/a}(R - e)$. But then $\{z_5, z_6, z_7\}$ is a triad in N; a contradiction as N is 4-connected.

Now assume that $\{z_1, z_2, z_3, z_4\} - a \subseteq G$. Then $G - e \subseteq P_1 - a$. By Lemma 2.21, either $e \in \mathrm{cl}_{N/a}(R - e)$ or $e \in \mathrm{cl}_{N/a}(G - e)$. If $e \in \mathrm{cl}_{N/a}(G - e)$, then $e \in \mathrm{cl}_N(P_1)$, and so $(P_1, \{\alpha, \beta\} \cup P_2)$ is a 3-separation in N; a contradiction. Therefore $e \in \mathrm{cl}_{N/a}(R - e)$. Since $(z_1, z_2, z_3, z_4, z_5, z_6, z_7)$ is a sequential ordering of P_1 in $N \backslash e$, it is easily checked that

$$(\{z_1, z_2, z_3, z_4\} - a, z_5, z_6, z_7, \{\alpha, \beta\} \cup P_2)$$

is a 3-sequence in $N \backslash e/a$. Therefore, by [5, Lemma 5.8], $(\{z_1, z_2, z_3, z_4\} - a, z_i, z_j, z_k, \{\alpha, \beta\} \cup P_2)$ is a 3-sequence in $N \backslash e/a$, where $\{z_i, z_j, z_k\} = \{z_5, z_6, z_7\}$. As $e \in \mathrm{cl}_{N/a}(R - e)$, it now follows that $G - e$, and therefore G, is sequential in N/a; a contradiction as (R, G) is a non-sequential 3-separation in N/a. Thus $Q_1 \cup Q_2 \not\subseteq R$ and so, by symmetry, (7.4.38) holds.

By Lemma 2.12, (7.4.38), and closure-equivalence, we may assume that $Q_1 \subseteq R$ and $Q_2 \subseteq G$. We may also assume that $\{z_1, z_2, z_3, z_4\} - a \subseteq R$. Then, by Lemma 2.10, $r_{N/a}(R - e) \geq 4$ and, as $|G| \geq 4$, we have $|\{z_5, z_6, z_7\} \cap G| \geq 1$. Now $|\{z_5, z_6, z_7\} \cap G| \neq 3$, otherwise $r_{N/a}(G - e) \geq 5$; a contradiction as $r(N/a) = 6$. If $|\{z_5, z_6, z_7\} \cap G| = 2$, then $r_{N/a}(G - e) \geq 4$ and, by Lemma 2.10, $r_{N/a}(R - e) \geq 5$; a contradiction. Hence $|\{z_5, z_6, z_7\} \cap G| = 1$ and $|\{z_5, z_6, z_7\} \cap R| = 2$. Thus $|G - e| = 3$ and $r_{N/a}(R) \geq 5$, so $G - e$ is a triad in $N \backslash e/a$ and hence in N/e. By Lemma 2.22, this implies $e \in \mathrm{cl}_{N/a}(G - e)$.

Now $G - e = Q_2 \cup z_j$ for some j in $\{5, 6, 7\}$. Suppose first that $j = i$ in the selection of a in (7.4.37). Then $a \notin \mathrm{cl}_N(P_2 \cup \{\alpha, \beta, z_i, e\})$. As $G \subseteq \{\alpha, \beta, z_i, e\}$, it follows that $3 = r_{N/a}(G) = r_N(G)$, so $e \in \mathrm{cl}_N(Q_2 \cup z_i)$. Thus $\{z_1, z_2, z_3\}$, which is 3-separating in $N \backslash e$, is also 3-separating in N; a contradiction. We may now assume that $j \neq i$. Then, by (7.4.36), $Q_1 \cup z_i$ is a triad of $N \backslash e$. Thus $Q_1 \cup z_i$ is a triad of $N \backslash e/a$ contained in $R - e$, contradicting Lemma 2.22. This completes the subcase that $|P_2| = 2$.

(iii)(b) $|P_2| = 3$. Then $|P_1| = 6$, so $r(P_1) = 5$, and P_2 is a triad in $N \backslash e$.

7.4.39. *We may assume that there is no triad T in $N \backslash e$ such that $T \subseteq P_2 \cup \{z_5, z_6\}$ and $|T \cap \{z_5, z_6\}| \geq 1$.*

Suppose there is such a triad T. If $|T \cap P_2| = 2$, then, as P_2 is a triad in $N \backslash e$, it follows that $P_2 \cup T$ is a 4-element cosegment in $N \backslash e$ that avoids α and β. Therefore, by Lemma 7.1, there is an element in $P_2 \cup T \cup e$ that does not expose any 3-separations in N^*; a contradiction. Thus $\{z_5, z_6\} \subseteq T$. Let $\{y\} = T \cap P_2$. Since $y \in \mathrm{cl}^*_{N \backslash e}(P_1)$, it follows by Lemma 2.9 that $P_1 \cup y$ and

$\{\alpha,\beta\}\cup P_1\cup y$ are 3-separating in $N\backslash e$. Therefore $(\{\alpha,\beta\}, P_1\cup y, P_2-y)$ is a flower in $N\backslash e$. Furthermore, as $\sqcap(\{\alpha,\beta\}, P_1)=1$ and $y\notin \mathrm{cl}(\{\alpha,\beta\}\cup P_1)$, it follows that $\sqcap(\{\alpha,\beta\}, P_1\cup y)=1$. Hence $(\{\alpha,\beta\}, P_1\cup y, P_2-y)$ is a flower in $N\backslash e$ of the form analyzed in the previous subcase. Thus (7.4.39) holds.

By Lemma 2.10, $\sqcap(\{z_1,z_2,z_3\}, P_2\cup\{\alpha,\beta\})=0$, so $\sqcap(\{z_1,z_2,z_3\}, P_2\cup\{\alpha,\beta,e\})\le 1$. From this, we deduce the following.

7.4.40. *There is an element a of $\{z_1,z_2,z_3\}$ such that $a\notin \mathrm{cl}_N(P_2\cup\{\alpha,\beta,e\})$.*

Consider N/a. By Lemma 2.13, $N\backslash e/a$ is 3-connected. Also, as $(z_1,z_2,z_3,z_4,z_5,z_6)$ is a sequential ordering of P_1 in $N\backslash e$, it follows, for each (i,j) in $\{(5,6),(6,5)\}$, that $(\{z_1,z_2,z_3,z_4\}-a, z_i, z_j, \{\alpha,\beta\}\cup P_2)$ is a 3-sequence in $N\backslash e/a$, where $\{z_i,z_j\}=\{z_5,z_6\}$; $z_i\in \mathrm{cl}^*_{N\backslash e/a}(\{z_1,z_2,z_3,z_4\}-a)$; and $z_j\in \mathrm{cl}^*_{N\backslash e/a}((\{z_1,z_2,z_3,z_4\}-a)\cup z_i)$. Now, as N/a is not sequentially 4-connected, it has a non-sequential 3-separation (R,G). By Lemma 2.12, we may assume that $\{\alpha,\beta\}\subseteq R$. Furthermore, by closure-equivalence, we may assume that either $\{z_1,z_2,z_3,z_4\}-a\subseteq R$ or $\{z_1,z_2,z_3,z_4\}-a\subseteq G$.

First assume that $\{z_1,z_2,z_3,z_4\}-a\subseteq R$. Then, by Lemma 2.10, $r_{N/a}(R-e)\ge 4$. If $|P_2\cap R|\ge 1$, then $r_{N/a}(R-e)\ge 5$ as P_2 is a triad in $N\backslash e/a$. This implies that $r_{N/a}(R)=5$ and $r_{N/a}(G)=3$, and so $|G\cap(P_2\cup\{z_5,z_6\})|=3$. But then $G-e$ is a triad in $N\backslash e/a$ and therefore a triad in $N\backslash e$, contradicting (7.4.39). Thus $|P_2\cap R|=0$ and so $P_2\subseteq G$. Since $r_{N/a}(P_2\cup\{z_5,z_6\})=5$ and $r_{N/a}(R)\ge 4$, it follows that $|G\cap\{z_5,z_6\}|\le 1$. If $|G\cap\{z_5,z_6\}|=0$, then $G=P_2\cup e$, so $G-e$ is a triad in $N\backslash e/a$. Therefore, by Lemma 2.22, $e\in \mathrm{cl}_{N/a}(G-e)$, contradicting the choice of a in (7.4.40). It now follows that $|G\cap\{z_5,z_6\}|=1$, and so $r_{N/a}(G-e)=4$ and $r_{N/a}(R-e)=4$. Thus $G-e$ is a 4-element cosegment in $N\backslash e/a$, and therefore also in $N\backslash e$. As $G-e$ avoids α and β, Lemma 7.1 implies that there is an element y in $G\cup e$ that does not expose any 3-separations in $M^*\backslash y$; a contradiction.

We may now assume that $\{z_1,z_2,z_3,z_4\}-a\subseteq G$.

7.4.41. $P_2\not\subseteq R$.

Suppose $P_2\subseteq R$. Then $G-e\subseteq P_1-a$. As P_2 is a triad of N/a, by Lemma 2.22, $e\in \mathrm{cl}_{N/a}(R-e)$. But then it is easily checked that G is sequential in N/a; a contradiction. Thus (7.4.41) holds.

7.4.42. $P_2\not\subseteq G$.

Suppose $P_2\subseteq G$. Then, by Lemma 2.10, $r_{N/a}(G-e)\ge 5$ and so $r_{N/a}(G)=5$ and $r_{N/a}(R)=3$. Since (R,G) is non-sequential, $|R\cap\{z_5,z_6\}|\ge 1$. If $|R\cap\{z_5,z_6\}|=2$, then $r_{N/a}(R)\ge 4$; a contradiction. So $|R\cap\{z_5,z_6\}|=$

1 and $|G \cap \{z_5, z_6\}| = 1$. But then $R - e$ is a triad in $N\backslash e/a$ and $P_2 \subseteq G - e$ is also a triad in $N\backslash e/a$, contradicting Lemma 2.22. Thus (7.4.42) holds.

7.4.43. *We may assume that* $|P_2 \cap R| = 1$.

Suppose that $|P_2 \cap R| = 2$. Let $P_2 \cap R = \{x, y\}$ and let $P_2 \cap G = \{z\}$. Since $R - e$ and $P_2 \cup \{\alpha, \beta\}$ are 3-separating sets in $N\backslash e/a$, it follows by uncrossing that their intersection, $\{\alpha, \beta, x, y\}$, is 3-separating in $N\backslash e/a$. Since the triad $\{z_1, z_2, z_3\}$ of $N\backslash e$ contains a, it follows that $\{\alpha, \beta, x, y\}$ is 3-separating in $N\backslash e$. Therefore $(\{\alpha, \beta\}, P_1 \cup z, P_2 - z)$ is a flower in $N\backslash e$. Also, as $\sqcap(\{\alpha, \beta\}, P_1) = 1$ and $z \notin \text{cl}(\{\alpha, \beta\} \cup P_1)$, it follows that $\sqcap(\{\alpha, \beta\}, P_1 \cup z) = 1$. Thus $(\{\alpha, \beta\}, P_1 \cup z, P_2 - z)$ is a flower in $N\backslash e$ of the form analyzed in the previous subcase. Hence (7.4.43) holds.

Let $P_2 \cap R = \{x\}$. By Lemma 2.10, $r_{N/a}(G - e) \geq 4$ and so $r_{N/a}(R) \in \{3, 4\}$. If $\{z_5, z_6\} \subseteq G$, then $R - e = \{\alpha, \beta, x\}$ and so $R - e$ is a triad in $N\backslash e/a$. Therefore, by Lemma 2.22, $e \in \text{cl}_{N/a}(R - e)$, contradicting our choice of a. Thus $\{z_5, z_6\} \not\subseteq G$. If $\{z_5, z_6\} \subseteq R$, then $r_{N/a}(R - e) \geq 5$; a contradiction. Therefore $|\{z_5, z_6\} \cap R| = 1$. Let $\{z_p\} = \{z_5, z_6\} \cap R$. Since $R - e$ and $(P_1 - a) \cup \{\alpha, \beta\}$ are 3-separating in $N\backslash e/a$, their intersection, $\{\alpha, \beta, z_p\}$, is 3-separating in $N\backslash e/a$ and so is a triad of $N\backslash e/a$. Thus, by Lemma 2.22, $e \in \text{cl}_{N/a}(\{\alpha, \beta, z_p\})$, but $e \notin \text{cl}_{N/a}((P_1 \cup P_2) - z_p)$. Therefore $\{\alpha, \beta, z_p, e\}$ is a cocircuit in N/a and so $\{\alpha, \beta, z_p, e\}$ is a cocircuit in N. Let $\{p, q\} = \{5, 6\}$. Then

7.4.44. $z_q \in \text{cl}_N((P_1 - z_q) \cup \{\alpha, \beta\})$.

If not, then $P_2 \cup z_q$ is a cosegment in $N\backslash e$, so $(P_2 - x) \cup z_q$ is a triad in $N\backslash e/a$ contained in G, contradicting Lemma 2.22. Hence (7.4.44) holds.

To complete the analysis, we now consider N/z_p. Since N/z_p is not sequentially 4-connected, it has a non-sequential 3-separation (R', G'). Then $r_{N/z_p}(R'), r_{N/z_p}(G') \in \{3, 4, 5\}$. By Lemma 2.12, we may assume that $\alpha, \beta \in R'$. Since $\{\alpha, \beta, z_p\}$ is a triad of $N\backslash e$, it follows by Lemma 2.14 that $N\backslash e/z_p$ is 3-connected. Furthermore, as $\{\alpha, \beta, z_p, e\}$ is a cocircuit of N and $\sqcap(\{z_1, z_2, z_3, z_4\}, P_2) = 0$ in N, we have

7.4.45. $\sqcap_{N/z_p}(\{z_1, z_2, z_3, z_4\}, P_2) = 0$.

The next result simplifies the remaining analysis.

7.4.46. *If* $|\{z_1, z_2, z_3, z_4\} \cap G'| \geq 2$ *and* $|P_2 \cap R'| = 2$*, then* $(\{\alpha, \beta\}, P_1 \cup (P_2 \cap G'), P_2 \cap R')$ *is a flower in* $N\backslash e$ *of the form analyzed in the previous subcase.*

To see this, first observe that $R' - e$ and $P_2 \cup \{\alpha, \beta\}$ are both 3-separating in $N\backslash e/z_p$. Thus their intersection, $\{\alpha, \beta\} \cup (P_2 \cap R')$, is also 3-separating in

$N\backslash e/z_p$. Furthermore, as $r(N\backslash e/z_p) = 6$ and $r_{N\backslash e/z_p}((P_1 - z_p) \cup (P_2 \cap G')) = 5$, we have $r_{N\backslash e/z_p}(\{\alpha, \beta\} \cup (P_2 \cap R')) = 3$. Therefore $r_{N\backslash e}(\{\alpha, \beta\} \cup (P_2 \cap R')) = 3$. As $r_{N\backslash e}(P_1 \cup (P_2 \cap G')) = 6$ and $\sqcap(\{\alpha, \beta\}, P_2 \cap R') = 1$, we have that $(\{\alpha, \beta\}, P_1 \cup (P_2 \cap G'), P_2 \cap R')$ is a flower in $N\backslash e$ of the form analyzed in the previous subcase. Thus (7.4.46) holds.

If $\{z_1, z_2, z_3, z_4\} \subseteq R'$, then, as $z_q \in \mathrm{cl}((P_1 - z_q) \cup \{\alpha, \beta\})$, we have $r_{N/z_p}(R' - e) \geq 5$. Since (R', G') is a non-sequential 3-separation of N/z_p, this implies that $P_2 \subseteq G'$. So G' contains a triad in $N\backslash e/z_p$. But $\{z_1, z_2, z_3\} \subseteq R'$ is also a triad in $N\backslash e/z_p$, contradicting Lemma 2.22. Thus $\{z_1, z_2, z_3, z_4\} \not\subseteq R'$.

If $\{z_1, z_2, z_3, z_4\} \subseteq G'$, then $|P_2 \cap G'| \geq 1$; otherwise, $\{z_1, z_2, z_3\} \subseteq G'$ and $P_2 \subseteq R'$, so $\{z_1, z_2, z_3\}$ and P_2 are triads in $N\backslash e/a$ that contradict Lemma 2.22. Also, $|P_2 \cap R'| \geq 1$; otherwise, by (7.4.45), $r_{N/z_p}(G') \geq 6$; a contradiction. Now $R' - e$ and $P_2 \cup \{\alpha, \beta\}$ are both 3-separating in $N\backslash e/z_p$, so their intersection, $\{\alpha, \beta\} \cup (P_2 \cap R')$, is also 3-separating in $N\backslash e/z_p$. Thus if $|P_2 \cap G'| = 2$, then $\{\alpha, \beta\} \cup (P_2 \cap R')$ is a triad in $N\backslash e/z_p$ contained in R'. As $\{z_1, z_2, z_3\}$ is a triad in $N\backslash e/z_p$ contained in G', we again contradict Lemma 2.22. Thus $|P_2 \cap G'| = 1$ and so $|P_2 \cap R'| = 2$, in which case, by (7.4.46), $N\backslash e$ has a flower of the form analyzed in the previous subcase.

By closure-equivalence, we may now assume that $|\{z_1, z_2, z_3, z_4\} \cap R'| = 2 = |\{z_1, z_2, z_3, z_4\} \cap G'|$. Suppose $r_{N/z_p}(\{\alpha, \beta\} \cup (\{z_1, z_2, z_3, z_4\} \cap R')) = 3$. Then $r_N(\{\alpha, \beta\} \cup (\{z_1, z_2, z_3, z_4\}) \leq 5$ so, by (7.4.44), $r_N(\{\alpha, \beta\} \cup P_1) \leq 5$; a contradiction. Thus $r_{N/z_p}(\{\alpha, \beta\} \cup (\{z_1, z_2, z_3, z_4\} \cap R')) \geq 4$. If $P_2 \subseteq G'$, then, by (7.4.45), $r_{N/z_p}(G') \geq 5$; a contradiction. If $|P_2 \cap G'| = 2$, then $|P_2 \cap R'| = 1$ and so $r_{N/z_p}(R') \geq 5$ and $r_{N/z_p}(G') \geq 4$, again a contradiction. The case $|P_2 \cap G'| = 1$ and $|P_2 \cap R'| = 2$ is covered by (7.4.46). Lastly, if $P_2 \subseteq R'$, then, by (7.4.45), $r_{N/z_p}(R') \geq 5$. Thus $r_{N/z_p}(R') = 5$ and $r_{N/z_p}(G') = 3$. But then $|G' - e| = 3$ and so $G' - e$ is a triad in $N\backslash e/z_p$. As $P_2 \cap R'$ is also a triad in $N\backslash e/z_p$, we contradict Lemma 2.22. This completes the argument in subcase (iii)(b), thereby completing the proof of (III) and the lemma. □

Theorem 7.5. *Let (A, B) be a non-sequential 3-separation in a 3-connected matroid M. Suppose that B is fully closed, A meets no triangle or triad of M, and if (X, Y) is a non-sequential 3-separation of M, then either $A \subseteq \mathrm{fcl}(X)$ or $A \subseteq \mathrm{fcl}(Y)$. If $|A| \leq 10$, then A contains an element whose deletion from M or M^* is 3-connected but does not expose any 3-separations.*

Proof. Suppose that $|A| \leq 10$. Since (A, B) is a non-sequential, $|A| \geq 4$ and $r(A), r^*(A) \geq 3$. If $r(A) = 3$ or $r^*(A) = 3$, then the theorem holds by

Lemma 7.2 and its dual. Thus we may assume that $r(A), r^*(A) \geq 4$. Now $\lambda_M(A) = 2$, so $2 = r(A) + r^*(A) - |A|$. Hence $|A| \geq 6$.

Let N be the clonal replacement of B by $\{\alpha, \beta\}$. By Lemma 4.13, N is 4-connected, and so, by Lemma 6.2, $M\backslash f$ and $M^*\backslash f$ are 3-connected for all f in $E(N) - \{\alpha, \beta\}$. As $6 \leq |A| \leq 10$, we have $8 \leq |E(N)| \leq 12$. Also, as $r(A) \geq 4$ and $r^*(A) \geq 4$, it follows that $r(N) \geq 4$ and, by Lemma 4.11, $r^*(N) \geq 4$. If $r(N) = 4$ or $r^*(N) = 4$, then, by Lemma 7.3, the theorem holds. Thus we may assume that $r(N), r^*(N) \geq 5$, and $10 \leq |E(N)| \leq 12$.

By Theorem 5.5, N has an element e not in $\{\alpha, \beta\}$ such that $N\backslash e$ or N/e is sequentially 4-connected. By duality, we may assume the former. By combining (I), (II), and (III) of Lemma 7.4, we get the theorem. □

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