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INTERTWINING CONNECTIVITY IN MATROIDS*

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Abstract. Let M be a matroid and let Q, R, S, and T be subsets of the ground set such that the smallest separation that separates Q from R has order k and the smallest separation that separates S from T has order ℓ. We prove that if $E(M)-(Q\cup R\cup S\cup T)$ is sufficiently large, then there is an element e of M such that, in one of $M\backslash e$ or M/e, both connectivities are preserved.

Key words. matroids, connectivity, intertwining connectivity

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1. Introduction.

Let M be a matroid with ground set $E(M)$. For any $X\subseteq E(M)$, define $\lambda_M(X) := r_M(X)+r_M(E(M)-X)-r(M)$. For disjoint subsets Q,R of $E(M)$, the *connectivity between Q and R* is

$$\kappa_M(Q,R) := \min\{\lambda_M(X) : Q\subseteq X\subseteq E(M)-R\}.$$

In the paper, we prove the following theorem.

THEOREM 1.1. *There is a function $c:\mathbb{N}^2\to\mathbb{N}$ with the following property. Let M be a matroid and Q, R, S, T, $F\subseteq E(M)$ sets of elements such that $Q\cap R = S\cap T=\emptyset$ and $F=E(M)-(Q\cup R\cup S\cup T)$. Let $k:=\kappa_M(Q,R)$ and $\ell:=\kappa(S,T)$. If $|F|\geq(2\ell+1)2^{2k+1}$, then there is an element $e\in F$ such that one of the following holds:*

(i) $\kappa_{M\backslash e}(Q,R)=k$ *and* $\kappa_{M\backslash e}(S,T)=\ell$;

(ii) $\kappa_{M/e}(Q,R)=k$ *and* $\kappa_{M/e}(S,T)=\ell$.

This theorem resolves a conjecture of Geelen [1]. It strengthens a theorem of Huynh and van Zwam [3], who prove the result for a class that includes all representable matroids but does not include all matroids.

The value that we get is unlikely to be tight. The $(k+1)\times(\ell+1)$ grid gives an example where the theorem fails with $|F|=2k\ell-\ell-k$. Perhaps this example is extremal?

CONJECTURE 1.2. *Theorem* 1.1 *holds with* $|F|=2k\ell-\ell-k+1$.

2. Proof of Theorem 1.1.

For all disjoint subsets Q, R of the ground set of a matroid M, Tutte [4] proved that there is a minor N of M with $E(N)=Q\cup R$ and such that $\kappa_M(Q,R)=\lambda_N(Q)$, which is a generalization of Menger's theorem to matroids. Equivalently, we have the following lemma.

LEMMA 2.1. *Let M be a matroid and Q,R be disjoint subsets of $E(M)$. For every $e\in E(M)-(Q\cup R)$ either $\kappa_{M\backslash e}(Q,R)=\kappa_M(Q,R)$ or $\kappa_{M/e}(Q,R)=\kappa_M(Q,R)$.*

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Let M be a matroid and Q, R be disjoint subsets of $E(M)$. Define $\sqcap_M(Q,R) := r_M(Q)+r_M(R)-r_M(Q\cup R)$. A partition (A,B) of $E(M)$ is *$Q-R$-separating of order $k+1$* if $Q\subseteq A$, $R\subseteq B$, and $\lambda_M(A)\le k$. Let $e\in E(M)-(Q\cup R)$. If $\kappa_{M\backslash e}(Q,R)=\kappa_M(Q,R)$, then e is *deletable with respect to* (Q,R); if $\kappa_{M/e}(Q,R)=\kappa_M(Q,R)$, then e is *contractible with respect to* (Q,R); and if e is both deletable and contractible with respect to (Q,R), then e is *flexible with respect to* (Q,R). Lemma 2.1 implies that for any $e\in E(M)-(Q\cup R)$ either e is deletable with respect to (Q,R) or e is contractible with (Q,R).

THEOREM 2.2 (see [3, Theorem 3.4]). *Let M be a matroid and Q, R be disjoint subsets of $E(M)$, let $k:=\kappa(Q,R)$, and let $F\subseteq E(M)-(Q\cup R)$ be a set of nonflexible elements. Then there are an ordering $(f_1,\dots,f_n)$ of F and a sequence $(A_1,\dots,A_n)$ of subsets of $E(M)$ such that*

(i) *A_i is $Q-R$-separating of order $k+1$ for each $i\in\{1,\dots,n\}$;*
(ii) *$A_i\subseteq A_{i+1}$ for each $i\in\{1,\dots,n\}$;*
(iii) *$A_i\cap F=\{f_1,\dots,f_i\}$ for each $i\in\{1,\dots,n\}$;*
(iv) *$f_i\in \mathrm{cl}(A_i-\{f_i\})\cap\mathrm{cl}(E(M)-A_i)$ or $f_i\in \mathrm{cl}^*(A_i-\{f_i\})\cap\mathrm{cl}^*(E(M)-A_i)$.*

THEOREM 2.3 (see [3, Lemma 3.6]). *Let M be a matroid and Q,R be disjoint subsets of $E(M)$, let $k:=\kappa(Q,R)$, and let $(U,E(M)-U)$ be a $Q-R$-separating set of order $k+1$. If $e\in E(M)-(U\cup R)$ is noncontradictable with respect to (Q,R), then e is also noncontradictable with respect to (U,R).*

First we prove that Theorem 1.1 holds for the case $|S|=|T|=\ell$.

LEMMA 2.4. *There is a function $c:\mathbb{N}^2\to\mathbb{N}$ with the following property. Let M be a matroid and Q, R, S, T, $F\subseteq E(M)$ sets of elements such that $Q\cap R=S\cap T=\emptyset$ and $F=E(M)-(Q\cup R\cup S\cup T)$. Let $k:=\kappa_M(Q,R)$ and $\ell:=\kappa_M(S,T)$. If $|S|=|T|=\ell$ and $|F|\ge(2\ell+1)2^{2k+1}$, then there is an element $e\in F$ such that one of the following holds:*

(i) *$\kappa_{M\backslash e}(Q,R)=k$ and $\kappa_{M\backslash e}(S,T)=\ell$;*
(ii) *$\kappa_{M/e}(Q,R)=k$ and $\kappa_{M/e}(S,T)=\ell$.*

Proof. If F contains some flexible element with respect to (Q,R) or (S,T), then we are done. So we may assume that each element in F is nonflexible with respect to (Q,R) and nonflexible with respect to (S,T). By Lemma 2.1 an element e in F is deletable (or contractible) with respect to (Q,R) if and only if e is contractible (or deletable) with respect to (S,T), for otherwise the lemma holds.

Let $(A_1,\dots,A_{(2\ell+1)2^{2k+1}})$ be the nested sequence of $Q-R$ separating sets from Theorem 2.2, let $(B_1,\dots,B_{(2\ell+1)2^{2k+1}})$ be their complements, and let $(f_1,\dots,f_{(2\ell+1)2^{2k+1}})$ be the corresponding ordering of F. Since $|S|=|T|=\ell$, there is a positive integer i such that $i+2^{2k+1}\le(2\ell+1)2^{2k+1}$ and such that $Q\cup R\cup S\cup T\subseteq A_i\cup B_{(2\ell+1)2^{2k+1}}$. Set

$$Q^{'}:=A_i,\ R^{'}:=B_{i+2^{2k+1}},\ F^{'}:=E(M)-(Q^{'}\cup R^{'}),$$
$$A^{'}_j:=A_{i+j},\ B^{'}_j:=B_{i+j},\ f^{'}_j:=f_{i+j},\ \text{for any } 1\le j\le 2^{2k+1}.$$

That is, $F^{'}=\{f^{'}_1,\dots,f^{'}_{2^{2k+1}}\}$. By duality and Lemma 2.3, each element in $F^{'}$ is nonflexible with respect to $(Q^{'},R^{'})$.

Let $(C_1,\dots,C_{2^{2k+1}})$ be the nested sequence of $S-T$ separating sets from Theorem 2.2 determined by the nonflexible-element set $F^{'}$ with respect to (S,T), let $(D_1,\dots,D_{2^{2k+1}})$ be their complements, and let $(g_1,\dots,g_{2^{2k+1}})$ be the corresponding ordering of $F^{'}$. By duality we may assume that g_1 is a deletable element with respect to (S,T). Then (i) $g_1\in\mathrm{cl}(C_1-\{g_1\})$ and (ii) g_1 is a contractible element with respect to (Q, R). By (i) and the fact that $C_1 - \{g_1\} \subseteq Q' \cup R'$ we see that $g_1 \in \mathrm{cl}(Q' \cup R')$. From (ii) we deduce that $g_1 \notin \mathrm{cl}(Q')$ and $g_1 \notin \mathrm{cl}(R')$. Therefore $\sqcap_M(Q' \cup \{g_1\}, R') = \sqcap_M(Q', R') + 1$. Assume that $g_1 = f'_j$. If $j \leq 2^{2k}$, then set $Q'' := A'_j$, $R'' := R'$; else if $j > 2^{2k}$, then set $Q'' := Q'$, $R'' := B'_{j-1}$. No matter which case happens, set $F'' := E(M) - (Q'' \cup R'')$. Evidently, $|F''| \geq 2^{2k}$ as $|F'| = 2^{2k+1}$. Replacing Q', R', F' with Q'', R'', F'', respectively, and repeating the above analysis $2k$ times, there are numbers j_1, j_2 with $2k+1 \leq j_1 \leq j_2 \leq 2^{2k+1}$ such that $\sqcap_M(A'_{j_1}, B'_{j_2}) \geq k+1$ or $\sqcap_{M^*}(A'_{j_1}, B'_{j_2}) \geq k+1$, a contradiction to the fact that $\lambda(A'_{j_1}) = k$. So the lemma holds. ∎

To prove Theorem 1.1 we still need the following lemma.

LEMMA 2.5 (see [2, Lemma 4.7]). *Let M be a matroid and S, T be disjoint subsets of $E(M)$. There exists sets $S_1 \subseteq S$, $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa(S, T)$.*

For convenience we restate Theorem 1.1 here.

THEOREM 2.6. *There is a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let M be a matroid and Q, R, S, T, $F \subseteq E(M)$ sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F = E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q, R)$ and $\ell := \kappa(S, T)$. If $|F| \geq (2\ell + 1)2^{2k+1}$, then there is an element $e \in F$ such that one of the following holds:*

(i) *$\kappa_{M\backslash e}(Q, R) = k$ and $\kappa_{M\backslash e}(S, T) = \ell$;*

(ii) *$\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = \ell$.*

Proof. By Lemma 2.5 there are sets $S_1 \subseteq S$, $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_M(S, T)$. Then Lemma 2.4 implies that there is an element $e_1 \in E(M) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_1 \in \{M\backslash e_1, M/e_1\}$ we have $\kappa_{M_1}(Q, R) = k$ and $\kappa_{M_1}(S_1, T_1) = \ell$. Since $\kappa_{M_1}(S_1, T_1) = \ell$ implies $\kappa_{M_1}(S, T) = \ell$, when $e_1 \in F$ the lemma holds. So we may assume that $e_1 \notin F$. That is, $e_1 \in (S \cup T) - (S_1 \cup T_1)$. Since $F \subseteq E(M_1) - (Q \cup R \cup S_1 \cup T_1)$, using Lemma 2.4 again there is an element $e_2 \in E(M_1) - (Q \cup R \cup S_1 \cup T_1)$ such that for some $M_2 \in \{M_1\backslash e_2, M_1/e_2\}$ we have $\kappa_{M_2}(Q, R) = k$ and $\kappa_{M_2}(S_1, T_1) = \ell$. Without loss of generality we may assume that $M_2 = M_1\backslash e_2$. Then $\kappa_{M\backslash e_2}(Q, R) = k$ and $\kappa_{M\backslash e_2}(S_1, T_1) = \ell$ as $\kappa_M(Q, R) = k$ and $\kappa_M(S_1, T_1) = \ell$. Thus, when $e_2 \in F$, the lemma holds. So we may assume that $e_2 \notin F$. Since $(S \cup T) - (S_1 \cup T_1)$ is finite, repeating the above analysis several times we can always find a minor with an element e such that (i) or (ii) holds. The theorem follows from this observation and the fact that the connectivity function is monotone under minors. ∎

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