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A Characterization of Tutte Invariants of 2-Polymatroids

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This paper develops a theory of Tutte invariants for 2-polymatroids that parallels the corresponding theory for matroids. It is shown that such 2-polymatroid invariants arise in the enumeration of a wide variety of combinatorial structures including matchings and perfect matchings in graphs, weak colourings in hypergraphs, and common bases in pairs of matroids. The main result characterizes all such invariants proving that, with some trivial exceptions, every 2-polymatroid Tutte invariant can be easily expressed in terms of a certain two-variable polynomial that is closely related to the Tutte polynomial of a matroid. © 1993 Academic Press, Inc.

1. INTRODUCTION

The theory of Tutte invariants for matroids had its origins within graph theory and, in particular, in the consideration of colouring and flow problems [1, 16–19]. The applications of this theory now extend into numerous diverse branches of combinatorics. These applications are surveyed in [3]. The purpose of this paper is to develop a corresponding theory for 2-polymatroids. The two highlights of this theory are that, just as for matroids, there is essentially a unique universal Tutte invariant for

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2-polymatroids; and this universal invariant has many interesting evaluations in a number of combinatorial contexts. Among these evaluations are some important ones for graphs that do not come from the Tutte polynomial for matroids.

Formally, a polymatroid is a normalized, increasing, submodular function on the power set of a set E, but, without loss of generality, one can think of a polymatroid as a multiset of flats in a matroid; in particular, a 2-polymatroid can be thought of as a multiset of lines, points, and loops in a matroid. If this matroid is free, then the 2-polymatroid is called Boolean. Such 2-polymatroids are essentially just graphs. The class of Boolean 2-polymatroids will feature prominently throughout this paper. Another way to obtain a 2-polymatroid is to add the rank functions of two matroids on E. Several other examples of 2-polymatroids appear in Section 2 and in the important paper of Lovász [11] (see also [12, Chap. 11]) which motivates the study of 2-polymatroids in general.

This paper is organized as follows: Section 2 contains a general discussion of 2-polymatroids and their properties; Section 3 defines Tutte invariants for 2-polymatroids and states the main result of the paper, Theorem 3.14. This theorem is proved in Section 5, while Section 4 describes some of the wide variety of examples of 2-polymatroid Tutte invariants.

2. Polymatroid-Theoretic Preliminaries

Let E be a finite set and let f be a function from the power set of E into the integers. Then f is *normalized* if $f(\emptyset)=0$; f is *increasing* if $f(A)\leqslant f(B)$ whenever $A\subseteq B\subseteq E$; and f is *submodular* if $f(A)+f(B)\geqslant f(A\cup B)+f(A\cap B)$ for all subsets A and B of E. If f is normalized, increasing, and submodular, then f is a *polymatroid on* E. We say that E is the *ground set* of f and $f(E)$ is the *rank* of f. Let k be a positive integer. Then the polymatroid f is a *k-polymatroid* if $f(e)\leqslant k$ for all elements e of E. A 1-polymatroid is a matroid. In regarding a matroid as a specialization of a polymatroid, it is convenient to identify matroids with their rank functions. We shall consistently follow this course.

Polymatroid Representation

Let f be a polymatroid on E and r be a matroid on the set S. Then f is *representable over* r if there is a function $\psi: E\to 2^S$ with the property that for all subsets A of E, $f(A)=r(\bigcup_{a\in A}\psi(a))$. Such a function is a *representation* of f in r. It is easily seen that if ψ is a representation of f in r, then the function $\psi': E\to 2^S$ defined by $\psi'(e)=cl(\psi(e))$ is also a representation of f in r. It follows that if f is representable over r, then a representation can be chosen so that the image of each element of the ground set of f is a flat of r. Conversely, if E is a multiset of flats of r, then the function $f'\colon 2^E \to \mathbb{Z}$ defined, for all $A \subseteq E$, by $f'(A) = r(\bigcup_{a \in A} a)$ is a polymatroid on E. It is a fundamental fact (see for, example, [7, 10, 13]) that every polymatroid can be obtained in this way. In other words, every polymatroid is representable over some matroid.

A consequence of the above discussion is that one loses no generality in picturing the elements of a 2-polymatroid as a multiset of lines, points, and loops of some matroid. It may aid the reader's intuition to observe that if distinct elements of a 2-polymatroid are represented by lines of a matroid, then the intersection of these lines may be nonempty. In the polymatroid, the intersection of the elements is, of course, always empty.

Let $\mathbb{F}$ be a field. Then a polymatroid is *representable* over $\mathbb{F}$ if it is representable over some matroid that is coordinatizable over $\mathbb{F}$.

Polymatroid Minors

Let f be a polymatroid on E, and let A be a subset of E. Then the *deletion* of A from f, denoted $f \backslash A$, is the polymatroid on $E - A$ that is defined, for all subsets X of $E - A$, by $(f \backslash A)(X) = f(X)$. The *contraction* of A from f, denoted f/A, is the polymatroid on $E - A$ defined, for all subsets X of $E - A$, by $(f/A)(X) = f(A \cup X) - f(A)$.

It is routinely verified that $f \backslash A$ and f/A are, indeed, polymatroids, and are k-polymatroids whenever f is. It is also easily seen that deletion and contraction commute, both with themselves and each other. In other words, if A and B are disjoint subsets of E, then $(f \backslash A) \backslash B = (f \backslash B) \backslash A = f \backslash (A \cup B)$; $(f/A)/B = (f/B)/A = f/(A \cup B)$; and $(f \backslash A)/B = (f/B) \backslash A$.

The polymatroid g is a *minor* of f if $g = (f \backslash A)/B$ for some disjoint subsets A and B of E. A class $\mathscr{F}$ of polymatroids is *minor-closed* if all minors of members of $\mathscr{F}$ also belong to $\mathscr{F}$.

The above definitions are all direct generalizations of standard ones for matroids and are uncontentious for polymatroids except possibly that of contraction. This definition is justified by the fact that contraction in a polymatroid f corresponds to contraction in any matroid over which f is represented. More precisely, we have the easily proved

(2.1) PROPOSITION. *Let f be a polymatroid on E, let r be a matroid on S, and let ϕ be a representation of f in r. For all subsets A of E, let $\phi(A) = \bigcup_{a \in A} \phi(a)$. Then the function ϕ' defined, for all $x \in E - A$, by $\phi'(x) = \phi(A \cup x) - \phi(A)$ is a representation of f/A in $r/\phi(A)$.*

Duality

One of the attractive features of matroid theory is that it has a satisfactory theory of duality. Let $\mathscr{C}$ be a class of structures with a well-defined notion of isomorphism. An *operation* on $\mathscr{C}$ is a function $*: \mathscr{C} \to \mathscr{C}$ with the property that c^* is isomorphic to d^* whenever c and d are isomorphic members of $\mathscr{C}$. (We write c^* for $*(c)$.) If $c^{**} = c$ for all c in $\mathscr{C}$, then $*$ is an *involution*. If $\mathscr{C}$ is a class of polymatroids, then $*$ *interchanges deletion and contraction* if, for any f in $\mathscr{C}$ and any element x of the ground set of f,

$$f^* \backslash x = (f/x)^*.$$

Kung [9] has shown that orthogonal duality is the only involution on the class of matroids which interchanges deletion and contraction.

Let $\mathscr{F}_k$ denote the class of k-polymatroids. For f in $\mathscr{F}_k$ with ground set E, define the *k-dual* of f, denoted f^*, by

$$f^*(A) = k\,|A| + f(E - A) - f(E)$$

for all subsets A of E. It is easily seen that k-duality is an involution on the class of k-polymatroids which interchanges deletion and contraction. Moreover, it is shown in [21] that k-duality is the unique such involution. Hence the definition of k-dual given here is the natural one to make to preserve the fundamental link between deletion, contraction, and duality that is of such basic importance in matroid theory.

Our primary interest is in the case $k = 2$, but we are also interested in matroid duality, that is, the case $k = 1$. The notation is ambiguous, since if r is a matroid, then r^* denotes both its 1-dual (that is, its orthogonal dual) and its 2-dual. Unless specified to the contrary, when we use the notation r^* we shall mean the 2-dual of r. Note that the only matroids whose 2-duals are also matroids are free matroids.

Not surprisingly, a number of properties of matroid duals have generalizations to k-duals. We briefly mention a few of these now. Let f be a k-polymatroid on E and consider the k-dual f^* of f. A trivial computation shows that

$$f(E) + f^*(E) = k\,|E|,$$

which reduces to a well-known fact in the case $k = 1$. Let A be a subset of E. Then A *spans* f if $f(A) = f(E)$, and A is a *k-matching* if $f(A) = k\,|A|$. A *perfect k-matching* is a k-matching which spans. In the cases $k = 2$ and $k = 1$ we specialize the language. A *matching* and a *perfect matching* are a 2-matching and a perfect 2-matching, respectively, in a 2-polymatroid. An *independent set* is a 1-matching and a *basis* is a perfect 1-matching in a matroid. A routine computation proves the following

(2.2) Proposition. *Let f be a k-polymatroid having ground set E. The* *subset A of E is a k-matching if and only if $E-A$ is a spanning set in the k-dual f^* of f. Moreover, f has a perfect k-matching if and only if f^* does.*

We now consider representability of duals. The straightforward arguments which establish the following assertions are omitted. Let f be a k-polymatroid on E. Then f is representable over a given field if and only if the k-dual of f is. If f is representable over a matroid r, then it does not, in general, follow that the k-dual of f is representable over the orthogonal dual of r. However, if there is a representation ϕ of f in r such that $\{\phi(e) : e \in E\}$ consists of disjoint subsets of the ground set of r, then the k-dual of f is representable over the orthogonal dual of r. Such a representation can always be found in a suitable parallel extension of r. It follows that if f is representable over r, then the k-dual of f is representable over some series extension of the orthogonal dual of r.

Boolean 2-Polymatroids

Let $G=(V, E)$ be a graph. In this paper we allow graphs to have *free loops*, that is, edges that are incident with no vertices. This terminology is due to Zaslavsky [22]. Define the set function f_G as follows: For any subset A of E,

$$f_G(A) = |V(A)|.$$

Here $V(A)$ denotes the set of vertices incident with at least one edge in A. It is well known, and easily seen, that f_G is a 2-polymatroid. A polymatroid is *Boolean* if it is representable over some free matroid. It is easily seen that the 2-polymatroid f is Boolean if and only if $f=f_G$ for some graph G. In this situation, we say that f is *represented* by the graph G.

It is routinely verified that if G_1 and G_2 are graphs and $f_{G_1} \cong f_{G_2}$, then, up to isolated vertices, $G_1 \cong G_2$. It follows that f_G carries almost all of the structure of G. This contrasts with the cycle matroid of G—which is, of course, also a 2-polymatroid—where much of the information about G is lost. This correspondence with graphs means that Boolean 2-polymatroids form a fundamental class. The correspondence also means that Theorem 3.14, which characterizes all 2-polymatroid Tutte invariants of Boolean 2-polymatroids, can be interpreted in a purely graph-theoretic way.

We now consider minors of Boolean 2-polymatroids. Let e be an edge of the graph G. It is immediate that $f_{G\backslash e} = f_G\backslash e$, where $G\backslash e$ is the graph obtained from G by deleting the edge e in the usual way. Consider contraction. Define the graph $G \square e$ as follows. Let V' be the set of vertices incident with e in G. We have $|V'| \in \{0, 1, 2\}$. Then $G \square e$ has edge set $E-e$ and vertex set $V-V'$. The vertices incident with an edge x in $G \square e$ are the vertices incident with x in G with those in V' removed. We can now characterize contraction by $f_{G \llcorner e} = f_G/e$. This fact follows easily from Proposition 2.1.

It now follows that the class of Boolean 2-polymatroids is minor-closed. A characterization of Boolean 2-polymatroids in terms of excluded minors is given in [15]. Note that, in general, the dual of a Boolean 2-polymatroid is not Boolean. Indeed, the class of Boolean 2-polymatroids with Boolean duals is rather trivial as the reader can readily verify.

Diagrams of 2-Polymatroids

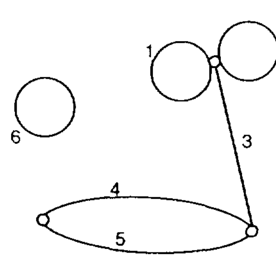
As with matroids, 2-polymatroids of low rank can be represented by diagrams. The conventions for such *geometric representations* naturally extend those for matroids. A Boolean 2-polymatroid can also be represented by a graph. Figure 2.1 gives both a geometric representation and a graphical representation of a 2-polymatroid f on the ground set $\{1, 2, 3, 4, 5, 6\}$. The following values of f should suffice to enable the reader to determine the conventions used: $f(\{6\})=0$, $f(\{1\})=f(\{2\})=f(\{1,2\})=1$, $f(\{3\})=f(\{4\})=f(\{5\})=f(\{4,5\})=f(\{1,3\})=2$, $f(\{3,4\})=3$. See Fig. 2.1.

We now fix some terminology for 2-polymatroids which is suggested by geometric representations. Let f be a 2-polymatroid on E. If x is an element of E, then x is a *point* of f if $f(\{x\})=1$, and x is a *line* of f if $f(\{x\})=2$. If x and y are points of f, they are *parallel* if $f(\{x,y\})=1$, and if x and y are lines of f, they are *parallel* if $f(\{x,y\})=2$. Also, x is a *loop* of f if $f(\{x\})=0$.

Assume that f is Boolean and is represented by a graph G. Then x is a loop of f if and only if x is a free loop of G, and x is a point of f if and only if x is a loop of G.

3. The Main Result

In defining Tutte invariants for 2-polymatroids, we shall be extending the definition of $T-G$ invariants for matroids. Therefore we begin this section by briefly reviewing the theory of $T-G$ invariants for matroids.



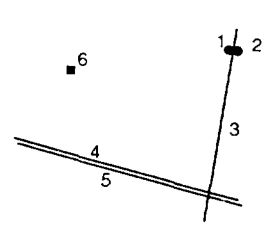


FIG. 1. A graphical and a geometric representation of a 2-polymatroid.

Let $\mathcal{M}$ be a class of matroids that is closed under isomorphism and the taking of minors. A function α on $\mathcal{M}$ that takes values in a field $\mathbb{F}$ is an *isomorphism invariant* if $\alpha(r_1)=\alpha(r_2)$ whenever $r_1 \cong r_2$. Several numbers that one can associate with a matroid r such as its number of bases, its number of independent sets, and its number of spanning sets obey the following two basic recursions:

(3.1) $\alpha(r)=\alpha(r\backslash e)\,\alpha(r\backslash(E-e))$ if e is a separator, that is, a loop or coloop of r; and

(3.2) for some fixed nonzero members j and k of $\mathbb{F}$, $\alpha(r)=j\alpha(r\backslash e)+k\alpha(r/e)$ if e is a nonseparator of r.

An isomorphism invariant on $\mathcal{M}$ that obeys (3.1) and (3.2) is called a *generalized* $T-G$ *invariant* on $\mathcal{M}$. Such an invariant for which both j and k are identically one is called simply a $T-G$ *invariant* on $\mathcal{M}$. There are many well-known important examples of generalized $T-G$ invariants; for instance, both the chromatic and flow polynomials are generalized $T-G$ invariants on the class of graphic matroids. One of the attractive features of these invariants is that they are all evaluations of a certain universal invariant. The formal statement of this result will require another definition. For a polymatroid f having ground set E, the *matroid rank generating function* is given by

$$s(f;u,v)=\sum_{X\subseteq E} u^{f(E)-f(X)}v^{|X|-f(X)}.$$

When f is a matroid r, one can easily check that an equivalent definition is

$$s(r;u,v)=\sum_{X\subseteq E} u^{r(E)-r(X)}v^{r^*(E)-r^*(E-X)},$$

where r^* denotes the orthogonal dual of r here. It is not difficult to show that the matroid rank generating function is a generalized $T-G$ invariant on any minor-closed class of matroids. Indeed, by extending a result of Brylawski [2], Oxley and Welsh [14] showed that every generalized $T-G$ invariant can be easily expressed in terms of this function.

(3.3) PROPOSITION. *Let α be a generalized $T-G$ invariant on a minor-closed class of matroids $\mathcal{M}$ and suppose that*

$$\alpha(U_{1,1})=x \qquad \textit{and} \qquad \alpha(U_{0,1})=y.$$

Then, for all r in $\mathcal{M}$ such that $|E|\geqslant 1$,

$$\alpha(r)=j^{|E|-r(E)}k^{r(E)}s\left(r;\frac{x}{k}-1,\frac{y}{j}-1\right).$$

This result is more commonly stated in terms of the *Tutte polynomial* of a matroid, this being $t(r; u, v) = s(r; u-1, v-1)$. However, the above form of the result is more easily generalized to 2-polymatroids.

We now discuss the extension of this theory to 2-polymatroids. A *2-polymatroid isomorphism invariant* is a function γ on the class of all 2-polymatroids such that

$$\gamma(f) = \gamma(g) \qquad \text{whenever} \quad f \cong g.$$

It was noted above that a generalized $T-G$ invariant for matroids obeys two basic recursions: a multiplicative recursion, which holds for those elements which are separators, and an additive recursion, which holds for those elements which are not. To mimic this approach for 2-polymatroids, we need first to decide when an element is a separator. Following Cunningham [5], we define an element e of a polymatroid f to be a *separator* of f if

$$f(e) + f(E-e) = f(E).$$

Now consider those elements which are not separators. If e is such an element of a 2-polymatroid f on a set E, and $*$ denotes 2-duality, then e obeys one of the following three conditions:

(i) $f(E-e) = f(E)$ and $f^*(E-e) = f^*(E) - 1$;

(ii) $f(E-e) = f(E) - 1$ and $f^*(E-e) = f^*(E)$; and

(iii) $f(E-e) = f(E)$ and $f^*(E-e) = f^*(E)$.

Elements obeying (iii) most closely resemble the nonseparator elements of a matroid, for if f is a matroid rank function, and $*$ denotes orthogonal duality, the elements obeying (i) are precisely the loops, while those obeying (ii) are precisely the coloops.

Conditions (i) and (ii) are clearly dual to each other, while (iii) is self-dual. It is straightforward to check that (i), (ii), and (iii) are equivalent to (i)′, (ii)′, and (iii)′, respectively, where (i)′–(iii)′ are as follows:

(i)′ $f(E-e) = f(E)$ and $f(e) = 1$;

(ii)′ $f(E-e) = f(E) - 1$ and $f(e) = 2$; and

(iii)′ $f(E-e) = f(E)$ and $f(e) = 2$.

Because the nonseparator elements of a 2-polymatroid are of three different types, we shall replace the single additive recursion (3.2) that holds for matroids by three potentially distinct additive recursions.

Let $\mathscr{K}$ be a class of 2-polymatroids that is closed under isomorphism and the taking of minors. Assume that $\mathscr{K}$ contains $U_{0,1}$, $U_{1,1}$, and $U_{2,1}$, the single-element 2-polymatroids of ranks zero, one, and two. Let γ be a

2-polymatroid isomorphism invariant defined on $\mathscr{K}$ and taking values in a field $\mathbb{F}$. Let

$$\gamma(U_{2,1}) = x, \qquad \gamma(U_{0,1}) = y, \qquad \text{and} \quad \gamma(U_{1,1}) = z. \tag{3.4}$$

The function γ is a *generalized Tutte invariant* for $\mathscr{K}$ over $\mathbb{F}$ if there are fixed elements a, b, c, d, m, and n of $\mathbb{F}$ such that, whenever f is a member of $\mathscr{K}$, the following recursions hold for all e in the ground set E of f:

$$\gamma(f) = \gamma(f\backslash(E-e))\,\gamma(f\backslash e) \qquad \text{if } \{e\} \text{ is a separator of } f; \tag{3.5}$$

$$\gamma(f) = \begin{cases} a\gamma(f\backslash e) + b\gamma(f/e) & \text{if } f(E-e) = f(E) \text{ and } f(e) = 1; \\ c\gamma(f\backslash e) + d\gamma(f/e) & \text{if } f(E-e) = f(E) - 1 \text{ and } f(e) = 2; \text{and} \\ m\gamma(f\backslash e) + n\gamma(f/e) & \text{if } f(E-e) = f(E) \text{ and } f(e) = 2. \end{cases} \tag{3.6}$$

Our task in this section is to determine precisely when a generalized Tutte invariant exists. Specifically, we shall determine all points $(x, y, z, a, b, c, d, m, n)$ in $\mathbb{F}^9$ for which such an invariant is well-defined.

Before considering some examples of generalized Tutte invariants, we note that one special type of such invariant, which we call simply a *Tutte invariant*, arises when we take both m and n to be identically one.

For an arbitrary 2-polymatroid f having ground set E, the *2-polymatroid rank generating function* $S(f; u, v)$ is defined by

$$S(f; u, v) = \sum_{X \subseteq E} u^{f(E) - f(X)} v^{2|X| - f(X)}.$$

One easily checks that

$$S(f; u, v) = \sum_{X \subseteq E} u^{f(E) - f(X)} v^{f^*(E) - f^*(E-X)}.$$

Clearly the 2-polymatroid and matroid rank generating functions are closely related. The precise link between these functions will be discussed in the next section.

Let $\mathscr{M}_2$ denote the class of all 2-polymatroids.

(3.7) PROPOSITION. *$S(f; u, v)$ is a Tutte invariant on $\mathscr{M}_2$ with $x = 1 + u^2$; $y = 1 + v^2$; $z = u + v$; $a = 1$; $b = v$; $c = u$; and $d = 1$.*

Proof. This is routine. To check (3.6), one breaks the summation up into two summations corresponding to those subsets which contain e and those which do not. ∎

Next we define another function for 2-polymatroids which we shall see is closely related to the rank generating function. For an arbitrary 2-polymatroid f having ground set E, let

$$U(f)=m^{|E|-f(E)}\sum_{X\subseteq E} c^{f(E)-f(X)}b^{2|X|-f(X)}(mn)^{f(X)-|X|},$$

where we suppose that neither m nor n is zero. By using the same technique as in the last proof, it is not difficult to prove the following result:

(3.8) PROPOSITION. *$U(f)$ is a generalized Tutte invariant on $\mathcal{M}_2$ provided that the following hold*:

$a=m$; $d=n$; $mx=mn+c^2$; $ny=mn+b^2$; $z=b+c$; $m\neq 0$; *and* $n\neq 0$.

One striking contrast between $U(f)$ and an arbitrary generalized $T-G$ invariant for matroids is that, for the latter, the four variables, x, y, j, and k, are independent whereas, for the former, the nine variables, a, b, c, d, m, n, x, y, and z, are clearly not. As we shall see, in the 2-polymatroid case, there is no generalized Tutte invariant for which the nine variables are independent.

Again straightforward manipulations give the following result and we shall not reproduce the proof.

(3.9) PROPOSITION. *Provided neither m nor n is zero,*

$$U(f)=m^{|E|-f(E)}(mn)^{f(E)/2}S(f;c(mn)^{-1/2},b(mn)^{-1/2}).$$

We remark here that the left-hand side of the last equation is well-defined for all fields $\mathbb{F}$. However, for the right-hand side to be meaningful, $\mathbb{F}$ must contain $(mn)^{1/2}$. Adjoining this element to $\mathbb{F}$ if necessary, we can do the calculations needed to evaluate the right-hand side in this extension field. By the proposition, the result of these calculations must be in $\mathbb{F}$. It should be noted that the extension field of $\mathbb{F}$ used here for relating U and S will not be needed elsewhere in the paper.

The main result of this section will be that, apart from some very special invariants, each of which is a monomial or is zero, U is the only generalized Tutte invariant on $\mathcal{M}_2$. Before explicitly stating and then proving this result, we shall introduce these special invariants. The first of these is the trivial invariant which is zero on all 2-polymatroids having nonempty ground sets.

For a 2-polymatroid f having ground set E, define

$$Q(f)=\begin{cases} y^{|E|-f(E)}z^{f(E)} & \text{if } f(E)<|E|;\\ z^{f(E)} & \text{if } f(E)=|E|;\\ x^{f(E)-|E|}z^{2|E|-f(E)} & \text{if } f(E)>|E|.\end{cases}$$

Another routine argument establishes the following:

(3.10) PROPOSITION. *Q is a generalized Tutte invariant on $\mathcal{M}_2$ provided that the following hold:*

$$z^2 = xy = ax + bz = cz + dy = mx + ny; \quad yz = az + by; \quad \textit{and} \quad xz = cx + dz.$$

The next invariant we consider will be important in our main result in spite of the fact that it is not a generalized Tutte invariant on $\mathcal{M}_2$. For a 2-polymatroid f on E, let

$$N(f) = \begin{cases} x^{f(E)-|E|} c^{2|E|-f(E)} & \text{if } f(e) = 2 \text{ for all } e \text{ in } E; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}_2^G$ denote the class of all Boolean 2-polymatroids. The fact that N is not a generalized Tutte invariant on $\mathcal{M}_2$ will follow from Theorem 3.14, the main result of this section.

(3.11) PROPOSITION. *N is a generalized Tutte invariant on $\mathcal{M}_2^G$ provided that the following hold:*

$$y = z = a = 0; \qquad mx = c^2; \qquad \textit{and} \quad x \neq 0.$$

Proof. Again this is routine, the key point being that if e is an element of a Boolean 2-polymatroid f and $\{e\}$ is not a separator of f, then f/e has an element e' such that $(f/e)(e') < 2$. This follows because, in the graph G corresponding to f, the element e corresponds to an edge; and, since $\{e\}$ is not a separator of f, at least one endpoint of e is incident with some other edge e'. In f/e, this element has rank less than two. ▮

In order to define certain other invariants, it will be convenient to introduce some further notation. For a 2-polymatroid f on E and for i in $\{0, 1, 2\}$, let

$$p_i = |\{e \in E : f(e) = i\}|$$

and let

$$q_i = |\{e \in E : f^*(e) = i\}|.$$

(3.12) LEMMA. $q_i = |\{e \in E : f(E) - f(E - e) = 2 - i\}|$.

For a 2-polymatroid f on a set E, let

$$I_1(f) = \begin{cases} y^{p_0} a^{p_1 - f(E)} m^{p_2} z^{f(E)} & \text{if } f(E) \leqslant p_1; \\ y^{p_0} z^{p_1} m^{p_1 + p_2 - f(E)} c^{f(E) - p_1} & \text{if } p_1 < f(E) < p_1 + p_2; \\ y^{p_0} z^{p_1} c^{p_1 + 2p_2 - f(E)} x^{f(E) - p_1 - p_2} & \text{if } p_1 + p_2 \leqslant f(E). \end{cases}$$

Also let $I_1^*(f) = I_1(f^*)$, so that

$$I_1^*(f)$$
$$= \begin{cases} x^{q_0} z^{q_1} b^{q_1 + 2q_2 + f(E) - 2|E|} y^{2|E| - f(E) - q_1 - q_2} & \text{if } f(E) \leqslant 2|E| - q_1 - q_2; \\ x^{q_0} z^{q_1} n^{q_1 + q_2 + f(E) - 2|E|} b^{2|E| - f(E) - q_1} & \text{if } 2|E| - q_1 - q_2 < f(E) \\ & \qquad < 2|E| - q_1; \\ x^{q_0} d^{q_1 + f(E) - 2|E|} n^{q_2} z^{2|E| - f(E)} & \text{if } 2|E| - q_1 \leqslant f(E). \end{cases}$$

It is not difficult to prove the following:

(3.13) PROPOSITION. (i) *I_1 is a generalized Tutte invariant on $\mathscr{M}_2$ provided that the following hold*:

$$b = d = n = 0; \qquad mx = c^2; \qquad ax = cz; \qquad \textit{and} \qquad mz = ac.$$

(ii) *I_1^* is a generalized Tutte invariant on $\mathscr{M}_2$ provided that the following hold*:

$$a = c = m = 0; \qquad ny = b^2; \qquad dy = bz; \qquad \textit{and} \qquad nz = bd.$$

Each of the invariants already encountered can be specialized by fixing the values of a, b, c, d, m, n, x, y, and z so that the conditions governing the original invariant are still met. If the invariant τ_2 is obtained from τ_1 in this way, then τ_2 is called an *evaluation* of τ_1. The next result, the main result of the paper, shows that every generalized Tutte invariant on $\mathscr{M}_2^G$ is an evaluation of one of the generalized Tutte invariants identified above.

(3.14) THEOREM. *Suppose that τ is a generalized Tutte invariant on $\mathscr{M}_2^G$ over an arbitrary field $\mathbb{F}$. Then one of the following occurs*:

(G1) *$a = m$; $d = n$; $mx = mn + c^2$; $ny = mn + b^2$; $z = b + c$; $m \neq 0$; $n \neq 0$; and τ is an evaluation of U*;

(G2) *$z^2 = xy = ax + bz = cz + dy = mx + ny$; $yz = az + by$; $xz = cx + dz$; and τ is an evaluation of Q*;

(G3) *$y = z = a = 0$; $mx = c^2$; $x \neq 0$; and τ is an evaluation of N*;

(G4) *$b = d = n = 0$; $mx = c^2$; $ax = cz$; $mz = ac$; and τ is an evaluation of I_1*;

(G5) *$a = c = m = 0$; $ny = b^2$; $dy = bz$; $nz = bd$; and τ is an evaluation of I_1^**;

(G6) *$x = y = z = 0$; and τ is zero on all nonempty 2-polymatroids.*

On combining this result with duality, we immediately obtain:

(3.15) COROLLARY. *Let τ be a generalized Tutte invariant on $\mathcal{M}_2$. Then one of* (G1), (G2), *and* (G4)–(G6) *holds.*

In fact, it is clear that the last corollary holds under the weaker hypothesis that τ is a generalized Tutte invariant on $\{f : f \text{ or } f^* \text{ is in } \mathcal{M}_2^G\}$.

If one restricts attention to Tutte invariants on $\mathcal{M}_2$, then many of the special invariants disappear:

(3.16) COROLLARY. *Suppose that τ is a Tutte invariant on $\mathcal{M}_2$. Then one of the following occurs:*

(T1) $a = d = 1$; $x = 1 + c^2$; $y = 1 + b^2$; $z = b + c$; *and* $\tau(f)$ *is an evaluation of* $S(f; c, b)$ *for all* f *in* $\mathcal{M}_2$;

(T2) $z^2 = xy = ax + bz = cz + dy = x + y$; $xz = cx + dz$; $yz = az + by$; *and* τ *is an evaluation of* Q;

(T3) $x = y = z = 0$ *and* τ *is zero on all nonempty 2-polymatroids.*

The six invariants in Theorem 3.14 may appear to belie the claim that there is essentially a unique universal generalized Tutte invariant on $\mathcal{M}_2^G$. However, we note that, for each of (G3)–(G6), at least three of the nine variables are identically zero. Of the remaining two invariants, (G2) is a monomial that conveys only the rank of the 2-polymatroid and the size of its ground set. Also note that (G6) is clearly trivial, while each of (G2)–(G5) can be determined immediately once the rank of f, the size of its ground set, the value of f on singletons, and the value of f on complements of singletons are known.

We defer the proof of Theorem 3.14 to Section 5.

4. COMBINATORIAL SIGNIFICANCE

In this section we assume that all evaluations of Tutte invariants take place in the complex numbers; that is, we take the field $\mathbb{F}$ to be $\mathbb{C}$. Our aim is to provide enough examples to demonstrate the widespread combinatorial significance of 2-polymatroid Tutte invariants. The examples given here are certainly not exhaustive.

First recall some basic properties of the 2-polymatroid rank generating function $S(f; u, v)$. For single-element 2-polymatroids we have $S(U_{2,1}; u, v) = u^2 + 1$; $S(U_{1,1}; u, v) = u + v$; and $S(U_{0,1}; u, v) = v^2 + 1$. Also, if e is an element of the 2-polymatroid f, then

$$S(f; u, v) = S(f \backslash e; u, v) + vS(f/e; u, v) \quad \text{if } f(E - e) = f(E) \text{ and } f(e) = 1;$$

$$S(f; u, v) = uS(f \backslash e; u, v) + S(f/e; u, v) \quad \text{if } f(E - e) = f(E) - 1 \text{ and } f(e) = 2;$$

$$S(f; u, v) = S(f \backslash e; u, v) + S(f/e; u, v) \quad \text{if } f(E - e) = f(E) \text{ and } f(e) = 2.$$

We begin by examining the connection between the matroid and 2-polymatroid rank generating functions. A routine computation shows that if $v \neq 0$, then the matroid rank generating function, $s(f; u, v)$, is a 2-polymatroid generalized Tutte invariant. In fact,

$$s(f; u, v) = v^{-f(E)/2} S(f; uv^{1/2}, v^{1/2}) \quad \text{if} \quad v \neq 0, \tag{4.1}$$

$$S(f; u, v) = v^{f(E)} s(f; uv^{-1}, v^2) \quad \text{if} \quad v \neq 0. \tag{4.2}$$

Now consider the case $v = 0$. Evidently

$$s(f; u, 0) = \sum_{X \subseteq E;\, f(X) = |X|} u^{f(E) - f(X)}, \tag{4.3}$$

$$S(f; u, 0) = \sum_{X \subseteq E;\, f(X) = 2|X|} u^{f(E) - f(X)}. \tag{4.4}$$

It is readily verified that $s(f; u, 0)$ is not a 2-polymatroid generalized Tutte invariant.

There is an interesting analogy between the role played by $s(f; u, 0)$ in matroid theory and that played by $S(f; u, 0)$ in the theory of 2-polymatroids. It follows immediately from (4.3) that if f is a matroid, then $s(f; u, 0)$ is, up to an obvious transformation, the generating function for the number of independent sets of f of each cardinality. Indeed,

$s(f; 0, 0)$ is the number of bases of f, and (4.5)

$s(f; 1, 0)$ is the number of independent sets of f. (4.6)

Now, for an arbitrary 2-polymatroid f, a similar situation holds except that one now considers matchings. Again, it is clear that $S(f; u, 0)$ is, up to an obvious transformation, the generating function for the number of matchings of f of each cardinality. Indeed,

$S(f; 0, 0)$ is the number of perfect matchings of f, and (4.7)

$S(f; 1, 0)$ is the total number of matchings of f. (4.8)

In fact, if f is the Boolean 2-polymatroid of a graph G, then

$u^{f(E)/2} S(f; u^{-1/2}, 0)$ is the matching generating polynomial of G, (4.9)

and if G has no isolated vertices, then

$i^{f(E)} S(f; -iu, 0)$ is the matching defect polynomial of G. (4.10)

For a good discussion of matching generating polynomials and matching defect polynomials of graphs, see [12, Sect. 8.5].

Now assume that $v \neq 0$. The meaning of some evaluations of $s(f; u, v)$ for matroids extends to 2-polymatroids. For example,

$$S(f; 0, 1) = s(f; 0, 1) \text{ is the number of spanning sets of } f. \tag{4.11}$$

Since, obviously,

$$S(f^*; u, v) = S(f; v, u), \tag{4.12}$$

where f^* denotes the 2-dual of f, (4.11) follows from (4.8) and Proposition 2.2. This is straightforward to verify directly. It is also easily seen that

$$2^{|E|} = S(f; 1, 1) = s(f; 1, 1). \tag{4.13}$$

In fact, for any point on the hyperbola $uv = 1$, $S(f; u, v)$ is easily computed. A routine induction argument shows that

$$S(f; 1/v, v) = (1 + v^2)^{|E|} v^{-f(E)}. \tag{4.14}$$

This also generalizes a property of Tutte polynomials of matroids (see, for example, [8, (2.15)]).

Another easily established fact is

$$S(f; -u, -v) = (-1)^{f(E)} S(f; u, v). \tag{4.15}$$

We now consider a somewhat less trivial evaluation. Suppose that every element of a 2-polymatroid f on E has, independently of all other elements, a probability $1-p$ of being deleted from f and assume that $0 < p < 1$. We call the resulting restriction minor $\omega(f)$ a *random subpolymatroid* of f. Let $\Pr(f)$ denote the probability that $\omega(f)$ has the same rank as f. If $E = \{e\}$, we have $\Pr(f) = p$ if $f(e) \in \{1, 2\}$, and $\Pr(f) = 1$ if $f(e) = 0$. Moreover, if $|E| > 1$, then

$$\Pr(f) = \begin{cases} (1-p)\Pr(f\backslash e) + p\Pr(f/e) & \text{if } f(E-e) = f(e) \text{ and } f(e) \in \{1, 2\}; \\ p\Pr(f/e) & \text{if } f(E-e) = f(E) - 1 \text{ and } f(e) = 2; \\ \Pr(f\backslash(E-e))\Pr(f\backslash e) & \text{otherwise.} \end{cases}$$

It is now easily seen—for example, by Proposition 3.9—that

$$\Pr(f) = (1-p)^{|E| - f(E)/2} p^{f(E)/2} S(f; 0, p^{1/2}(1-p)^{-1/2}). \tag{4.16}$$

This also extends a result for matroids [3, Example 2.12].

Now let G be a graph, let $M(G)$ and f_G denote its cycle matroid and associated Boolean 2-polymatroid, respectively, and let $\omega(G)$ be the subgraph obtained by independently deleting edges with probability $1-p$. By taking f to be $M(G)$ in (4.16), we obtain a result noted in [14], namely, that if G is connected, then

$(1-p)^{|E|-r(M(G))/2}p^{r(M(G))/2}S(M(G);\ 0,\ p^{1/2}(1-p)^{-1/2})$ is the probability that $\omega(G)$ is connected. (4.17)

Another easy consequence of (4.16) is that if G has no isolated vertices, then

$(1-p)^{|E|-f_G(E)/2}p^{f_G(E)/2}S(f_G;0,p^{1/2}(1-p)^{-1/2})$ is the probability that $\omega(G)$ has no isolated vertices. (4.18)

Helgason [7] defined the *characteristic polynomial* $P(f;\lambda)$ of a polymatroid f by

$$P(f;\lambda)=\sum_{X\subseteq E}(-1)^{|X|}\lambda^{f(E)-f(X)}.$$

Evidently, if f is a 2-polymatroid,

$$P(f;\lambda)=(-1)^{f(E)}s(f;\ -\lambda,\ -1)=i^{f(E)}S(f;\ -i\lambda,\ i). \tag{4.19}$$

The combinatorial significance of characteristic polynomials is not restricted to 2-polymatroids. For example, it is shown in [7] that characteristic polynomials enumerate colourings of a hypergraph via a polymatroid associated with the hypergraph (see also below). In [20] it is shown that the critical problem of Crapo and Rota [4] extends to polymatroids representable over finite fields. Critical exponents are determined by an evaluation of the characteristic polynomial of the polymatroid.

Let G be a graph and f_G be its associated Boolean polymatroid. A set of vertices of G is *stable* if no edge has both endpoints in the set. Suppose each vertex of G is chosen independently of all the others with probability p, and let $A(G,p)$ denote the probability that the chosen set of vertices is stable. It is shown in [6] that if G has no isolated vertices, then

$$A(G,p)=\sum_{X\subseteq E}(-1)^{|X|}p^{f_G(X)}.$$

Hence,

$$A(G,p)=p^{f_G(E)}P(f_G;p^{-1})=(ip)^{f_G(E)}S(f_G;\ -ip^{-1},\ i). \tag{4.20}$$

A number of polynomials of graphs related to $A(G,p)$ have been studied. For references the reader is referred to the introduction to [6].

We now consider a class of polymatroids which generalize cycle matroids of graphs. We first fix terminology. A *hypergraph* is a triple $H=(V,\ E,\ \psi_H)$, where V and E are finite sets whose members are called *vertices* and *edges*, respectively, and ψ_H is a function from E into the power set of V. A *component* of H is a minimal nonempty subset V' of V with the property that if e is an edge of H, then either $\psi_H(e) \cap V' = \emptyset$ or $\psi_H(e) \subseteq V'$. Evidently the components of H partition V. For a subset A of E, let $\kappa(A)$ denote the number of components of the hypergraph $(V, A, \psi_H|_A)$. It is well known that the function $g_H: 2^E \to \mathbb{Z}$, defined, for all subsets A of E, by

$$g_H(A) = |V| - \kappa(A),$$

is a polymatroid. If H is, in fact, a graph, then g_H is just the cycle matroid of the graph. As noted earlier, some properties of characteristic polynomials of cycle matroids of graphs generalize. In particular, if H has no isolated vertices, then the number of proper weak k-colourings of H is equal to $k^{\kappa(E)}P(g_H; k)$.

The hypergraph H is a k-hypergraph if $|\psi(e)| \leqslant k$ for each edge e. Evidently, g_H is a 2-polymatroid if and only if H is a 3-hypergraph. Assume then, that H is a 3-hypergraph. A *triangular cactus* of H is a matching of g_H. A triangular cactus can be pictured as a tree-like or, more generally, a forest-like structure based on triangles. (For a discussion of triangular cacti in a slightly less general setting, see [12, Sect. 11.3].) It follows from (4.16), that

$S(g_H; 1, 0)$ is the number of triangular cacti of H, and (4.21)

$S(g_H; 0, 0)$ is the number of spanning triangular cacti of H. (4.22)

Assume that H is *connected*, that is, $\kappa(E) = 1$, and let $\omega(H)$ denote the hypergraph obtained by independently deleting edges with probability $1-p$. It follows from (4.16) that

$(1-p)^{|E| - g_H(E)/2} p^{g_H(E)/2} S(g_H; 0, p^{1/2}(1-p)^{-1/2})$ is the probability that $\omega(H)$ is connected. (4.23)

This generalizes (4.17).

We conclude this section with some observations on sums of rank functions of matroids. If r_1 and r_2 are matroids on a common ground set E, then it is well known (see, for example, [12, p. 410]) that $r_1 + r_2$ is a 2-polymatroid. Also, a set is independent in both r_1 and r_2 if and only if it is a matching in $r_1 + r_2$. It follows from (4.7), (4.8), and (4.11), respectively, that

$S(r_1 + r_2; 1, 0)$ is the number of common independent sets of r_1 and r_2; (4.24)

FIG. 5.1. The 2-polymatroid g_1.

$S(r_1+r_2;0,1)$ is the number of common spanning sets of r_1 and r_2; and (4.25)

$S(r_1+r_2;0,0)$ is the number of common bases of r_1 and r_2. (4.26)

Also, a straightforward argument proves that, for any matroid r,

$$S(r+r;u,v)=s(r;u^2,v^2). \tag{4.27}$$

5. PROOF OF THEOREM 3.14

We shall begin by considering a number of examples of Boolean 2-polymatroids. Each example is accompanied by a figure with the same number. This figure gives a geometric representation of the 2-polymatroid being considered along with a corresponding graph. In each case, τ is computed in two ways using the recursions in (3.6) and one deduces an identity that must be satisfied by a, b, c, d, m, n, x, y, and z if τ is to be well defined. The theorem will be proved by determining precisely which functions satisfy all these identities.

(5.1) EXAMPLE. By removing the point first, we find that $\tau(g_1)=ax+bz$. On the other hand, removing the line first gives $\tau(g_1)=cz+dy$. Hence

$$ax+bz=cz+dy. \tag{I1}$$

(5.2) EXAMPLE. By removing the line first, we find that $\tau(g_2)=mz^2+ny^2$. We can also determine $\tau(g_2)$ by removing a point first. Equating these two expressions and using the fact that $\tau(g_1)=cz+dy$, we obtain

$$mz^2+ny^2=a(b+c)\,z+(b^2+ad)\,y. \tag{I2}$$

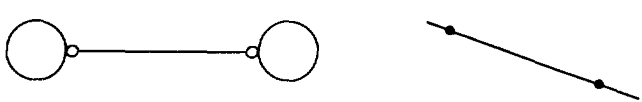


FIG. 5.2. The 2-polymatroid g_2.

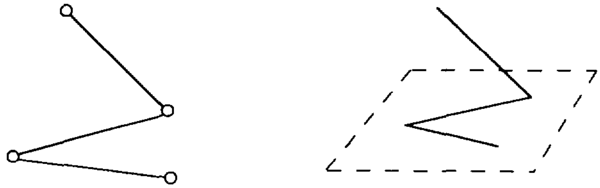


FIG. 5.2*. The 2-polymatroid g_3^*.

(5.2)* EXAMPLE. By using 2-polymatroid duality, we obtain from (I2) that

$$mx^2 + nz^2 = d(b+c)\,z + (c^2 + ad)\,x. \tag{I2)*}$$

The next example will be presented in more detail since the procedure used for it is typical of that used on all of the remaining examples. Two elements in the polymatroid are chosen so that no automorphism of the polymatroid maps one to the other. These two elements are then removed, first in one order and then in the other. Some cancellation of like terms will result and what is left is an identity which must be obeyed by a, b, c, d, m, n, x, y, and z if τ is to be well-defined. In the calculations below, we shall abbreviate $\tau(f)$ as simply (f). Each polymatroid will be represented geometrically, an element e for which $f(e) = 0$ being written as $\blacksquare$.

(5.3) EXAMPLE. On deleting and contracting the line 1 from g_3, we obtain

$$(g_3) = (\,=\!\!\bullet\!\!=\,) = m(\overset{3}{-\!\bullet\!-}) + n(\blacksquare\;\blacksquare).$$

Now, on deleting and contracting 3 from the polymatroid in the first term, we obtain

$$(g_3) = ma(-) + mb(\bullet) + n(\blacksquare\;\blacksquare).$$

If instead, we first remove 3 from g_3 and then remove 1 from the resulting deletion, we obtain

$$\begin{aligned}(g_3) &= a(=) + b(\bullet\!\bullet)\\ &= am(-) + an(\blacksquare) + b(\bullet\!\bullet).\end{aligned}$$

Therefore, on cancelling like terms in the two expressions for (g_3) and using the fact that $(-) = x$, $(\blacksquare) = y$, and $(\bullet) = z$, we obtain that

$$mbz + ny^2 = any + b[a(\bullet) + b(\blacksquare)],$$

FIG. 5.3. The 2-polymatroid g_3.

that is,

$$mbz + ny^2 = any + abz + b^2y.$$

Rewriting the last equation, we obtain the identity

$$y(ny - an - b^2) = bz(a - m). \tag{I3}$$

The next seven examples proceed similarly to the above so most of the details are omitted.

(5.4) EXAMPLE. From Fig. 5.4, we deduce that

$$0 = z(bc + dy - ad - bz). \tag{I4}$$

(5.5) EXAMPLE. From Fig. 5.4, we deduce that

$$mdz + nyz = bdy + adz + cny. \tag{I5}$$

(5.6) EXAMPLE. Removing 1 and 2 in the two possible orders and using the fact that $\tau(g_1) = cz + dy$, we obtain

$$mxz + nyz = acx + bcz + adz + bdy. \tag{I6}$$

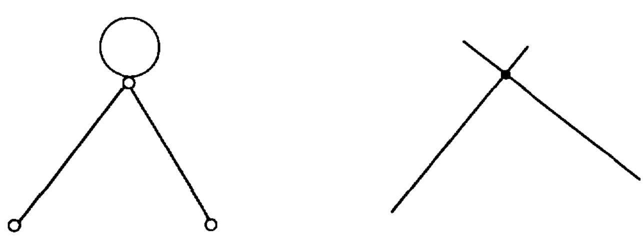


FIG. 5.4. The 2-polymatroid g_4.

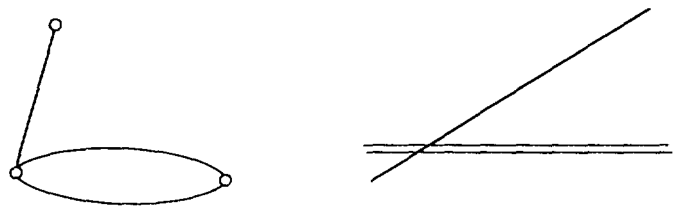


FIG. 5.5. The 2-polymatroid g_5.

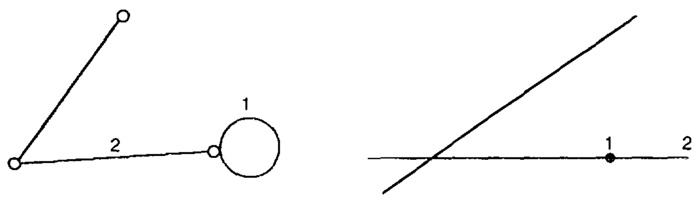


FIG. 5.6. The 2-polymatroid g_6.

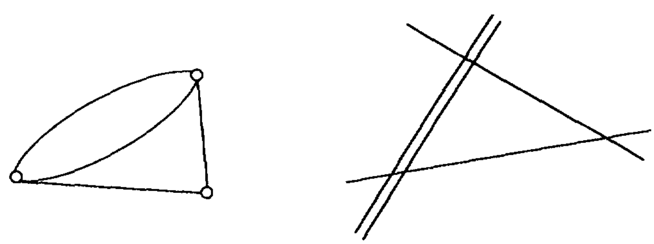


FIG. 5.7. The 2-polymatroid g_7.

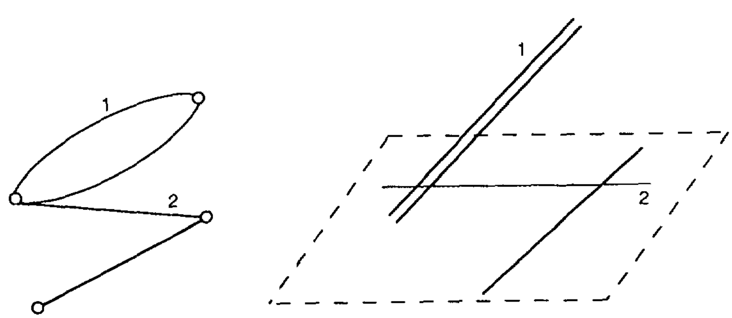


FIG. 5.8. The 2-polymatroid g_8.

(5.7) EXAMPLE. From Fig. 5.7, we deduce that

$$n(m-a)(yz-az-by)=0. \tag{I7}$$

(5.8) EXAMPLE. Evaluating $\tau(g_8)$ by removing the lines 1 and 2 in the two possible orders, we obtain the identity

$$n(m-a)(z^2-xy)=0. \tag{I8}$$

(5.9) EXAMPLE. Evaluating $\tau(g_9)$ by removing 1 and 2 in the two possible orders gives the identity

$$yz(ny-an-b^2)=b(a-m)(ax+bz).$$

But, by (I3),

$$y(ny-an-b^2)=bz(a-m).$$

Hence

$$b(a-m)(z^2-ax-bz)=0. \tag{I9}$$

(5.10) EXAMPLE. Evaluating $\tau(g_{10})$ by removing the elements 1 and 2 in the two possible orders gives the identity

$$(md-an)(z^2-ax-bz)=0. \tag{I10}$$

(5.11) EXAMPLE. Evaluating $\tau(g_{11})$ by removing the elements 1 and 2 in the two possible orders gives the identity

$$b(m-a)(az+by)=y^2(an+b^2-ny).$$

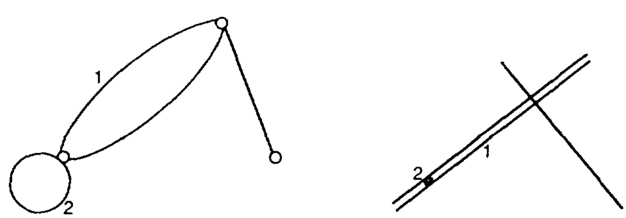


FIG. 5.9. The 2-polymatroid g_9.

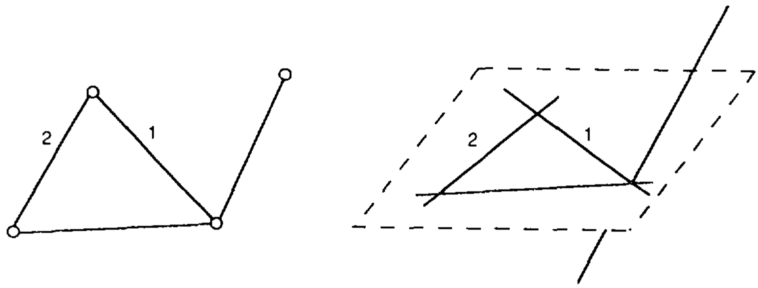


FIG. 5.10. The 2-polymatroid g_{10}.

But, by (I3),

$$byz(m-a) = y^2(an + b^2 - ny).$$

Hence

$$b(m-a)(yz - az - by) = 0. \tag{I11}$$

The rest of the proof of Theorem 3.14 will use a sequence of lemmas to show that the various identities noted in the above examples imply that a generalized Tutte invariant on $\mathcal{M}_2^G$ must satisfy one of (G1)–(G6). The reader is unlikely to be surprised by the fact that the proofs of these lemmas involve a lot of case analysis. The first four lemmas pick out some special cases for separate treatment.

(5.12) LEMMA. *If* $y = z = 0$, *then* (G3) *or* (G6) *holds.*

Proof. If $x = 0$, then τ is zero on all nonempty 2-polymatroids, and (G6) holds. If $x \neq 0$, then, by (I1), $ax = 0$, so $a = 0$. Thus (I2)* implies that $mx = c^2$. Hence (G3) holds. ∎

(5.13) LEMMA. *If* $n = d = b = 0$, *then* (G2), (G4), *or* (G6) *holds.*

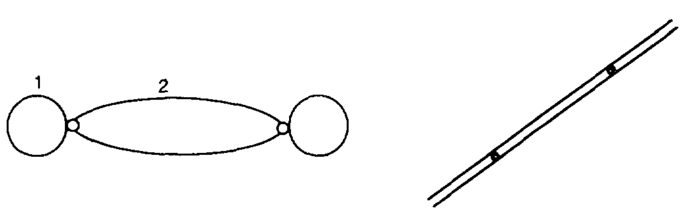


FIG. 5.11. The 2-polymatroid g_{11}.

Proof. If $x = z = 0$, then (G2) or (G6) holds. Thus we may assume that at least one of x and z is nonzero. By (I1),

$$ax = cz; \tag{E1.1}$$

by (I6),

$$mxz = acx; \tag{E1.2}$$

by (I2),

$$mz^2 = acz; \tag{E1.3}$$

and, by (I2)*,

$$mx^2 = c^2x. \tag{E1.4}$$

Substituting from (E1.1) into (E1.2) gives

$$mxz = c^2z. \tag{E1.5}$$

As x or z is nonzero, (E1.2) and (E1.3) imply that $mz = ac$. Moreover, (E1.4) and (E1.5) imply that $mx = c^2$. Hence (G4) holds. ∎

(5.14) LEMMA. *If $a = m$ and $d = n \neq 0$, then* (G1), (G3), (G5), *or* (G6) *holds.*

Proof. Substituting into (I5) and (I3), we obtain

$$dy(b + c - z) = 0 \tag{E2.1}$$

and

$$y(ny - mn - b^2) = 0. \tag{E2.2}$$

Suppose that

$$z = b + c \tag{E2.3}$$

and

$$ny = mn + b^2. \tag{E2.4}$$

Then substituting into (I1) gives

$$mx + b(b + c) = c(b + c) + mn + b^2.$$

Hence

$$mx = mn + c^2. \tag{E2.5}$$

Thus if m is nonzero, then (G1) holds. We may now suppose that $m=0$. Then, by (E2.5), $c=0$. Thus, by (I1) and (E2.4), $bz=dy$ and $ny=b^2$. Moreover, by (E2.3), $z=b$. Since $n=d$, it follows that $nz=bd$ and so (G5) holds.

We may now suppose that at least one of (E2.3) and (E2.4) fails. Then, since $d\neq 0$, (E2.1) and (E2.2) imply that $y=0$. If $z=0$, then, by Lemma 5.12, (G3) or (G6) holds. Hence we may assume that $z\neq 0$. By (I2), $mz^2=a(b+c)z$, and so $mz=m(b+c)$. Thus either (i) $z=b+c$, or (ii) $z\neq b+c$ and $m=0$. In case (i), by (I4),

$$bc-ad-b(b+c)=0.$$

Hence, as $a=m$, $n=d$, and $y=0$,

$$b^2+mn=ny.$$

This contradicts our assumption that one of (E2.3) and (E2.4) fails. Hence we may assume that (ii) holds. Then, by (I1), since $a=m=y=0$ and $z\neq 0$,

$$b=c.$$

Moreover, by (I6), $bcz=0$ so $b^2z=0$, and hence $b=0$ and $c=0$. Thus, by (I2)*, $n=0$; a contradiction. ▮

(5.15) LEMMA. *Suppose that*

$$z^2=ax+bz \tag{E3.1}$$

and

$$yz=az+by. \tag{E3.2}$$

Then

$$a(z^2-xy)=0, \tag{E3.3}$$

$$z[z^2-(mx+ny)]=0, \tag{E3.4}$$

$$a[xz-(cx+dz)]=0, \tag{E3.5}$$

and

$$mz(z^2-xy)=0. \tag{E3.6}$$

Proof. By (E3.1) and (E3.2), respectively,

$$yz^2=axy+byz$$

and

$$yz^2 = az^2 + byz.$$

Combining these two equations gives

$$a(z^2 - xy) = 0,$$

which is (E3.3).

Next we note that, by (I6), (E3.1), and (E3.2),

$$\begin{aligned} mxz + nyz &= c(ax + bz) + d(az + by) \\ &= cz^2 + dyz. \end{aligned}$$

Thus

$$z(mx + ny) = z(cz + dy),$$

and so, by (E3.1) and (I1),

$$z(mx + ny) = z^3, \tag{E3.7}$$

giving (E3.4).

Grouping the terms on the right-hand side of (I6) differently, we obtain from (E3.7) that

$$z^3 = a(cx + dz) + b(cz + dy).$$

Thus, by (I1) and (E3.1),

$$z^3 = a(cx + dz) + bz^2.$$

But, by (E3.1) again,

$$z^3 = axz + bz^2.$$

Combining the last two equations gives (E3.5).

Finally, by (I2), (E3.2), (I1), and (E3.1), we have

$$\begin{aligned} mz^2 + ny^2 &= b(az + by) + a(cz + dy) \\ &= byz + az^2 \\ &= z(by + az) \\ &= z^2y. \end{aligned}$$

Thus

$$mz^3 + ny^2z = z^3y.$$

Moreover, by (E3.7),

$$mxyz + ny^2z = z^3y.$$

Combining the last two equations gives (E3.6). ▌

With the above preliminaries, we now begin the main part of the case analysis for the proof of Theorem 3.14. We focus attention on (I9) and (I11). We shall assume that $a \neq m$. Under that assumption, the following three cases, which will be considered in the next three lemmas, exhaust all the possibilities:

(i) $b \neq 0$;

(ii) $b = 0$, $n \neq 0$; and

(iii) $b = n = 0$.

(5.16) LEMMA. *If* $a \neq m$ *and* $b \neq 0$, *then* (G2), (G3), *or* (G6) *holds.*

Proof. By (I9) and (I11), we have

$$z^2 = ax + bz \tag{E4.1}$$

and

$$yz = az + by. \tag{E4.2}$$

Thus, by Lemma 5.15, (G2) holds provided that neither a nor z is zero. If $z = 0$, then by (E4.2), $by = 0$ so $y = 0$ and, by Lemma 5.12, (G3) or (G6) holds. Hence we may assume that $z \neq 0$ and that $a = 0$. Then, by (E3.6), since $a \neq m$, we obtain that

$$z^2 = xy. \tag{E4.3}$$

Thus

$$cz^2 = cxy$$

so

$$cz^2 + dyz = cxy + dyz.$$

Hence

$$z(cz + dy) = y(cx + dz).$$

Therefore, by (E4.3), (E4.1), and (I1),

$$z(xy) = z(z^2) = z(cz + dy) = y(cx + dz).$$

Thus, as $y \neq 0$,

$$xz = cx + dz$$

and it follows by Lemma 5.15 that (G2) holds. ∎

(5.17) LEMMA. *If* $a \neq m$, $b = 0$, *and* $n \neq 0$, *then* (G2), (G3), *or* (G6) *holds.*

Proof. By (I7) and (I8),

$$yz = az + by = az \tag{E5.1}$$

and

$$z^2 = xy. \tag{E5.2}$$

By Lemma 5.12, we may assume that y or z is nonzero, since otherwise (G3) or (G6) holds. But, by (E5.2), if $y = 0$, then $z = 0$. Hence we may assume $y \neq 0$. Since n is also nonzero, it follows by (I3) that $y = a$.

If $z = 0$, then, by (E5.2), $x = 0$. Thus, by (I1), $dy = 0$. Hence, by (I2), $ny^2 = 0$; a contradiction. Therefore we may assume that $z \neq 0$. By (E5.2), since $y = a$ and $b = 0$,

$$z^2 = xy = ax = ax + bz.$$

Combining this with (E5.1), we see that both (E3.1) and (E3.2) hold. Since neither a nor z is zero, (G2) holds by Lemma 5.15. ∎

(5.18) LEMMA. *If* $a \neq m$ *and* $b = n = 0$, *then* (G2), (G3), (G4), *or* (G6) *holds.*

Proof. By (I5),

$$dz = 0.$$

If $d = 0$, then, by Lemma 5.13, (G2), (G4), or (G6) holds. Thus we may assume that $d \neq 0$. Hence $z = 0$. By (I2) and (I1),

$$0 = ady = a(cz + dy) = a(ax + bz) = a^2x.$$

Thus $ax = 0$, so, by (I1), $dy = 0$. Hence, as $d \neq 0$, we have $y = 0$. Thus, by Lemma 5.12, (G3) or (G6) holds. ∎

Having just dealt with all the possibilities when $a \neq m$, we now assume that $a = m$. Then, by (I3), (I5), and (I10), we have

$$y(ny - an - b^2) = 0, \tag{E6.1}$$

$$y(nc + bd - nz) = 0, \tag{E6.2}$$

and

$$m(d-n)(z^2-ax-bz)=0. \tag{E6.3}$$

The following three cases, which will be treated in the next three lemmas, exhaust all the possibilities when $a=m$:

(i) $y=0$;

(ii) $y\neq 0$ and $b=0$; and

(iii) $y\neq 0$ and $b\neq 0$.

(5.19) LEMMA. *If* $a=m$ *and* $y=0$, *then* (G1), (G2), (G3), (G4), (G5), *or* (G6) *holds.*

Proof. If $z=0$, then, by Lemma 5.12, (G3) or (G6) holds. Hence we may assume that $z\neq 0$.

Suppose first that $m=0$. Then $a=0$ and, by (I1), $bz=cz$, so, as $z\neq 0$,

$$b=c.$$

By (I6), $bcz=0$. Hence, as $z\neq 0$, we obtain $b=c=0$. Substituting into (I2)* gives that $n=0$. Then $ny=b^2$, $dy=bz$, and $nz=bd$. Thus (G5) holds.

We may now suppose that $m\neq 0$. Then, by (I2),

$$mz^2=a(b+c)\,z,$$

so, as $a=m\neq 0$ and $z\neq 0$,

$$z=b+c. \tag{E7.1}$$

Next we distinguish the following three cases:

(I) $n=d\neq 0$;

(II) $n=d=0$; and

(III) $n\neq d$.

In case (I), by Lemma 5.14, (G1), (G3), (G5), or (G6) holds. In case (II), by (I4),

$$bc-bz=0.$$

Substituting from (E7.1), we obtain $b^2=0$, so $b=0$ and hence, by Lemma 5.13, one of (G2), (G4), or (G6) holds. In case (III), by (E6.3),

$$z^2=ax+bz. \tag{E7.2}$$

Hence, by (I1), $z^2 = cz$, so, as $z \neq 0$,

$$c = z. \tag{E7.3}$$

Thus, by (E7.1),

$$b = 0.$$

Therefore, by (I4), $zad = 0$, so

$$d = 0.$$

Thus, by (I2)*,

$$mx^2 + nz^2 = c^2x. \tag{E7.4}$$

But, by (E7.2), as $a = m$ and $b = 0$,

$$z^2 = mx.$$

Substituting this into the first term of (E7.4) and using (E7.3), we obtain

$$xz^2 + nz^2 = xz^2.$$

Hence $n = 0$. But, since $d = 0$, we have $n = d$; a contradiction. ∎

(5.20) LEMMA. *If* $a = m$, $y \neq 0$, *and* $b = 0$, *then* (G1), (G2), (G4), *or* (G6) *holds.*

Proof. As $y \neq 0$ and $b = 0$, (E6.1), (E6.2), and (E6.3) imply that

$$n(y - a) = 0, \tag{E8.1}$$

$$n(c - z) = 0, \tag{E8.2}$$

and

$$m(d - n)(z^2 - ax) = 0. \tag{E8.3}$$

Moreover, by (I4),

$$dz(y - a) = 0. \tag{E8.4}$$

Suppose first that $m = 0$. Then, by (E8.1), since $a = m$ and $y \neq 0$, we obtain that $n = 0$. Moreover, by (E8.4), $ydz = 0$. Again, as $y \neq 0$, we have $dz = 0$. If $d = 0$, then $n = b = d = 0$, and so, by Lemma 5.13, (G2), (G4), or (G6) holds. If $d \neq 0$, then $z = 0$ and so, by (I1), $dy = 0$; a contradiction.

We may now assume that $m \neq 0$. By (I2) and (I1),

$$mz^2 + ny^2 = acz + ady = a(cz + dy) = a^2x.$$

But, by (E8.1), $ny = an$. Hence, as $a = m$,

$$mz^2 + mny = m^2x.$$

Since $m \neq 0$, it follows that

$$z^2 + ny = mx. \tag{E8.5}$$

Moreover, by (I6),

$$mxz + nyz = acx + adz.$$

Thus, by (E8.1),

$$mxz + anz = acx + adz,$$

so, as $a = m \neq 0$,

$$xz + nz = cx + dz. \tag{E8.6}$$

We shall break the rest of the proof into two cases: (I) $n = 0$; and (II) $n \neq 0$. Assume that (I) holds. Then we may suppose that $d \neq 0$; otherwise $n = b = d = 0$ and so (G2), (G4), or (G6) holds. By (E8.5),

$$z^2 = ax = ax + bz. \tag{E8.7}$$

Moreover, by (E8.4),

$$z(y - a) = 0.$$

If $z = 0$, then, by (E8.7), $ax = 0$ and so, by (I1), $dy = 0$; a contradiction. Thus we may suppose that $z \neq 0$. Hence $y = a$. Then

$$yz = az = az + by.$$

Hence, by (E8.7), both (E3.1) and (E3.2) hold. Therefore, by Lemma 5.15, since neither a nor z is zero, (G2) holds.

In case (II), by (E8.1) and (E8.2),

$$y = a \tag{E8.8}$$

and

$$c = z. \tag{E8.9}$$

Thus, by (E8.6),

$$nz = dz. \tag{8.10}$$

If $n = d$, then, since $a = m \neq 0$ and $b = 0$, (E8.5), (E8.8), and (E8.9) imply that $mx = mn + c^2$, $ny = mn + b^2$, and $z = b + c$; that is, (G1) holds. Thus we may assume that $n \neq d$. Then, by (E8.10), $z = 0$. Thus, by (E8.3), $amx = 0$, so $x = 0$. But now (E8.5) implies that $ny = 0$; a contradiction. ∎

(5.21) LEMMA. *If* $a = m$, $y \neq 0$, *and* $b \neq 0$, *then* (G1), (G2), (G3), (G5), *or* (G6) *holds.*

Proof. By (E6.1) and (E6.2), respectively,

$$ny = an + b^2 \tag{E9.1}$$

and

$$nz = nc + bd. \tag{E9.2}$$

By (E9.1), since $b \neq 0$, it follows that $n \neq 0$. We may also assume that $n \neq d$ otherwise, by Lemma 5.14, (G1), (G3), (G5), or (G6) holds.

Now rewriting (I2) we have

$$mz(z - b - c) = y(b^2 + md - ny). \tag{E9.3}$$

But, by (E9.1),

$$b^2 - ny = -an.$$

Substituting into (E9.3) and using the fact that $a = m$, we obtain

$$mz(z - b - c) = y(md - an) = my(d - n).$$

Hence

$$mz(nz - nc - nb) = mny(d - n).$$

But, by (E9.2), $nz - nc = bd$. Thus

$$mzb(d - n) = mny(d - n).$$

Therefore, as $d \neq n$,

$$mbz = mny. \tag{E9.4}$$

The rest of the proof will be broken into the two cases: (I) $m \neq 0$; and (II) $m = 0$.

In case (I), by (E9.4) and (E6.3),

$$bz = ny \tag{E9.5}$$

and

$$z^2 = ax + bz. \tag{E9.6}$$

Moreover, by (E9.1),

$$nyz = anz + b^2z,$$

so, by (E9.5),

$$nyz = anz + bny.$$

Since $n \neq 0$, it follows that

$$yz = az + by. \tag{E9.7}$$

But neither n nor y is zero, so, by (E9.5), $z \neq 0$. Hence, by (E9.6), (E9.7), and Lemma 5.15, (G2) holds.

In case (II), by (E9.1),

$$ny = b^2. \tag{E9.8}$$

Moreover, by (I5) and (I6),

$$ncy = bcz. \tag{E9.9}$$

If $c = 0$, then, by (I1) and (E9.2), $dy = bz$ and $nz = bd$. Thus (G5) holds. Hence we may assume that $c \neq 0$. Then, by (E9.8) and (E9.9),

$$b^2c = ncy = bcz.$$

So, as neither b nor c is zero,

$$b = z. \tag{E9.10}$$

By (I2)* and (E9.2), respectively, we have

$$nz^2 = bdz + cdz + c^2x$$

and

$$nz^2 = ncz + bdz.$$

Thus

$$ncz = cdz + c^2x.$$

Hence, as $c \neq 0$,

$$nz = cx + dz. \tag{E9.11}$$

But, by (E9.2) and (E9.10),

$$nz = nc + bd = nc + dz,$$

so, as $c \neq 0$,

$$n = x. \tag{E9.12}$$

Thus, by (E9.11),

$$xz = cx + dz$$

By (E9.8), (E9.10), and (E9.12),

$$z^2 = xy.$$

Moreover, one now easily checks that

$$z^2 = ax + bz = mx + ny = cz + dy$$

and

$$yz = az + by.$$

We conclude that (G2) holds. ▮

This completes the proof of Theorem 3.14. ▮

It is natural to try to extend this theorem to k-polymatroids when $k \geqslant 3$. But, in view of the lengthy case analysis needed in the above proof, such an extension is unlikely to be easy to prove.

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