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Tutte Invariants for 2-Polymatroids

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Introduction

This paper describes a theory of Tutte invariants for 2-polymatroids that parallels the corresponding theory for matroids. The paper is a slightly informal exposition of the main results of [13] and contains no proofs. In particular, it shows that 2-polymatroid invariants obeying deletion-contraction recursions arise in the enumeration of many combinatorial structures including matchings and perfect matchings in graphs, colourings in hypergraphs, and common bases in pairs of matroids. The main result is that, just as for matroids, there is a two-variable polynomial that is essentially the universal Tutte invariant for 2-polymatroids.

Section 1 of the paper presents some basic facts about polymatroids. Section 2 summarizes the theory of Tutte-Grothendieck invariants for matroids which we are seeking to generalize, and Section 3 describes this generalization. The graph and matroid terminology used throughout will follow Bondy and Murty [1] and Oxley [11], respectively.

1. Polymatroids

We begin with an example. Let M be the rank-3 matroid that is represented geometrically in Figure 1. Now pick some set of flats of M, say the lines that are labelled 1,2,3, and 4 and the points labelled 5,6,7,8, and 9. Let $E = \{1, 2, \ldots, 9\}$ and, for each subset X of E, let $f(X)$ be the rank in M of the union of the flats that are labelled by members of X. So, for example,

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This paper is a summary of the main results of [13], which has been submitted for publication elsewhere.

$$f(\{1\}) = f(\{2\}) = f(\{3\}) = f(\{4\}) = 2,$$

$$f(\{5\}) = f(\{6\}) = f(\{7\}) = f(\{8\}) = f(\{9\}) = 1,$$

$$f(\{1,5\}) = f(\{1,5,6\}) = 2, f(\{1,2\}) = 3, \text{ and so on.}$$

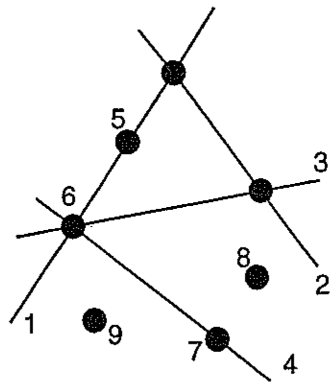

Figure 1

Then the pair (E, f) is an example of a polymatroid.

Next consider slightly modifying this example by allowing the set E to be a multiset of flats of M. This amounts geometrically to adding flats in parallel to existing flats as shown, for instance, in Figure 2. In that case, $E = \{1, 1', 2, 3, 3', 4, 5, 5', 5'', 6, 7, 7', 8, 9\}$ and, for example, $f(\{1, 1'\}) = 2$,

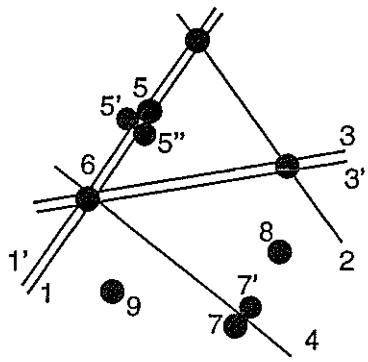

Figure 2

$f(\{1, 5, 5'\}) = 2$, and so on. Again, (E, f) is an example of a polymatroid.

Formally, a *polymatroid* (E, f) consists of a finite set E and a function $f : 2^E \to \mathbb{Z}$ such that

(i) $f(\emptyset) = 0$;

(ii) if $X \subseteq Y$, then $f(X) \leq f(Y)$; and

(iii) if $X, Y, \subseteq E$, then $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$.

But we find it easier to think of polymatroids geometrically as in the above examples. Indeed, every polymatroid arises as a multiset of flats of some matroid in the manner described there [7, 8, 10].

This paper will focus on 2-polymatroids, where, for a positive integer k, a *k-polymatroid* is a polymatroid (E, f) such that $f(\{e\}) \leq k$ for all e in E. Thus a 1-polymatroid is just the rank function of a matroid, and both of the examples looked at earlier are 2-polymatroids. Geometrically, every 2-polymatroid can be viewed as consisting of a multiset of lines, points, and loops of some matroid.

Two well-known classes of 2-polymatroids will receive the most attention here. The members of the first class arise from graphs in the following way. Let G be a graph having edge set E and, for all $X \subseteq E$, let $f_G(X)$ be $|V(X)|$, the number of vertices of G that are incident with some edge in X. It is not difficult to check that (E, f_G) is indeed a 2-polymatroid. Comparing this 2-polymatroid with the more familiar cycle matroid $M(G)$ of G, we note that the rank of X in $M(G)$ is $|V(X)| - k(X)$ where $k(X)$ is the number of components of the induced graph $G[X]$. Moreover, unlike $M(G)$, the 2-polymatroid (E, f_G) uniquely determines G up to the possible presence of isolated vertices.

Our second fundamental class of examples of 2-polymatroids arises from matroids. Let M_1 and M_2 be matroids on a common ground set E and, for all $X \subseteq E$, let $f(X) = r_1(X) + r_2(X)$ where r_i is the rank function of M_i. Since each of (E, r_1) and (E, r_2) is a 1-polymatroid, it is easy to show that $(E, r_1 + r_2)$ is a 2-polymatroid.

2. Tutte-Grothendieck invariants for matroids

Much of the motivation for our results on 2-polymatroid isomorphism invariants derives from the well-established theory of Tutte-Grothendieck invariants for matroids. This theory, which grew out of work of Tutte [14] for graphs, is reviewed in detail in [3]. We now briefly summarize some of the relevant aspects of the theory.

Let $\mathfrak{M}$ be a class of matroids that is closed under isomorphism and the taking of minors. A function ψ on $\mathfrak{M}$ taking values in a field K is an *isomorphism invariant* if $\psi(M) = \psi(N)$ whenever $M \cong N$.

Several numbers that one can associate with a matroid M such as its number of bases, its number of independent sets, and its number of spanning sets obey the following two basic recursions:

(2.1) $\psi(M) = \psi(M\backslash e)\psi(M|\{e\})$ *if e is a separator (a loop or coloop) of M; and*

(2.2) *for some fixed non-zero members σ and τ of K,*

$$\psi(M) = \sigma\psi(M\backslash e) + \tau\psi(M/e)$$

if e is not a separator of M.

An isomorphism invariant on $\mathfrak{M}$ that obeys (2.1) and (2.2) is called a *generalized T-G invariant* on $\mathfrak{M}$. There are many well-known important examples of such invariants; for instance, the chromatic polynomial is a generalized T-G invariant on the class of graphic matroids as is the flow polynomial. One particularly attractive feature of these invariants is that they are all evaluations of a certain universal invariant. To state this result formally, we shall need another definition. For a matroid M having ground set E and rank function r, the *(matroid) rank generating function* is given by

(2.3) $s(M;u,v) = \sum_{X\subseteq E} u^{r(E)-r(X)}v^{|X|-r(X)}$,

or, equivalently,

(2.4) $s(M;u,v) = \sum_{X\subseteq E} u^{r(E)-r(X)}v^{r^*(E)-r^*(E-X)}$.

It is not difficult to show that this function is a generalized T-G invariant with $\sigma = \tau = 1$. Moreover, for the two single-element matroids, $U_{1,1}$ and $U_{0,1}$, which consist of a single coloop and a single loop, respectively,

$$s(U_{1,1};u,v) = u+1 \text{ and } s(U_{0,1};u,v) = v+1.$$

These matroids are distinguished here because they are the only irreducible matroids with respect to the operations in (2.1) and (2.2).

Extending a result of Brylawski [2], Oxley and Welsh [12] proved that every generalized T-G invariant is easily expressible in terms of the rank-generating function:

(2.5) THEOREM. *Let $\mathfrak{M}$ be a class of matroids that is closed under isomorphism and the taking of minors. If ψ is a generalized T-G invariant from $\mathfrak{M}$ into a field K such that $\psi(U_{1,1}) = x$ and $\psi(U_{0,1}) = y$, then, for all M in $\mathfrak{M}$,*

$$\psi(M) = \sigma^{|E(M)|-r(M)}\tau^{r(M)}s(M;\frac{x}{\tau}-1,\frac{y}{\sigma}-1).$$

This result is more usually stated in terms of the Tutte polynomial $t(M;x,y)$, where

$$t(M;x,y) = s(M;x-1,y-1).$$

However, the above form of the result extends more naturally to 2-polymatroids. Some well-known basic evaluations of the rank generating function are as follows:

(2.6) $s(M;1,0)$ is the number of independent sets of M;

(2.7) $s(M;0,0)$ is the number of bases of M; and

(2.8) $s(M;0,1)$ is the number of spanning sets of M.

3. Isomorphism invariants for 2-polymatroids

Our approach to developing a theory of isomorphism invariants for 2-polymatroids will be to try to mimic the corresponding theory for matroids. But there are several potential problems that one needs to solve.

Firstly, what does it mean for an element e to be a separator in a 2-polymatroid (E,f)? Here we follow Cunningham [4] and define e to be a *separator* if $f(e)+f(E-e)=f(E)$. It should be noted that whereas the separators in a matroid are of just two types, loops and points, those in a 2-polymatroid are of three types: loops, points, and lines.

Next we need to define deletion and contraction in a 2-polymatroid (E,f). Deletion is straightforward; we define it in terms of restriction of f: if $A \subseteq E$ and $X \subseteq E-A$, then

$$(f\backslash A)(X)=f(X).$$

For contraction, we again look to matroids. If r_M is the rank function of a matroid M on E and $A \subseteq E$, then the rank function of M/A is defined by

$$r_{M/A}(X)=r_M(X\cup A)-r_M(A)$$

for all $X \subseteq E-A$. This suggests defining contraction in a 2-polymatroid (E,f) analogously, that is,

$$(f/A)(X)=f(X\cup A)-f(A)$$

for all $X \subseteq E-A$ [6]. It is not difficult to show that $(E-A, f/A)$ is indeed a 2-polymatroid. Moreover, this definition of contraction is consistent with the matroid definition in another sense. If the polymatroid f is represented as a multiset E of flats of a matroid M and $A \subseteq E$, then f/A has a natural representation as a multiset of flats of $M/\left(\cup_{a\in A} a\right)$.

We now have analogues for 2-polymatroids of two of the three fundamental matroid operations of deletion, contraction, and the taking of duals. The basic link between these operations in the matroid case is

$$M^*\backslash e=(M/e)^*. \tag{3.1}$$

Whittle [15] proposed that duality for 2-polymatroids should be an idempotent operation for which the analogue of (3.1) always holds. Moreover, he showed that if this occurs, that is, if, for all 2-polymatroids (E,f) and all e in E, $(f^*)^*=f$ and $f^*\backslash e=(f/e)^*$, then the dual (E,f^*) of (E,f) must be defined by

(3.2) $$f^*(X) = 2|X| + f(E - X) - f(E)$$

for all $X \subseteq E$. The last equation should be compared with the formula for the usual dual of a matroid rank function r, which is given by

$$r^*(X) = |X| + r(E - X) - r(E).$$

Next we consider the elements in a 2-polymatroid that are not separators. Such elements obey one of the following three conditions, where the dual f^* of f is its 2-polymatroid dual:

(i) $f(E - e) = f(E)$ and $f^*(E - e) = f^*(E) - 1$;

(ii) $f(E - e) = f(E) - 1$ and $f^*(E - e) = f^*(E)$; and

(iii) $f(E - e) = f(E)$ and $f^*(E - e) = f^*(E)$.

Note that if f is a matroid rank function and f^* denotes the rank function of the dual matroid, then those elements obeying (i) and (ii) above are precisely the loops and coloops, respectively, of the matroid. Conditions (i), (ii), and (iii) are equivalent to (i)′, (ii)′, and (iii)′, respectively, where (i)′ – (iii)′ are as follows:

(i)′ $f(E - e) = f(E)$ and $f(e) = 1$;

(ii)′ $f(E - e) = f(E) - 1$ and $f(e) = 2$; and

(iii)′ $f(E - e) = f(E)$ and $f(e) = 2$.

In view of the existence of these three different types of nonseparator elements in 2-polymatroids, the definition of a generalized Tutte invariant for 2-polymatroids, which we shall give next, will allow three distinct variants on the fundamental deletion-contraction recursion.

Let $\mathfrak{N}$ be a class of 2-polymatroids that is closed under isomorphism, deletion, and contraction. Assume that $\mathfrak{N}$ contains $U_{0,1}, U_{1,1}$, and $U_{2,1}$, the single-element 2-polymatroids of ranks zero, one, and two which correspond to a loop, a point, and a line. An isomorphism invariant ψ on $\mathfrak{N}$ is a *generalized Tutte invariant* for $\mathfrak{N}$ if, whenever f is a member of $\mathfrak{N}$ having ground set E and $e \in E$, $\psi(f) \in \mathbb{C}[x, y, z, a, b, c, d, m, n]$ where

(3.3) $$\psi(U_{2,1}) = x, \ \psi(U_{0,1}) = y, \ \text{ and } \psi(U_{1,1}) = z;$$

(3.4) $\psi(f) = \psi(f\backslash(E - e))\psi(f\backslash e)$ if e is a separator of f;

and

(3.5)

$$\psi(f) = \begin{cases} a\psi(f\backslash e) + b\psi(f/e) & \text{if } f(E-e) = f(E) \text{ and } f(e) = 1; \\ c\psi(f\backslash e) + d\psi(f/e) & \text{if } f(E-e) = f(E) - 1 \text{ and } f(e) = 2; \\ m\psi(f\backslash e) + n\psi(f/e) & \text{if } f(E-e) = f(E) \text{ and } f(e) = 2. \end{cases}$$

An important example of such an invariant is the *2-polymatroid rank generating function* which is defined as follows:

$$(3.6) \qquad S(f;u,v) = \sum_{X \subseteq E} u^{f(E)-f(X)} v^{2|X|-f(X)}$$

or, equivalently,

$$(3.7) \qquad S(f;u,v) = \sum_{X \subseteq E} u^{f(E)-f(X)} v^{f^*(E)-f^*(E-X)}.$$

Indeed, it is not difficult to check that, $S(f;u,v)$ is a generalized Tutte invariant on the class of all 2-polymatroids having

$x = 1+u^2,\ y = 1+v^2,\ z = u+v,\ m = n = 1,\ a = d = 1,\ b = v,$ and $c = u.$

The reader should note the similarity between (2.4) and (3.7), the difference being that, in the first, the duality is matroid duality while, in the second, it is 2-polymatroid duality. One striking difference occurs here between $S(f;u,v)$ and an arbitrary generalized T-G invariant for matroids. For the latter, the four parameters, $x, y, \sigma,$ and τ are independent. But, for the former the nine parameters, $x, y, z, m, n, a, b, c,$ and d, are far from being independent. A natural question here is whether some such dependence is forced for all generalized Tutte invariants for 2-polymatroids. Our main result will answer this. Before presenting it, we look at certain interesting evaluations of $S(f;u,v)$ for the two special classes of 2-polymatroids distinguished earlier. We begin with the analogues of (2.6)–(2.8).

Recall that, for a graph $G, f_G(X) = |V(X)|$ for all $X \subseteq E(G)$. It is not difficult to see that

(3.8) $S(f_G;1,0)$ is the number of matchings of G.

Moreover, if G has no isolated vertices, then

(3.9) $S(f_G;0,0)$ is the number of perfect matchings of G; and

(3.10) $S(f_G;0,1)$ is the number of subsets of $E(G)$ that cover $V(G)$.

Now suppose that r_1 and r_2 are the rank functions of matroids M_1 and M_2 on E. Then

(3.11) $S(r_1+r_2;1,0)$ is the number of common independent sets of M_1 and M_2;

(3.12) $S(r_1+r_2;0,0)$ is the number of common bases of M_1 and M_2; and

(3.13) $S(r_1+r_2;0,1)$ is the number of common spanning sets of M_1 and M_2.

Generalizing the above, we note that, for an arbitrary 2-polymatroid (E,f),

(3.14) $S(f;1,0)$ is the number of matchings of (E,f), and

(3.15) $S(f;0,1)$ is the number of spanning sets of (E,f),

where a *matching* is a set X such that $f(X) = 2|X|$, while a *spanning set* is a set Y for which $f(Y) = f(E)$.

The 2-polymatroid rank generating function is closely related to the matroid rank generating function. Indeed, if $s(f;u,v)$ is defined for a 2-polymatroid f by simply replacing r by f in (2.3), then

(3.16) $S(f;u,v) = v^{f(E)}s(f;uv^{-1},v^2)$ provided $v \neq 0$.

The last observation may suggest that $S(f;u,v)$ contains little more information than $s(f;u,v)$. In practice, however, several of the more interesting evaluations of $S(f;u,v)$ arise when $v = 0$. For instance, if G is a graph, then

(3.17) $u^{f_G(E)/2}\, S\left(f_G;u^{-1/2},0\right)$ is the matching generating polynomial $\sum_{k\geq 0} m_k u^k$ of G where m_k is the number of k-edge matchings of G.

If the graph G has n isolated vertices and $i = \sqrt{-1}$, then

(3.18) $u^n\, i^{f_G(E)}\, S\left(f_G;-iu,0\right)$ is the matching defect polynomial of G (Lovász and Plummer [9]).

Among the interesting properties of $S(f;u,v)$ that are easily proved are the following:

(3.19) $S\left(f^*;u,v\right) = S(f;v,u)$;

(3.20) $S(f;1,1) = 2^{|E|}$;

(3.21) $S(f;-u,-v) = (-1)^{f(E)}S(f;u,v)$; and

(3.22) $S(f;\frac{1}{u},u) = \left(1+u^2\right)^{|E|}u^{-f(E)}$ provided $u \neq 0$.

The matching generating and matching defect polynomials are just two examples of several single-variable polynomials that arise as special cases of $S(f;u,v)$. For example, if G is a graph without isolated vertices and $\omega(G)$ is a random subgraph of G obtained by independently deleting each edge of G with probability $1-p$, then

(3.23) $(1-p)^{|E|-f_G(E)/2}\, p^{f_G(E)/2}\, S\left(f_G;0,p^{1/2}(1-p)^{-1/2}\right)$ is the probability that $\omega(G)$ has no isolated vertices.

Another evaluation of $S(f;u,v)$ is the stability polynomial $A(G;p)$ of a graph G, a polynomial that has been studied by a number of authors (see, for example, Farr [5]). For G having no isolated vertices, $A(G;p)$ is defined as follows. Suppose that the vertices of G are chosen independently, each with probability p. Then $A(G;p)$ is the probability that the chosen set of vertices is stable. Farr showed that $A(G;p) = \sum_{X\subseteq E}(-1)^{|X|}\,p^{f_G(X)}$. Hence

(3.24) $A(G;p) = (ip)^{f_G(E)}\, S\left(f_G; -ip^{-1}, i\right).$

Finally we note that $S(f;u,v)$ has several important applications to hypergraphs. For instance,

(3.25) $i^{f(E)}\ S(f;-i\lambda,i) = \sum\limits_{X\subseteq E} (-1)^{\,|X|}\ \lambda^{f(E)-f(X)} = P(f;\lambda)$ where $P(f;\lambda)$ is the characteristic polynomial of f (Helgason [7]), which enumerates colourings of a hypergraph in the same way that the chromatic polynomial enumerates colourings in a graph.

Evidently 2-polymatroid rank generating functions arise in a wide variety of contexts. The next theorem, the main result of the paper, indicates why these functions are so pervasive by noting that $S(f;u,v)$ is essentially the universal Tutte invariant for 2-polymatroids.

(3.26) THEOREM. *Let ψ be a generalized Tutte invariant on the class of all 2-polymatroids and suppose that at most two of $x, y, z, a, b, c, d, m,$ and n are identically zero. Then one of the following occurs:*

(i) $a = m; d = n; mx = mn + c^2; ny = mn + b^2; z = b + c; m \neq 0; n \neq 0;$ *and for all 2-polymatroids f,* $\psi(f) = m^{|E|-f(E)/2}\ S\left(f; \frac{c}{(mn)^{1/2}}, \frac{b}{(mn)^{1/2}}\right);$

(ii) $z^2 = xy = ax + bz = cz + dy = mx + ny;\ yz = az + by;\ xz = cx + dz;$ *and, for all 2-polymatroids f,* $\psi(f) = Q(f)$ *where*

$$Q(f) = \begin{cases} y^{|E|-f(E)} z^{f(E)} & \text{if } f(E) \leq |E|; \\ x^{f(E)-|E|} z^{f^*(E)} & \text{if } f(E) \geq |E|. \end{cases}$$

Of the two functions arising here, $S(f;u,v)$ is, by now, quite familiar. The other function Q is basically trivial. The only information it conveys about (E,f) is the cardinality of E and the value of $f(E)$. Thus, in the 2-polymatroid case, just as in the matroid case, there is essentially a unique universal Tutte invariant.

The proof of the theorem involves looking at a number of small 2-polymatroids. For each of these, one evaluates ψ in two different ways. For instance, if (E,f) is represented geometrically by a single point on a line, then, on deleting and contracting the line, we get

$$\begin{aligned} \psi(\bullet\!\!/) &= c\psi(\bullet) + d\psi(U_{0,1}) \\ &= cz + dy. \end{aligned}$$

On the other hand, deleting and contracting the point gives

$$\begin{aligned} \psi(\bullet\!\!/) &= a\psi(/) + b\psi(\bullet) \\ &= ax + bz. \end{aligned}$$

Thus, for ψ to be well-defined,

$$cz + dy = ax + bz.$$

By looking at several other examples, one obtains a number of other relations between the nine variables involved. A detailed case analysis of these relations leads eventually to the result. In fact, one can drop the restriction that at most two of x, y, z, a, b, c, d, m, and n are zero for, in so doing, one merely admits six more trivial invariants each of which is a monomial conveying very limited information.

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