Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Combinatorial Theory, Series B 92 (2004) 257–293
Journal of Combinatorial Theory Series B
http://www.elsevier.com/locate/jctb

The structure of the 3-separations of 3-connected matroids

James Oxley,[a,1] Charles Semple,[b,2] and Geoff Whittle[c,2]

[a] *Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana, USA*
[b] *Department of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand*
[c] *School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand*

Received 13 August 2003

Abstract

Tutte defined a k-separation of a matroid M to be a partition (A,B) of the ground set of M such that $|A|,|B|\geqslant k$ and $r(A)+r(B)-r(M)<k$. If, for all $m<n$, the matroid M has no m-separations, then M is n-connected. Earlier, Whitney showed that (A,B) is a 1-separation of M if and only if A is a union of 2-connected components of M. When M is 2-connected, Cunningham and Edmonds gave a tree decomposition of M that displays all of its 2-separations. When M is 3-connected, this paper describes a tree decomposition of M that displays, up to a certain natural equivalence, all non-trivial 3-separations of M.

MSC: 05B35

Keywords: 3-Connected matroid; Tree decomposition; 3-Separation; Tutte connectivity

1. Introduction

One of Tutte's many important contributions to matroid theory was the introduction of the general theory of separations and connectivity [10] defined in the abstract. The structure of the 1-separations in a matroid is elementary. They induce a partition of the ground set which in turn induces a decomposition of the

E-mail addresses: oxley@math.lsu.edu (J. Oxley), c.semple@math.canterbury.ac.nz (C. Semple), geoff.whittle@mcs.vuw.ac.nz (G. Whittle).
[1] James Oxley was supported by the National Security Agency.
[2] Charles Semple and Geoff Whittle were supported by the New Zealand Marsden Fund.

0095-8956/\$ - see front matter
doi:10.1016/j.jctb.2004.03.006

matroid into 2-connected components [11]. Cunningham and Edmonds [1] considered the structure of 2-separations in a matroid. They showed that a 2-connected matroid M can be decomposed into a set of 3-connected matroids with the property that M can be built from these 3-connected matroids via a canonical operation known as 2-sum. Moreover, there is a labelled tree that gives a precise description of the way that M is built from the 3-connected pieces.

Because of the above decompositions, for many purposes in matroid theory, it is possible to restrict attention to 3-connected matroids. For example, a matroid is representable over a field if and only if the 3-connected components of its 2-sum decomposition are representable over that field. For some time, it was felt that 3-connectivity sufficed to eliminate most degeneracies caused by low connectivity. Indeed, Kahn [7] conjectured that, for each prime power q, there is an integer $\mu(q)$ such that every 3-connected matroid has at most $\mu(q)$ inequivalent $GF(q)$-representations.

Unfortunately, examples are given in [9] to show that Kahn's Conjecture is false for all fields with at least seven elements. While the existence of such counter-examples is disappointing, it is encouraging that all known counterexamples are of two very specific types, each of which has many mutually interacting 3-separations. This encourages the belief that a version of Kahn's Conjecture could be recovered for matroids whose 3-separations are controlled in some way. This motivates a study of the structure of the 3-separations in a matroid, and this paper is the outcome of that study.

Loosely speaking, the main theorem of this paper, Theorem 9.1, says that, associated with a 3-connected matroid M having at least nine elements, there is a labelled tree T with the property that, up to a certain equivalence, all "non-sequential" 3-separations of M are displayed by T. There are three important features of this theorem that require discussion at this stage.

The first is that, in contrast with the above-mentioned result of Cunningham and Edmonds for 2-separations in a matroid, we do not give a decomposition of M into more highly connected parts. This is because, in general, it is not possible to decompose a matroid across a 3-separation in a reasonable way. To see this, consider the non-representable Vámos matroid V_8 (see [8, pp. 511]). This matroid has a number of 3-separations, but there is no reasonable way to see V_8 as a "3-sum" of smaller more-connected parts. Having said this, we do believe that Theorem 9.1 can be used to obtain a decomposition result for *representable* matroids that would be similar in flavour to that of Cunningham and Edmonds' 2-sum decomposition. The components of such a decomposition would be *sequentially* 4-*connected* in the sense of [5].

A 3-separation (A, B) of M is *sequential* if either A or B can be ordered $(c_1, c_2, \dots, c_n)$ such that, for all $i \in \{3, 4, \dots, n\}$, the partition $(\{c_1, c_2, \dots, c_i\}, E(M) - \{c_1, c_2, \dots, c_i\})$ is a 3-separation. A second feature of Theorem 9.1 is that we make no attempt to display sequential 3-separations. To see the necessity for this, consider the matroid $P = PG(r-1, q)$ for some prime power q and positive integer $r \geqslant 3$. If L is a line of P, then $(L, E(P) - L)$ is a 3-separation of P. But the structure of the lines of P is very complex, at least as complex as the structure of P itself, and there is clearly no reasonable way of displaying all these lines in a tree-like way.

The third feature of our main result is that we only display 3-separations up to an equivalence. To illustrate the need for this, let P_1 and P_2 be two distinct planes of $PG(3,q)$ and let $M = PG(3,q)|(P_1 \cup P_2)$. Let $A = P_1 - P_2$, $B = P_1 \cap P_2$, and $C = P_2 - P_1$. Evidently B is a line of M. Moreover, it is easily seen that, for every subset B' of B, the partition $(A \cup B', C \cup (B - B'))$ is a 3-separation of M. There are 2^{q+1} distinct such 3-separations and there is clearly no reasonable way of displaying all of them. Furthermore, there is a quite natural sense in which all these 3-separations are equivalent.

Note that both sequential and equivalent 3-separations can appear more complicated than the ones in the examples given above. But, from a structural point of view, the existence of sequential and equivalent 3-separations is not problematic. They can be characterised using a straightforward extension of the closure operator, which is discussed in Section 3. Moreover, the interacting 3-separations in the counterexamples to Kahn's Conjecture given in [9] are mutually inequivalent, non-sequential 3-separations.

We now discuss the structure of the paper in more detail. Two 3-separations (A, B) and (C, D) *cross* if all intersections $A \cap C, A \cap D, B \cap C$, and $B \cap D$ are non-empty. Considering the structure of a collection of mutually crossing 3-separations leads to the notion of a flower, defined in Section 4. Essentially, a flower is a cyclically ordered partition of the ground set of a matroid that "displays" a collection of 3-separations of M. Understanding the structure of flowers turns out to be crucial, and the bulk of the paper is devoted to this. In Section 4, it is shown that the flowers in matroids are of five specific types. Two flowers are equivalent if they display, up to equivalence of 3-separations, the same collection of non-sequential 3-separations. Sections 5 and 6 develop an understanding of flower equivalence that enables us to give, in Section 7, a precise characterisation of equivalent flowers of different types. There is a natural partial order on flowers induced by the non-sequential 3-separations that they display. Theorem 8.1, the main result of Section 8, shows that all non-sequential 3-separations of a matroid interact with a maximal flower in a coherent way. At last, in Section 9, we introduce the notion of a partial 3-tree, which is a tree associated with a matroid M, some of whose vertices are labelled by members of a partition of the ground set of M. A flower can be thought of as a special type of 3-tree and, just as with flowers, partial 3-trees display certain 3-separations of M. Theorem 9.1 then shows that a maximal partial 3-tree displays, up to equivalence of 3-separations, all non-sequential 3-separations of M.

Finally, we note that discussions with James Geelen over several years were crucial to the evolution of many of the ideas that are fundamental in this paper. Indeed, if it were not for these early discussions, it is likely that this paper would never have come to fruition.

2. Preliminaries

Any unexplained matroid terminology used here will follow Oxley [8]. A 2-connected matroid is also commonly referred to as a *connected matroid* or as a *non-separable matroid*. A *partition* of a set S is an ordered collection $(S_1, S_2, \ldots, S_n)$ of subsets of S so that each element of S is in exactly one of the subsets S_i. Note that we are allowing the sets S_i to be empty.

Connectivity: Tutte [10] considered matroid connectivity as part of a general theory of separations. We recall aspects of that theory now. Let M be a matroid on ground set E. The *connectivity function* λ of M is defined, for all subsets A of E, by $\lambda(A) = r(A) + r(E - A) - r(M)$. The set A or the partition $(A, E - A)$ is *k-separating* if $\lambda(A) < k$. The partition $(A, E - A)$ is a *k-separation* if it is k-separating and $|A|, |E - A| \geqslant k$. For $n \geqslant 2$, the matroid M is *n-connected* if it has no $(n - j)$-separations for all j with $1 \leqslant j \leqslant n - 1$. A k-separating set A, or k-separating partition $(A, E - A)$, or k-separation $(A, E - A)$ is *exact* if $\lambda(A) = k - 1$.

The connectivity functions of a matroid M and its dual M^* are equal. Moreover, the connectivity function of M is submodular, that is, $\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E$. This means that if X and Y are k-separating, and one of $X \cap Y$ or $X \cup Y$ is not $(k-1)$-separating, then the other must be k-separating. Specialising to 3-connected matroids, we have the following:

Lemma 2.1. *Let M be a* 3*-connected matroid, and let X and Y be* 3*-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geqslant 2$, then $X \cup Y$ is* 3*-separating.*
(ii) *If $|E(M) - (X \cup Y)| \geqslant 2$, then $X \cap Y$ is* 3*-separating.*

We apply Lemma 2.1 many times in this paper and, rather than constantly referring to the lemma by name, we call such applications *uncrossings*.

Segments, cosegments and fans: Let M be a 3-connected matroid. A subset S of $E(M)$ is a *segment* if each 3-element subset of S is a triangle. Equivalently, given that M is 3-connected, S is a segment if $r(S) \leqslant 2$. The subset S is a *cosegment* if each 3-element subset of S is a triad. Equivalently, S is a cosegment if S is a segment of M^*.

Let $(s_1, s_2, \ldots, s_n)$ be an ordering of the elements of S. Then $(s_1, s_2, \ldots, s_n)$ is a *fan* if

(i) for all $i \in \{1, 2, \ldots, n-2\}$, the triple $\{s_i, s_{i+1}, s_{i+2}\}$ is either a triangle or a triad, and
(ii) if $i \in \{1, 2, \ldots, n-2\}$ and $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad, while if $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, then $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

Note that a fan in M is also a fan in M^*. Note also that, according to the above definitions, any set in M of size at most two is trivially both a segment and a cosegment, while any ordered set of size at most two is trivially a fan. This is somewhat non-standard, but allowing such structures gains considerable economy in the statement and proofs of a number of the theorems in this paper. The following result is straightforward.

Lemma 2.2. *Let S be a set in a* 3*-connected matroid. If S has an ordering* $(s_1, s_2, \ldots, s_n)$ *such that, for all* $i \in \{1, 2, \ldots, n-2\}$, *the triple* $\{s_i, s_{i+1}, s_{i+2}\}$ *is* 3*-separating, then either* $(s_1, s_2, \ldots, s_n)$ *is a fan, or S is a segment or a cosegment.*

Local connectivity: For subsets X and Y in M, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined by $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. Evidently, $\sqcap(Y, X) = \sqcap(X, Y)$. Note that if (X, Y) is a partition of $E(M)$, then $\sqcap(X, Y) = \lambda_M(X)$. If M is a representable matroid and we view it as a restriction of a projective geometry P, then the modularity of P means that $\sqcap(X, Y)$ is the rank of the intersection of the closures, in P, of X and Y. The next elementary lemma is just a restatement of Lemma 8.2.10 of [8].

Lemma 2.3. *Let* X_1, X_2, Y_1, *and* Y_2 *be subsets of the ground set of a matroid M. If* $X_1 \supseteq Y_1$ *and* $X_2 \supseteq Y_2$, *then* $\sqcap(X_1, X_2) \geqslant \sqcap(Y_1, Y_2)$.

The next lemma summarises some useful basic properties of the local connectivity function.

Lemma 2.4. *Let A, B, C, and D be subsets of the ground set of a matroid M. Then the following hold*:

(i) $\sqcap(A \cup B, C \cup D) + \sqcap(A, B) + \sqcap(C, D) = \sqcap(A \cup C, B \cup D) + \sqcap(A, C) + \sqcap(B, D)$.
(ii) $\sqcap(A \cup B, C) + \sqcap(A, B) = \sqcap(A \cup C, B) + \sqcap(A, C)$.
(iii) $\sqcap(A \cup B, C) + \sqcap(A, B) \geqslant \sqcap(A, C) + \sqcap(B, C)$.
(iv) *If* $\{X, Y, Z\}$ *is a partition of the ground set of M, then*

$$\lambda(X) + \sqcap(Y, Z) = \lambda(Z) + \sqcap(X, Y).$$

Hence $\sqcap(X, Y) = \sqcap(Y, Z)$ *if and only if* $\lambda(X) = \lambda(Z)$.
(v) *When A and B are disjoint,*

$$\lambda_{M/A}(B) = \lambda_M(B) - \sqcap(A, B).$$

Proof. By definition,

$$\begin{aligned}
&\sqcap(A \cup B, C \cup D) + \sqcap(A, B) + \sqcap(C, D)\\
&\quad = r(A \cup B) + r(C \cup D) - r(A \cup B \cup C \cup D)\\
&\qquad + r(A) + r(B) - r(A \cup B) + r(C) + r(D) - r(C \cup D)\\
&\quad = r(A) + r(B) + r(C) + r(D) - r(A \cup B \cup C \cup D).
\end{aligned}$$

Thus, by symmetry, (i) holds. Part (ii) follows immediately from (i) by putting $D = \emptyset$. By Lemma 2.3, $\sqcap(A \cup C, B) \geqslant \sqcap(B, C)$. Part (iii) now follows from (ii). The first part of (iv) follows from (ii) by observing that $\lambda(X) = \sqcap(X, Y \cup Z)$, and $\lambda(Z) = \sqcap(X \cup Y, Z)$. The second part of (iv) is an immediate consequence of the first part. Finally, (v) follows by writing both sides in terms of r_M. $\square$

If follows from the definition of $\sqcap$ that $r(\mathrm{cl}(X)\cap\mathrm{cl}(Y))\leqslant\sqcap(X,Y)$. This immediately implies the following:

Lemma 2.5. *Let M be a 3-connected matroid with at least four elements, let X and Y be subsets of $E(M)$, and let $Z=\mathrm{cl}(X)\cap\mathrm{cl}(Y)$. If $\sqcap(X,Y)=2$, then Z is a segment; if $\sqcap(X,Y)=1$, then $|Z|\leqslant 1$; and if $\sqcap(X,Y)=0$, then $Z=\emptyset$.*

The last lemma in this section gives a useful relation between $\sqcap_M(X,Y)$ and $\sqcap_{M^*}(X,Y)$. The straightforward proof is omitted.

Lemma 2.6. *Let X and Y be disjoint sets in a matroid M. Then*

$$\sqcap_M(X,Y)+\sqcap_{M^*}(X,Y)=\lambda(X)+\lambda(Y)-\lambda(X\cup Y).$$

In particular, if X, Y, and $X\cup Y$ are exactly 3-separating, then

$$\sqcap_M(X,Y)+\sqcap_{M^*}(X,Y)=2.$$

3. Sequential and equivalent 3-separations

Let A be a set in a matroid M. The *coclosure* $\mathrm{cl}^*(A)$ of A is the closure of A in M^*. If $\mathrm{cl}^*(A)=A$, then A is *coclosed* in M. If A is closed in both M and M^*, then A is *fully closed*. The *full closure* of A, denoted $\mathrm{fcl}(A)$, is the intersection of all fully closed sets containing A. Since the intersection of fully closed sets is clearly fully closed, the full closure is a well-defined closure operator. It is easily seen that one way of obtaining the full closure of a set A is to take $\mathrm{cl}(A)$, and then $\mathrm{cl}^*(\mathrm{cl}(A))$ and so on until neither the closure nor the coclosure operator adds new elements. It is just as easily seen that the elements of $\mathrm{fcl}(A)$ can be ordered $(a_1,a_2,\dots,a_n)$ such that, for all $i\in\{1,2,\dots,n\}$, either $a_i\in\mathrm{cl}(A\cup\{a_1,a_2,\dots,a_{i-1}\})$ or $a_i\in\mathrm{cl}^*(A\cup\{a_1,a_2,\dots,a_{i-1}\})$. We remark that $\mathrm{fcl}(A)$ has also been denoted by $\mathrm{ccl}(A)$ and called the *complete closure* of A [6].

We say that $x\in\mathrm{cl}^{(*)}(A)$ if either $x\in\mathrm{cl}(A)$ or $x\in\mathrm{cl}^*(A)$. Note that we do not regard $\mathrm{cl}^{(*)}$ as an operator (if we did it would not be a closure operator); rather it is just a convenient shorthand. The following easy lemma holds for k-separating sets for arbitrary k, but, in this paper, our only interest is in the case $k=3$.

Lemma 3.1. *Let (A,B) be exactly 3-separating in a matroid M.*

(i) *For $e\in E(M)$, the partition $(A\cup\{e\},B-\{e\})$ is 3-separating if and only if $e\in\mathrm{cl}^{(*)}(A)$.*
(ii) *For $e\in B$, the partition $(A\cup\{e\},B-\{e\})$ is exactly 3-separating if and only if e is in exactly one of $\mathrm{cl}(A)\cap\mathrm{cl}(B-\{e\})$ and $\mathrm{cl}^*(A)\cap\mathrm{cl}^*(B-\{e\})$.*
(iii) *The elements of $\mathrm{fcl}(A)-A$ can be ordered $(a_1,a_2,\dots,a_n)$ so that $A\cup\{a_1,a_2,\dots,a_i\}$ is 3-separating for all $i\in\{1,2,\dots,n\}$.*

We can use the full closure operator to define an equivalence on 3-separating sets as follows. Let M be a 3-connected matroid. Let A and B be exactly 3-separating sets of M. Then A is *equivalent* to B if $\mathrm{fcl}(A) = \mathrm{fcl}(B)$. Let the partitions (A_1, A_2) and (B_1, B_2) be exactly 3-separating in M. Then (A_1, A_2) is equivalent to (B_1, B_2) if, for some ordering (C_1, C_2) of $\{B_1, B_2\}$, we have A_1 is equivalent to C_1, and A_2 is equivalent to C_2.

Let X be an exactly 3-separating set of a 3-connected matroid M. Then X is *sequential* if it has an ordering $(x_1, x_2, \dots, x_n)$ such that $\{x_1, x_2, \dots, x_i\}$ is 3-separating for all $i \in \{1, 2, \dots, n\}$. Let (X, Y) be exactly 3-separating in M. Then (X, Y) is *sequential* if either X or Y is a sequential 3-separating set. Another easy argument proves the following.

Lemma 3.2. *Let X be an exactly 3-separating set of a 3-connected matroid M. Then X is sequential if and only if $\mathrm{fcl}(E(M) - X) = E(M)$.*

To test if (A_1, A_2) is equivalent to (B_1, B_2), we must show that the sets $\{\mathrm{fcl}(A_1), \mathrm{fcl}(A_2)\}$ and $\{\mathrm{fcl}(B_1), \mathrm{fcl}(B_2)\}$ are equal. To see this, let P be the set of points of a finite projective plane, and let L_1 and L_2 be distinct lines of the plane. Then $(L_1, P - L_1)$ is not equivalent to $(L_2, P - L_2)$, even though $\mathrm{fcl}(P - L_1) = \mathrm{fcl}(P - L_2) = P$. For non-sequential 3-separations, we can simplify things somewhat.

Lemma 3.3. *Let (A_1, A_2) be a non-sequential 3-separation of a 3-connected matroid M and let (B_1, B_2) be a 3-separation of M. Then (A_1, A_2) is equivalent to (B_1, B_2) if and only if $\mathrm{fcl}(A_1) = \mathrm{fcl}(B_1)$ or $\mathrm{fcl}(A_1) = \mathrm{fcl}(B_2)$.*

Proof. In one direction the lemma is trivial. For the other direction, assume that $\mathrm{fcl}(A_1) = \mathrm{fcl}(B_1) = X$. Then, by Lemma 3.1, X is 3-separating. Set $Y = E(M) - X$. Since (A_1, A_2) is not sequential, Y is non-empty. If $|Y| \leqslant 2$, then X is not fully closed, so $|Y| \geqslant 3$ and hence Y is exactly 3-separating. By Lemma 3.1, $A_2 \subseteq \mathrm{fcl}(Y)$ and $B_2 \subseteq \mathrm{fcl}(Y)$. Since the full closure operator is a closure operator and Y is a subset of both A_2 and B_2, we now have $\mathrm{fcl}(A_2) = \mathrm{fcl}(Y) = \mathrm{fcl}(B_2)$, so that (A_1, A_2) is indeed equivalent to (B_1, B_2). □

The next two lemmas note some further elementary properties of 3-separating sets. Part (ii) of the first of these follows by Lemma 2.4(v).

Lemma 3.4. *Let (X, Y) be exactly 3-separating in a 3-connected matroid M.*

(i) *If (X, Y) is non-sequential, then $|X|, |Y| \geqslant 4$.*
(ii) *For $y \in Y$, if $y \in \mathrm{cl}^*(X)$, then X is 2-separating in $M \backslash y$; and if $y \in \mathrm{cl}(X)$, then X is 2-separating in M/y.*

Lemma 3.5. *Let A and B be disjoint 3-separating sets in a 3-connected matroid M. If $\mathrm{fcl}(A)$ does not contain B and $|B| \geqslant 3$, then $\mathrm{fcl}(A) - B$ is 3-separating and $\mathrm{fcl}(\mathrm{fcl}(A) - B) = \mathrm{fcl}(A)$.*

Proof. Evidently, $\mathrm{fcl}(A) - B$ is the intersection of the 3-separating sets $\mathrm{fcl}(A)$ and $E(M) - B$. Thus, by uncrossing, if $|B - \mathrm{fcl}(A)| \geqslant 2$, then $\mathrm{fcl}(A) - B$ is certainly 3-separating. Now suppose that $|B - \mathrm{fcl}(A)| = 1$, say $B - \mathrm{fcl}(A) = \{b\}$. As $\mathrm{fcl}(A) \not\supseteq B$, it follows that $|E(M) - \mathrm{fcl}(A)| \geqslant 4$ so $|E(M) - (\mathrm{fcl}(A) \cup B)| \geqslant 3$. Hence, by uncrossing, $\mathrm{fcl}(A) \cap B$, which equals $B - \{b\}$, is 3-separating. As $|B| \geqslant 3$, it follows that $b \in \mathrm{cl}^{(*)}(B - \{b\})$ so $b \in \mathrm{fcl}(B - \{b\}) \subseteq \mathrm{fcl}(\mathrm{fcl}(A)) = \mathrm{fcl}(A)$; a contradiction. We conclude that $\mathrm{fcl}(A) - B$ is 3-separating. The final assertion of the lemma follows from the fact that $A \subseteq \mathrm{fcl}(A) - B \subseteq \mathrm{fcl}(A)$. $\square$

4. Flowers

The complexity in characterising 3-separations in a matroid is caused by the fact that they can cross, in other words, it is possible to have 3-separations (A_1, A_2) and (B_1, B_2) such that each of the four sets of the form $A_i \cap B_j$ is non-empty. However, if each of these sets has at least two elements, it follows from Lemma 2.1 that each is 3-separating. We now have an ordered partition into four 3-separating sets, $(A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_2, A_2 \cap B_1)$ such that, in this cyclic order, the union of any consecutive pair is 3-separating.

This is an example of a structure that turns out to be fundamental. Let n be a positive integer and M be a 3-connected matroid. Then $(P_1, P_2, \ldots, P_n)$ is a *flower* in M with *petals* $P_1, P_2, \ldots, P_n$ if $(P_1, P_2, \ldots, P_n)$ is a partition of $E(M)$, each P_i has at least two elements and is 3-separating, and each $P_i \cup P_{i+1}$ is 3-separating, where all subscripts are interpreted modulo n. Observe that flowers have only been defined in 3-connected matroids. In what follows, whenever we refer to a flower, it will be implicit that this flower occurs in a 3-connected matroid.

The purpose of this section is to characterise flowers by describing which unions of petals are 3-separating and by specifying the local connectivity between petals. Let Φ be a flower $(P_1, P_2, \ldots, P_n)$. We say that Φ is an *anemone* if $\bigcup_{s \in S} P_s$ is 3-separating for every subset S of $\{1, 2, \ldots, n\}$; and Φ is a *daisy* if, for all i and k in $\{1, 2, \ldots, n\}$, the set $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{i+k}$ is 3-separating and no other union of petals is 3-separating. Thus a flower is an anemone if all unions of petals are 3-separating, and it is a daisy if a union of petals is 3-separating if and only if the petals are consecutive in the cyclic order.

Note that, if $n \leqslant 3$, the concepts of daisy and anemone coincide, but if $n \geqslant 4$, then a flower cannot be both a daisy and an anemone. A *trivial* flower has just the one petal, namely $E(M)$. A flower with two petals is just a 3-separating partition. Genuine structure emerges in flowers with at least three petals. The 3-petal case presents certain difficulties as we shall see.

Amongst anemones, we distinguish three different types according to the behaviour of the local connectivity function. For $n \geqslant 3$, an anemone $(P_1, P_2, \ldots, P_n)$ is called

(i) a *paddle* if $\sqcap(P_i, P_j) = 2$ for all distinct $i, j \in \{1, 2, \ldots, n\}$;

(ii) a *copaddle* if $\sqcap(P_i, P_j) = 0$ for all distinct $i, j \in \{1, 2, \ldots, n\}$; and
(iii) *spike-like* if $n \geqslant 4$, and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.

Similarly, we distinguish two different types of daisies. Specifically, a daisy $(P_1, P_2, \ldots, P_n)$ is called

(i) *swirl-like* if $n \geqslant 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j; and
(ii) *Vámos-like* if $n = 4$ and $\sqcap(P_i, P_j) = 1$ for all consecutive i and j, while $\{\sqcap(P_1, P_3), \sqcap(P_2, P_4)\} = \{0, 1\}$.

Finally, we say that a flower is *unresolved* if $n = 3$, and $\sqcap(P_i, P_j) = 1$ for all distinct $i, j \in \{1, 2, 3\}$. At this stage, we could define an unresolved flower to be both spike-like and swirl-like. But we will see in Section 6 that, due to the presence of additional structure, some unresolved flowers are best viewed as spike-like and others as swirl-like.

The next theorem is the main result of this section.

Theorem 4.1. *If $\Phi = (P_1, P_2, \ldots, P_n)$ is a flower, then Φ is either an anemone or a daisy. Moreover, if $n \geqslant 3$, then Φ is either a paddle, a copaddle, spike-like, swirl-like, Vámos-like, or is unresolved.*

Before turning to the proof of Theorem 4.1, we illustrate the types of flowers with some generic examples. We first note that there is a straightforward connection between flowers in M and M^*, which follows from Lemma 2.6.

Proposition 4.2. *If $\Phi = (P_1, P_2, \ldots, P_n)$ is a flower, then it is also a flower in M^*. Moreover, for $n \geqslant 3$,*

(i) *if Φ is either spike-like, swirl-like, Vámos-like, or unresolved, then Φ has the same type in M^* as in M; and*
(ii) *if Φ is a paddle in M, then Φ is a copaddle in M^*.*

What follows is an informal description that may help the reader's intuition for different types of flowers. To visualise a flower geometrically, it is useful to think of a collection of lines in projective space. These lines can be thought of as lines of attachment of the 3-separating sets that form the petals of the flower. We obtain a paddle by gluing the petals along a single common line. Fig. 1 represents a 5-petal paddle in which each petal is a plane with sufficient structure to make the overall matroid 3-connected. The resulting matroid, whose points have been suppressed in the figure, has rank 7. In general, the rank of a paddle is $\sum_{i=1}^{n} r(P_i) - 2(n-1)$. We obtain a copaddle by choosing a collection $\{L_1, L_2, \ldots, L_n\}$ of lines placed as freely as possible in rank $2(n-1)$. Thus, if any one of the lines is deleted, the remaining lines are mutually skew. Each petal P_i is attached to the line L_i and the overall rank of M is $\sum_{i=1}^{n} r(P_i) - 2$. For all $n \geqslant 3$, we obtain a paddle in $M(K_{3,n})$ by taking the

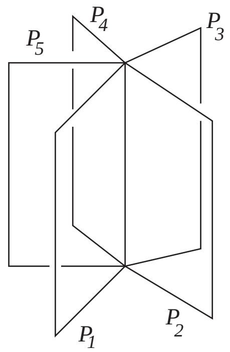

Fig. 1. A representation of a rank-7 paddle.

petals to be the 3-element bonds in $K_{3,n}$. In this case, the common line of attachment for the petals contains no elements of the matroid. Clearly, the same flower is a copaddle in $M^*(K_{3,n})$.

For a spike-like flower, consider a set $\mathcal{L} = \{L_1, L_2, \dots, L_n\}$ of copunctual lines placed as freely as possible in rank n. Again these lines are the lines of attachment for the petals and the overall rank of M is $\sum_{i=1}^{n} r(P_i) - n$. The terminology arises from the connection with a class of matroids called spikes. If two points are chosen from each of the lines in $\mathcal{L}$ so that each chosen point is on no other line in $\mathcal{L}$, then the matroid induced by this set of points is an example of a spike. Spikes turn out to be a fundamental class of matroids (see, for example, [2,3,4,9,12]).

Consider a swirl-like flower. Choose an independent set $\{p_1, p_2, \dots, p_n\}$ in a projective space and let L_i be the line spanned by $\{p_i, p_{i+1}\}$. Since subscripts are interpeted modulo n, the line L_n is spanned by $\{p_n, p_1\}$. Using these lines as lines of attachment for the petals gives a swirl-like flower with overall rank $\sum_{i=1}^{n} r(P_i) - n$. An example of such a flower has been given in Fig. 2. In that figure, four planes have been attached to lines of a rank-4 matroid to produce a rank-8 matroid. The points p_1, p_2, p_3, and p_4, which may or may not be in the matroid, have been indicated but the other points of the matroid, all of which lie on one of the planes P_1, P_2, P_3, or P_4, have been suppressed. If, in the general construction above, exactly two points are chosen from each of the lines in $\{L_1, L_2, \dots, L_n\}$ so that each chosen point is on exactly one such line, the matroid induced by this set of points is an example of a swirl. As with spikes, swirls have turned out to be important in recent work in matroid theory (see, for example [4,9]).

Finally, consider a Vámos-like flower. There is a group of non-representable matroids with eight elements amongst which is the Vámos matroid (see, for example [8, p. 511]) that share a common feature: their ground sets can be partitioned into four lines L_1, L_2, L_3, L_4 such that (L_1, L_2, L_3, L_4) is a Vámos-like flower as described above. More general Vámos-like flowers can be formed by using these lines to glue

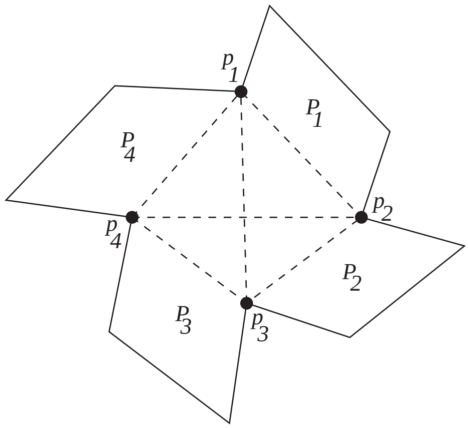

Fig. 2. A representation of a rank-8 swirl-like flower.

on larger 3-separating sets. Intuitively, any matroid with a Vámos-like flower is not representable over any field, and we shall prove this in Corollary 6.2.

We now turn to the proof of Theorem 4.1, which will follow from a sequence of lemmas. We show first that any consecutive union of petals in any flower must be 3-separating.

Lemma 4.3. *Let* $(P_1, P_2, \ldots, P_n)$ *be a flower. Then, for all i and k in* $\{1, 2, \ldots, n\}$, *the set* $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{i+k}$ *is* 3-*separating.*

Proof. We argue by induction on k. Since $(P_1, P_2, \ldots, P_n)$ is a flower, the result holds for $k \in \{1, 2\}$. Now let $k \geqslant 3$, and assume that the result holds for $k-1$. Then $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{i+k-1}$ and $P_{i+k-1} \cup P_{i+k}$ are 3-separating, and their intersection P_{i+k-1} contains at least two elements so we see by uncrossing that their union $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_{i+k}$ is 3-separating as required. □

Lemma 4.4. *Let* $\Phi = (P_1, P_2, \ldots, P_n)$ *be a flower. Then* Φ *is either an anemone or a daisy.*

Proof. The result is trivial if $n \leqslant 3$. Assume $n \geqslant 4$. By Lemma 4.3, all consecutive sets of petals are 3-separating. If no other union of petals is 3-separating, then Φ is a daisy.

Assume that Φ is not a daisy. Then there is a non-consecutive set of petals whose union P is 3-separating. Evidently, P contains a pair P_i and P_j of non-consecutive petals with the property that $i < j$ and no petal between P_i and P_j is contained in P. There is at least one petal not contained in $P \cup P_i \cup P_{i+1} \cup \cdots \cup P_j$, otherwise P is the union of a consecutive set of petals. Uncrossing P and $P_i \cup P_{i+1} \cup \cdots \cup P_j$ now shows that $P_i \cup P_j$ is 3-separating. Thus Φ has a non-consecutive pair of petals that is 3-separating.

We now show that the union of every pair of petals is 3-separating. Consider the non-consecutive pair P_i and P_j such that $P_i \cup P_j$ is 3-separating. We begin by showing that $P_i \cup P_{j-1}$ is 3-separating. Since $P_i \cup P_j$ and $P_{j-1} \cup P_j$ are both 3-separating and their intersection has at least two elements, uncrossing implies that $P_i \cup P_{j-1} \cup P_j$ is 3-separating. Furthermore, by Lemma 4.3, $P_i \cup P_{i+1} \cup \cdots \cup P_{j-1}$ is 3-separating. The set P_{j+1} has at least two elements and avoids the last two 3-separating sets and so, by uncrossing again, $P_i \cup P_{j-1}$ is 3-separating. By repeatedly applying this argument, we deduce that the union of every pair of petals of Φ containing P_i is 3-separating. It follows that if $n = 4$, then the union of every pair of petals of Φ is 3-separating. Hence, we may assume that $n \geqslant 5$.

By repeating the argument of the last paragraph with P_i replaced by P_j, we get that the union of every pair of petals of Φ containing P_j is 3-separating. Since $n \geqslant 5$, there is at most one petal in $\{P_1, P_2, \dots, P_n\} - \{P_i, P_j\}$ that is adjacent to both P_i and P_j in the original ordering. If there is such a petal, call it P_k. For all petals P_m with $m \neq k$, the argument of the last paragraph implies that the union of every pair of petals containing P_m is 3-separating. It follows that the union of every pair of petals of Φ is 3-separating. Therefore, any circular ordering of the petals is a flower and it follows, from Lemma 4.3, that all unions of petals are 3-separating and hence that Φ is an anemone. $\square$

We show next that every anemone is a paddle, a copaddle, is spike-like, or is unresolved. We begin with a preliminary lemma that holds for all flowers.

Lemma 4.5. *Let $(P_1, P_2, \dots, P_n)$ be a flower. Then $\sqcap(P_i, P_{i+1}) = \sqcap(P_j, P_{j+1})$ for all i,j.*

Proof. Choose $k = \max\{\sqcap(P_i, P_{i+1}) : i \in \{1, 2, \dots, n\}\}$. We lose no generality in assuming that $\sqcap(P_1, P_2) = k$ and that $n \geqslant 3$. It suffices to show that $\sqcap(P_2, P_3) = k$. Now $E - (P_2 \cup P_3)$ is 3-separating, so

$$\lambda(E - (P_2 \cup P_3)) = 2 = \lambda(P_3).$$

Thus, by Lemma 2.4(iv) and Lemma 2.3,

$$k \geqslant \sqcap(P_3, P_2) = \sqcap(P_2, E - (P_2 \cup P_3)) \geqslant \sqcap(P_2, P_1) = k.$$

Hence, $\sqcap(P_3, P_2) = k$, so $\sqcap(P_2, P_3) = k$. $\square$

Lemma 4.6. *Let $n \geqslant 3$, and let $\Phi = (P_1, P_2, \dots, P_n)$ be an anemone in a matroid M. Then Φ is a paddle, a copaddle, is spike-like, or is unresolved.*

Proof. Since an anemone is a flower relative to any circular ordering of the petals, it follows from Lemma 4.5 that there is a constant k such that $\sqcap(P_i, P_j) = k$ for all distinct i,j. Since M is 3-connected, $k \in \{0, 1, 2\}$. If $k = 2$, then Φ is a paddle; if $k = 1$, then Φ is spike-like or, when $n = 3$, is unresolved; and, if $k = 0$, then Φ is a copaddle. $\square$

We now work towards showing that if a flower is a daisy, then it is swirl-like or Vámos-like.

Lemma 4.7. *Let $n \geqslant 4$ and $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower of a matroid M on E.*

(i) *If $\sqcap(P_j, P_{j+1}) = 2$ for some $j \in \{1, 2, \ldots, n\}$, then Φ is a paddle.*
(ii) *If $\sqcap(P_j, P_{j+1}) = 0$ for some $j \in \{1, 2, \ldots, n\}$, then Φ is a copaddle.*
(iii) *If Φ is a daisy, then $\sqcap(P_i, P_{i+1}) = 1$ for all $i \in \{1, 2, \ldots, n\}$.*

Proof. Suppose that $\sqcap(P_j, P_{j+1}) = 2$ for some $j \in \{1, 2, \ldots, n\}$. Then, by Lemma 4.5, $\sqcap(P_i, P_{i+1}) = 2$ for all $i \in \{1, 2, \ldots, n\}$. Hence $\sqcap(P_1, P_2) = 2 = \lambda_M(P_1)$. By Lemma 2.4(v), $\lambda_{M/P_2}(P_1) = \lambda_M(P_1) - \sqcap(P_1, P_2)$, so $\lambda_{M/P_2}(P_1) = 0$. Similarly, $\lambda_{M/P_2}(P_3) = 0$. Thus, by submodularity, $\lambda_{M/P_2}(P_1 \cup P_3) = 0$ so, by Lemma 2.4(v) again,

$$\lambda_M(P_1 \cup P_3) = \sqcap(P_1 \cup P_3, P_2) \leqslant \lambda_M(P_2) = 2.$$

Hence $P_1 \cup P_3$ is 3-separating so Φ is an anemone. Thus, by Lemma 4.6, Φ is a paddle and (i) is proved.

Part (ii) follows from (i) by duality since, by Proposition 4.2, $(P_1, P_2, \ldots, P_n)$ is a flower in M^*, and, by Lemma 2.6, $\sqcap_{M^*}(P_j, P_{j+1}) = 2 - \sqcap_M(P_j, P_{j+1})$. Finally, if Φ is a daisy, then it is neither a paddle nor a copaddle, so (iii) follows from (i) and (ii) using Lemma 4.5. $\square$

Lemma 4.8. *Let $n \geqslant 5$, and let $(P_1, P_2, \ldots, P_n)$ be a daisy of a matroid with ground set E. If $\sqcap(P_s, P_t) = 0$ for some non-consecutive s and t, then $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j.*

Proof. Since $P_i \cup P_{i+1}$ is 3-separating and contains at least two elements,

$$2 = \sqcap(P_i \cup P_{i+1}, E - (P_i \cup P_{i+1}))$$

for all i. Now taking A, B, and C equal to P_{i+1}, P_i, and $E - (P_i \cup P_{i+1})$, respectively, we get from Lemma 2.4(ii) that

$$2 = \sqcap(P_i, E - P_i) + \sqcap(P_{i+1}, E - (P_i \cup P_{i+1})) - \sqcap(P_i, P_{i+1}).$$

Since P_i is 3-separating, we deduce that

$$\sqcap(P_{i+1}, E - (P_i \cup P_{i+1})) = \sqcap(P_i, P_{i+1}) = 1.$$

Thus, for all $j \notin \{i, i+1, i+2\}$, by Lemma 2.3,

$$1 = \sqcap(P_{i+1}, E - (P_i \cup P_{i+1})) \geqslant \sqcap(P_{i+1}, P_{i+2} \cup P_j) \geqslant \sqcap(P_{i+1}, P_{i+2}) = 1.$$

Therefore $\sqcap(P_{i+1}, P_{i+2} \cup P_j) = 1$ for all $j \notin \{i, i+1, i+2\}$. By symmetry, if $j \notin \{i+3, i+2, i+1\}$, then $\sqcap(P_{i+2}, P_{i+1} \cup P_j) = 1$. By Lemma 2.4(ii),

$$\sqcap(P_{i+1}, P_{i+2} \cup P_j) + \sqcap(P_{i+2}, P_j) = \sqcap(P_{i+1} \cup P_j, P_{i+2}) + \sqcap(P_{i+1}, P_j).$$

Therefore

$$\sqcap(P_{i+1}, P_j) = \sqcap(P_{i+2}, P_j) \tag{1}$$

for all $j \notin \{i, i+1, i+2, i+3\}$.

Now we know that $\sqcap(P_s, P_t) = 0$ for some s and t that are non-consecutive. By relabelling if necessary, we may assume that $\{s, t\} = \{1, k\}$ and that k is chosen so that, among all such pairs involving 1, we have $k - 1 \leqslant n + 1 - k$ and $k - 1$ is minimized. If $k > 3$, then $k \leqslant n - 2$ so $1 \notin \{k-2, k-1, k, k+1\}$ and $\sqcap(P_1, P_{k-1}) = \sqcap(P_1, P_k)$ by (1). This contradicts the choice of k. Thus $\sqcap(P_1, P_3) = 0$. Therefore, by (1), $\sqcap(P_1, P_4) = 0$ since $n > 4$. By repeatedly applying (1), we obtain that $\sqcap(P_1, P_g) = 0$ for all g with $3 \leqslant g \leqslant n-1$, that is, $\sqcap(P_1, P_g) = 0$ for all g such that 1 and g are non-consecutive. Hence if $\sqcap(P_i, P_j) = 0$ for some j, then $\sqcap(P_i, P_h) = 0$ for all h such that i and h are non-consecutive. As $\sqcap(P_g, P_1) = 0$ for all g such that 1 and g are non-consecutive, we may apply the observation of the last sentence to deduce that $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j such that $i \notin \{2, n\}$. In particular, $\sqcap(P_c, P_2) = \sqcap(P_d, P_n)$ for all c and d such that c and 2 are non-consecutive and d and n are non-consecutive. It follows that $\sqcap(P_i, P_j) = 0$ for all non-consecutive i and j with $i \in \{2, n\}$, and the lemma holds. $\square$

The next lemma is an immediate consequence of Lemma 4.3.

Lemma 4.9. *If* $(P_1, P_2, \ldots, P_n)$ *is a flower and* $i \in \{1, 2, \ldots, n\}$, *then* $(P_1, P_2, \ldots, P_{i-1}, P_i \cup P_{i+1} \cup \cdots \cup P_n)$ *is a flower.*

Lemma 4.10. *Let* $n \geqslant 4$ *and let* $\Phi = (P_1, P_2, \ldots, P_n)$ *be a daisy of a matroid* M. *Then* Φ *is either swirl-like or Vámos-like.*

Proof. Set $P_4' = P_4 \cup P_5 \cup \cdots \cup P_n$. By Lemma 4.9, (P_1, P_2, P_3, P_4') is a flower. As $P_1 \cup P_3$ is not 3-separating in M, this flower is a daisy, so, by Lemma 4.7, $\sqcap(P_1, P_2) = \sqcap(P_3, P_4') = 1$. Assume that

$$\sqcap(P_1, P_3) + \sqcap(P_2, P_4') \geqslant 2.$$

Then

$$\begin{aligned} r(P_1 \cup P_3) + r(P_2 \cup P_4') &\leqslant r(P_1) + r(P_2) + r(P_3) + r(P_4') - 2 \\ &= r(P_1 \cup P_2) + r(P_3 \cup P_4') \\ &= r(M) + 2. \end{aligned}$$

Thus $P_1 \cup P_3$ is 3-separating, contradicting the fact that Φ is a daisy. Therefore

$$\sqcap(P_1, P_3) + \sqcap(P_2, P_4') \leqslant 1.$$

As $\sqcap(P_2, P_4) \leqslant \sqcap(P_{,}P_4')$, at least one of $\sqcap(P_1, P_3)$ and $\sqcap(P_2, P_4)$ is 0 and the other is at most 1. In the case that $n = 4$, it follows immediately that Φ is either swirl-like or Vámos-like. In the case that $n \geqslant 5$, it follows from Lemma 4.8 that Φ is swirl-like. □

Proof of Theorem 4.1. By Lemma 4.4, Φ is either an anemone or a daisy. Say $n \geqslant 3$. Assume that Φ is an anemone. Then, by Lemma 4.6, Φ is either a paddle, a copaddle, spike-like, or unresolved. Assume that Φ is not an anemone. Then $n \geqslant 4$ and Φ is a daisy, so it follows from Lemma 4.10 that Φ is either swirl-like or Vámos-like. □

We will often seek to verify that a partition of the elements of a matroid is a flower of a certain type. The following is an economical way to check this.

Lemma 4.11. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a partition of the ground set of a* 3*-connected matroid M, where $n \geqslant 4$ and $|P_i| \geqslant 2$ for all i.*

(i) *If $P_i \cup P_{i+1}$ is* 3*-separating for each $i \in \{1, 2, \ldots, n-1\}$, then Φ is a flower.*
(ii) *Assume that Φ is a flower with $n \geqslant 5$ and that i, j, and k are elements of $\{1, 2, \ldots, n\}$ such that j and k are distinct and non-consecutive.*
    (a) *If $\sqcap(P_i, P_{i+1}) = 2$, then Φ is a paddle.*
    (b) *If $\sqcap(P_i, P_{i+1}) = 1$ and $\sqcap(P_j, P_k) = 1$, then Φ is spike-like.*
    (c) *If $\sqcap(P_i, P_{i+1}) = 1$ and $\sqcap(P_j, P_k) = 0$, then Φ is a swirl-like.*
    (d) *If $\sqcap(P_i, P_{i+1}) = 0$, then Φ is a copaddle.*

Proof. Since $P_2 \cup P_3$ and $P_3 \cup P_4$ are 3-separating, we see by uncrossing that $P_2 \cup P_3 \cup P_4$ is 3-separating. By repeating this argument, we deduce that $P_2 \cup P_3 \cup \cdots \cup P_{n-1}$ is 3-separating. Hence $P_n \cup P_1$ is 3-separating. Thus the union of each consecutive pair of P_i's is 3-separating. Another easy uncrossing argument shows that each P_i is 3-separating. This establishes (i). Part (ii) follows by combining Theorem 4.1 and Lemma 4.7. □

The next lemma gives one more useful fact about flowers.

Lemma 4.12. *If $(P_1, P_2, \ldots, P_n)$ is a flower with $n \geqslant 4$, then $\mathrm{cl}(P_i \cup P_{i+1})$ and $\mathrm{cl}(P_{i+1} \cup P_{i+2})$ form a modular pair of flats whose intersection is spanned by P_{i+1}.*

Proof. Evidently, we may assume that $i = 1$. Since each of $P_1 \cup P_2, P_2 \cup P_3$, $P_1 \cup P_2 \cup P_3$, and P_2 is exactly 3-separating,

$$\lambda(P_1 \cup P_2) + \lambda(P_2 \cup P_3) = \lambda(P_1 \cup P_2 \cup P_3) + \lambda(P_2). \tag{2}$$

By submodularity,

$$r(P_1 \cup P_2) + r(P_2 \cup P_3) \geqslant r(P_1 \cup P_2 \cup P_3) + r(P_2)$$

and

$$r(E - (P_1 \cup P_2)) + r(E - (P_2 \cup P_3)) \geqslant r(E - (P_1 \cup P_2 \cup P_3)) + r(E - P_2).$$

On summing the last two inequalities and comparing the result with (2), we deduce that both inequalities must be equations, and the lemma follows. $\square$

5. Equivalent flowers

Let Φ be a flower of a matroid M recalling that, whenever we refer to a flower, it is implicit that the underlying matroid is 3-connected. We say that Φ *displays* a 3-separating set X or a 3-separation (X, Y) if X is a union of petals of Φ. Now let Φ_1 and Φ_2 be flowers of M. Then $\Phi_1 \preccurlyeq \Phi_2$ if every non-sequential 3-separation displayed by Φ_1 is equivalent to one displayed by Φ_2. Evidently, $\preccurlyeq$ is a quasi order on the collection of flowers of M. We will say that Φ_1 and Φ_2 are *equivalent* flowers if $\Phi_1 \preccurlyeq \Phi_2$ and $\Phi_2 \preccurlyeq \Phi_1$. Thus equivalent flowers display, up to equivalence of 3-separations, exactly the same non-sequential 3-separations.

The *order* of a flower Φ is the minimum number of petals in a flower equivalent to Φ. Thus, a flower has order 1 if it displays no non-sequential 3-separations, so that it is equivalent to the flower with one petal consisting of all elements of M. A flower has order 2 if it displays a single non-sequential 3-separation.

Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower. The flower Φ' is obtained from Φ by an *elementary move* if it is obtained in one of the following ways:

(0) Φ' is obtained by an arbitrary permutation of the petals of Φ in the case that Φ is an anemone or is obtained from Φ by a cyclic shift or a reversal of the order of the petals of Φ in the case that Φ is a daisy.

(1) $|P_2| \geqslant 3$, there is an element $a \in P_2$ such that $a \in \mathrm{cl}^{(*)}(P_1)$, and $\Phi' = (P_1 \cup \{a\}, P_2 - \{a\}, P_3, \ldots, P_n)$.

(2) $|P_2| = 2$, there is an element $a \in P_2$ such that $a \in \mathrm{cl}^{(*)}(P_1)$, and $\Phi' = (P_1 \cup P_2, P_3, \ldots, P_n)$.

(3) $|P_1| \geqslant 4$, and P_1 has a 2-element subset $\{a, b\}$ such that $b \in \mathrm{cl}^{(*)}(P_2)$ and $a \in \mathrm{cl}^{(*)}(P_2 \cup \{b\})$, and $\Phi' = (P_1 - \{a, b\}, \{a, b\}, P_2, \ldots, P_n)$.

Note that, given moves of Type 0, we lose no generality in defining the other moves with reference only to petals P_1, P_2, P_3. In what follows, when we refer to the moves needed to effect a certain change, we shall usually omit explicit reference to Type-0 moves. The main goal of this section is to prove the following characterisation of equivalent flowers.

Theorem 5.1. *Two flowers of order at least* 3 *are equivalent if and only if one can be obtained from the other by a sequence of elementary moves.*

Note that Theorem 5.1 does not hold for flowers of order less than 3. For example, let M have a single non-sequential 3-separation (A, B), where $B = \{b_1, b_2, b_3, b_4\}$. It is easily seen that such a matroid exists. Then $\Phi = (A, \{b_1, b_2\}, \{b_3, b_4\})$ is a flower equivalent to the 2-petal flower $\Phi' = (A, B)$. But Φ' cannot be obtained from

Φ by a sequence of elementary moves. A similar example can be given for flowers of order 1.

Theorem 5.1 will follow from a sequence of lemmas in which we develop further structural properties of flowers. An element e of M is *loose* in the flower Φ if $e \in \mathrm{fcl}(P_i) - P_i$ for some petal P_i of Φ. An element that is not loose is *tight*. The petal P_i is *loose* if all elements in P_i are loose. A *tight* petal is one that is not loose, that is, one that contains at least one tight element. A flower of order at least 3 is *tight* if all of its petals are tight. A flower of order 2 or 1 is *tight* if it has two petals or one petal, respectively. The next lemma is an immediate consequence of Lemma 3.1(i).

Lemma 5.2. *Let* $(P_1, \{a,b\}, P_3, \ldots, P_n)$ *be a flower where* $a \in \mathrm{cl}^{(*)}(P_1)$. *Then* $b \in \mathrm{cl}^{(*)}(P_1 \cup \{a\})$, *so that* $\{a,b\} \subseteq \mathrm{fcl}(P_1)$, *and* $\{a,b\}$ *is a loose petal of* Φ.

It follows that elementary moves of Types 1–3 can be seen as ways of moving loose elements from one petal to another or of adding or removing loose petals.

Lemma 5.3. *Let* $\Phi = (P_1, P_2, \ldots, P_n)$ *be a flower of order at least* 2 *of a matroid* M, *and suppose that* Φ' *is obtained from* Φ *by an elementary move. Then* Φ *and* Φ' *are equivalent and an element is loose in* Φ *if and only if it is loose in* Φ'.

Proof. Evidently, $n \geqslant 2$. It is clear that moves of Type 0 satisfy the lemma. Consider moves of Type 1. Say that $|P_2| \geqslant 3$, that $a \in P_2$, and that $a \in \mathrm{cl}^{(*)}(P_1)$. Let $\Phi' = (P_1 \cup \{a\}, P_2 - \{a\}, P_3, \ldots, P_n)$.

We first show that Φ' is a flower. If $n = 2$, this is immediate and, if $n = 3$, it is easy. Assume that $n \geqslant 4$. Consider consecutive pairs of sets in the partition Φ. The only unions of such pairs that are not unions of consecutive pairs of petals of Φ are $P_n \cup (P_1 \cup \{a\})$ and $(P_2 - \{a\}) \cup P_3$. By Lemma 4.11, we only have to check that the former set is 3-separating. But this holds since $a \in \mathrm{cl}^{(*)}(P_n \cup P_1)$. Thus Φ' is a flower. Moreover, $a \in \mathrm{cl}^{(*)}(P_2 - \{a\})$.

We now show that Φ and Φ' are equivalent. Let (S,T) be a non-sequential 3-separation. Say that (S,T) is displayed by Φ, where $P_1 \subseteq S$. Now $a \in \mathrm{cl}^{(*)}(S)$, and $(S \cup \{a\}, T - \{a\})$ is a 3-separation that is equivalent to (S,T) and is displayed by Φ'. A similar argument shows that if (S,T) is displayed by Φ', then it is equivalent to a 3-separation that is displayed by Φ. Thus Φ and Φ' are equivalent.

We now consider the loose elements. Since $a \in \mathrm{cl}^{(*)}(P_1)$, we see that $\mathrm{fcl}(P_1 \cup \{a\}) = \mathrm{fcl}(P_1)$. Similarly, $\mathrm{fcl}(P_2) = \mathrm{fcl}(P_2 - \{a\})$. It follows easily from these observations that the loose elements of Φ and Φ' are the same.

Now consider Type-2 moves. Assume that $P_2 = \{a,b\}$, where $a \in \mathrm{cl}^{(*)}(P_1)$, and let $\Phi' = (P_1 \cup \{a,b\}, P_3, \ldots, P_n)$. By Lemma 4.9, Φ' is a flower. We now show that Φ and Φ' are equivalent. Let (S,T) be a non-sequential 3-separation of M. Since Φ is a refinement of Φ', it is immediate that if (S,T) is displayed by Φ', then it is displayed by Φ. Assume that (S,T) is displayed by Φ, where $P_1 \subseteq S$. By Lemma 5.2,

$\{a,b\} \subseteq \mathrm{fcl}(P_1)$. Hence (S,T) is equivalent to $(S \cup \{a,b\}, T - \{a,b\})$ and the latter 3-separation is displayed by Φ'. Thus Φ and Φ' are equivalent.

Consider loose elements of Φ and Φ'. Since Φ' is equivalent to Φ and Φ has order at least two, $n \geqslant 3$. We know that $\{a,b\} \subseteq \mathrm{fcl}(P_1)$. From this, it follows that Φ and Φ' have the same loose elements as long as all elements of $\mathrm{fcl}(\{a,b\})$ are loose in Φ'. Clearly, elements of $\mathrm{fcl}(\{a,b\})$ that are not in P_1 are loose in Φ'. But it is easily seen that $\{a,b\} \subseteq \mathrm{fcl}(P_3)$, so $\mathrm{fcl}(P_3) \cap P_1 \supseteq \mathrm{fcl}(\{a,b\}) \cap P_1$. Thus the elements of $\mathrm{fcl}(\{a,b\})$ are indeed loose in Φ' as required.

Consider a move of Type 3. Say that $|P_1| \geqslant 4$, that $\{a,b\} \subseteq P_1$, that $b \in \mathrm{cl}^{(*)}(P_2)$, and that $a \in \mathrm{cl}^{(*)}(P_2 \cup \{b\})$. Then $P_2 \cup \{a,b\}$ is 3-separating. Let $\Phi' = (P_1 - \{a,b\}, \{a,b\}, P_2, \dots, P_n)$. Uncrossing $E - (P_2 \cup \{a,b\})$ and P_1 shows that the intersection of these two sets, $P_1 - \{a,b\}$, is 3-separating. Then each set in the partition Φ' is 3-separating. An analogous argument shows that $P_n \cup (P_1 - \{a,b\})$ is 3-separating. It follows that each union of a consecutive pair of sets in Φ' is 3-separating. Hence Φ' is a flower. We now observe that Φ is obtained from Φ' by a Type-2 move. Hence Φ and Φ' are equivalent and have the same sets of loose elements. □

We will say that the flower Φ_1 is *move-equivalent* to the flower Φ_2 if Φ_2 can be obtained from Φ_1 by a sequence of elementary moves.

Lemma 5.4. *Let M be a 3-connected matroid. Then move-equivalence is an equivalence relation on the set of flowers of M of order at least 2.*

Proof. The relation of move-equivalence is certainly reflexive and transitive. Assume that Φ_1 is a flower of M of order at least 2. If Φ_2 is obtained from Φ_1 by a move of Type 2, then Φ_2 is obtained from Φ_1 by a move of Type 3, provided Φ_2 has at least 2 petals, which it does as Φ_2 has order at least 2. Moreover, if Φ_2 is obtained from Φ_1 by a move of Type 1, then Φ_1 is obtained from Φ_2 by a move of Type 0 followed by a move of Type 1. Thus move-equivalence is also symmetric and is hence an equivalence relation. □

We now work towards showing that every flower of order at least 3 is move-equivalent to a tight flower.

Lemma 5.5. *Let $\Phi = (P_1, P_2, \dots, P_n)$ be a flower and suppose $i \in \{1, 2, \dots, n-2\}$.*

(i) *If $x \in \mathrm{cl}(P_1 \cup P_2 \cup \cdots \cup P_i) - (P_1 \cup P_2 \cup \cdots \cup P_i)$ and $x \notin P_n$, then $x \in \mathrm{cl}(P_i)$.*
(ii) *If $x \in \mathrm{cl}(P_1 \cup P_2 \cup \cdots \cup P_i) - (P_1 \cup P_2 \cup \cdots \cup P_i)$, then $x \in \mathrm{cl}(P_1) - P_1$ or $x \in \mathrm{cl}(P_i) - P_i$.*

Proof. Assume the hypotheses of (i) hold. Then $P_1 \cup P_2 \cup \cdots \cup P_i \cup \{x\}$ and $P_i \cup P_{i+1} \cup \cdots \cup P_{n-1}$ are 3-separating, and their union avoids P_n, so, by uncrossing, their intersection, $P_i \cup \{x\}$, is 3-separating. Thus $x \in \mathrm{cl}^{(*)}(P_i)$. If $x \in \mathrm{cl}^*(P_i)$, then $x \in \mathrm{cl}^*(P_1 \cup P_2 \cup \cdots \cup P_i)$. By Lemma 3.1(ii), this contradicts the fact that

$x \in \mathrm{cl}(P_1 \cup P_2 \cup \cdots \cup P_i)$. Hence $x \in \mathrm{cl}(P_i)$. Part (ii) follows from part (i) using symmetry. $\square$

We omit the statement of the obvious dual of Lemma 5.5, which we shall also use in what follows.

Lemma 5.6. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower. Let a be an element of* $\mathrm{cl}^{(*)}(P_1) \cap P_i$ *for some $i > 1$.*

(i) *If $|P_i| \geqslant 3$, then $(P_1 \cup \{a\}, P_2, \ldots, P_{i-1}, P_i - \{a\}, P_{i+1}, \ldots, P_n)$ is a flower Φ' that is move-equivalent to Φ via a sequence of Type-*1 *moves. Moreover,* $\mathrm{fcl}(P_j{}') = \mathrm{fcl}(P_j)$ *for every petal $P_j{}'$ of Φ'.*
(ii) *If $P_i = \{a, b\}$, then Φ is move-equivalent to*

$$(P_1 \cup \{a\}, P_2, \ldots, P_{i-1} \cup \{b\}, P_{i+1}, \ldots, P_n).$$

Proof. Consider (i). If $i = 2$ or $i = n$, the result follows from a single Type-1 move. Otherwise, by Lemma 5.5(i) or its dual, $a \in \mathrm{cl}^{(*)}(P_{i-1})$. Assume that $|P_i| \geqslant 3$. Then we can use a Type-1 move to obtain a flower equivalent to Φ by taking a out of P_i and adding it to P_{i-1}. This process can clearly be repeated, until a eventually arrives at P_1 as required. This establishes the first part of (i). For the second part, observe that, as both P_1 and $P_1 \cup \{a\}$ are 3-separating, we have $\mathrm{fcl}(P_1) = \mathrm{fcl}(P_1 \cup \{a\})$. Similarly, $\mathrm{fcl}(P_i - \{a\}) = \mathrm{fcl}(P_i)$. The rest of (i) follows from these observations. Consider (ii). By a single Type-2 move, we can add $\{a, b\}$ to P_{i-1} and delete the petal P_i. But, now we can apply (i) to move a to the petal P_1. $\square$

We call a move of the type described in Lemma 5.6(i) a *Type*-1a *move* and a move of the type described in Lemma 5.6(ii) a *Type*-2a *move*.

Lemma 5.7. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower of order at least* 3. *Then Φ is move-equivalent to a tight flower.*

Proof. Assume that Φ is not tight having P_n, say, as a loose petal. We show that, using only Types-1a and 2a moves, we can transform Φ to a move-equivalent flower with fewer petals. Neither of these moves increases the number of petals, so if, at any stage, we have the opportunity to use a Type-2a move, then we have reduced the number of petals. Assume that we never have the opportunity to use such a move.

We now describe a sequence of flowers obtained by using only Type-1a moves. By a sequence of such moves, we may add elements to P_1 to obtain the flower $\Phi_1 = (P_1^1, P_2^1, \ldots, P_n^1)$, where $P_1^1 = \mathrm{fcl}(P_1)$ and $P_i^1 = P_i - \mathrm{fcl}(P_1)$ for all $i > 1$. Also, by Lemma 5.6(i), $\mathrm{fcl}(P_1^i) = \mathrm{fcl}(P_i)$. Now repeat this process with successive petals. After k iterations, we will have a flower Φ_k with the following properties: Φ_k is equivalent to Φ; for each i, $\mathrm{fcl}(P_i^k) = \mathrm{fcl}(P_i)$, and $P_n^k = P_n - (\mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_2^1) \cup \cdots \cup \mathrm{fcl}(P_i^{i-1}))$. Thus $P_n^k = P_n - (\mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_2) \cup \cdots \cup \mathrm{fcl}(P_i))$. In particular, after $n-1$ iterations, we have

$$P_n^{n-1} = P_n - (\mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_2) \cup \cdots \cup \mathrm{fcl}(P_{n-1})).$$

But, as P_n is loose, the above set is empty so that Φ_{n-1} is not a well-defined flower. This contradiction shows that Φ is move-equivalent to a flower with fewer petals. The result now follows easily. $\square$

Lemma 5.8. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower of order at least* 3, *and let T be the set of tight elements of Φ.*

(i) *If Φ' is move-equivalent to Φ, then there is a bijection α between the tight petals of Φ and those of Φ' such that $P \cap T = \alpha(P) \cap T$ for every tight petal P of Φ.*
(ii) *If P is a petal of Φ, then $|P \cap T| \neq 1$.*
(iii) *If P is a tight petal of Φ, then* $\mathrm{fcl}(P \cap T) = \mathrm{fcl}(P)$.

Proof. Part (i) is easily seen to hold if Φ' is obtained from Φ by a single elementary move. Thus it holds if Φ' is obtained by a sequence of such moves. Consider (ii). We prove that a tight petal contains at least two tight elements. By (i) and Lemma 5.7, we lose no generality in assuming that Φ is a tight flower. From this, it follows that if we perform a sequence of moves of Type 1a or Type 2a, we will never have the opportunity to perform a move of Type 2a, as this decreases the number of petals. To complete the proof of (ii), it suffices to show that $|P_n \cap T| \geqslant 2$. Perform the sequence of moves on Φ as described in the proof of Lemma 5.7. In this case, Φ_{n-1} is a well-defined flower and it is tight. Moreover,

$$P_n^{n-1} = P_n - (\mathrm{fcl}(P_1) \cup \mathrm{fcl}(P_2) \cup \cdots \cup \mathrm{fcl}(P_{n-1})) = P_n \cap T.$$

Since Φ_{n-1} is a well-defined flower, $|P_n^{n-1}| \geqslant 2$ so P_n meets T in at least two elements. This establishes (ii).

Consider (iii). Reversing the moves used in (ii) gives a sequence of elementary moves that transforms Φ_{n-1} to Φ. If, for some i, an element is added to P_n^i in going from Φ_i to Φ_{i-1}, then that element is in $\mathrm{cl}^{(*)}(P_n^i)$. It follows that $P_n \subseteq \mathrm{fcl}(P_n \cap T)$. Hence $\mathrm{fcl}(P_n) \subseteq \mathrm{fcl}(P_n \cap T)$. Thus (iii) holds when $P = P_n$ and, by symmetry, it holds in general. $\square$

The proof of the next lemma uses Lemmas 5.5(ii), 5.6, and 5.8 within a straightforward induction argument. We omit the details.

Lemma 5.9. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight flower of order at least* 3.

(i) *If $1 \leqslant j \leqslant n-2$, then*

$$\mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_j) - (P_1 \cup P_2 \cup \cdots \cup P_j) \subseteq (\mathrm{fcl}(P_1) - P_1) \cup (\mathrm{fcl}(P_j) - P_j)$$

and every element of $(\mathrm{fcl}(P_1) - P_1) \cup (\mathrm{fcl}(P_j) - P_j)$ *is loose.*

(ii) *If* $2 \leqslant j \leqslant n-1$, *then* $P_1 \cup P_2 \cup \cdots \cup P_j$ *is a non-sequential* 3-*separating set*. *If*, *in addition*, $j \leqslant n-2$, *then* $(P_1 \cup P_2 \cup \cdots \cup P_j, P_{j+1} \cup P_{j+2} \cup \cdots \cup P_n)$ *is a non-sequential* 3-*separation*.

At last, we can prove the main result of this section.

Proof of Theorem 5.1. By Lemma 5.3, two flowers are equivalent if one can be obtained from the other by a sequence of elementary moves. To prove the converse, let $\Phi = (P_1, P_2, \ldots, P_n)$ and $\Psi = (O_1, O_2, \ldots, O_m)$ be equivalent flowers of order at least 3. By Lemma 5.7, we may assume that Φ and Ψ are both tight flowers. We may also assume that Φ has at least as many petals as Ψ. Let T be the set of tight elements of Φ.

Assume that Φ has at least four petals. Let s and t be tight elements of Φ that are in different petals of Φ. Then there is a consecutive pair of petals P_i and P_{i+1} of Φ such that $s \in P_i \cup P_{i+1}$ and $t \in E - (P_i \cup P_{i+1})$. By Lemma 5.9, $t \notin \mathrm{fcl}(P_i \cup P_{i+1})$ and $s \notin \mathrm{fcl}(E - (P_i \cup P_{i+1}))$. Thus, if (S, T) is a 3-separation equivalent to $(P_i \cup P_{i+1}, E - (P_i \cup P_{i+1}))$ and $s \in S$, then $t \in T$. But Ψ displays some 3-separation equivalent to $(P_i \cup P_{i+1}, E - (P_i \cup P_{i+1}))$. This shows that s and t are in different petals of Ψ. It follows that Ψ has n petals and there is a bijection α between the petals of Φ and those of Ψ such that $\alpha(P_i) \cap T = P_i \cap T$. Moreover, we may assume that the petals of Ψ are labelled so that $P_i \cap T = O_i \cap T$. This is immediate if Ψ is an anemone while, if Ψ is a daisy, it follows from the fact that a union of two petals is 3-separating if and only if the petals are consecutive, so consecutive petals in Φ must map to consecutive petals in Ψ.

Assume that Φ has three petals. Then, since Φ has at least as many petals as Ψ and Ψ has order at least 3, Ψ also has three petals. Also Φ must have at least two petals that are non-sequential regarded as 3-separating sets; otherwise Φ displays at most one non-sequential 3-separation contradicting the fact that it has order 3. Assume, without loss of generality, that P_1 and P_2 are not sequential although P_3 may be sequential. By using elementary moves, we may also assume that $\mathrm{fcl}(P_1) \cap P_3 = \mathrm{fcl}(P_2) \cap P_3 = \emptyset$. This is because (P_1, P_2, P_3) is move-equivalent to $(\mathrm{fcl}(P_1), P_2 - \mathrm{fcl}(P_1), P_3 - \mathrm{fcl}(P_1))$ which, in turn, is move-equivalent to $(\mathrm{fcl}(P_1), (P_2 - \mathrm{fcl}(P_1)) \cup (P_3 \cap \mathrm{fcl}(P_2 - \mathrm{fcl}(P_1))), P_3 - \mathrm{fcl}(P_1) - \mathrm{fcl}(P_2 - \mathrm{fcl}(P_1)))$. We may further assume that $\Psi = (O_1, O_2, O_3)$, where $\mathrm{fcl}(O_i) = \mathrm{fcl}(P_i)$ for i in $\{1, 2\}$ and $\mathrm{fcl}(O_1) \cap O_3 = \mathrm{fcl}(O_2) \cap O_3 = \emptyset$. Say that s is a tight element of Φ. Assume that $s \in P_1$. Since s is tight, $s \notin \mathrm{fcl}(P_2)$. Assume that $s \notin O_1$. Then, since $\mathrm{fcl}(O_1) = \mathrm{fcl}(P_1)$, we have $s \in \mathrm{fcl}(O_1)$, so $s \in O_2$. But $\mathrm{fcl}(O_2) = \mathrm{fcl}(P_2)$ contradicting the fact that $s \notin \mathrm{fcl}(P_2)$. This proves that $s \in O_1$. The same argument shows that if $s \in P_2$, then $s \in O_2$. Assume that $s \in P_3$. Then $s \notin \mathrm{fcl}(P_1)$ by assumption, so $s \notin O_1$. Similarly, $s \notin O_2$. Hence $s \in O_3$. Therefore, in this case too, we have $P_i \cap T = O_i \cap T$ for all i. Moreover, it is easily seen that every element of T is tight in Ψ. By reversing the roles of Φ and Ψ in the argument above, we conclude that T is the set of tight elements of Ψ.

Now consider P_1 and O_1. By Lemma 5.8(iii), $\mathrm{fcl}(P_1) = \mathrm{fcl}(P_1 \cap T)$, and $\mathrm{fcl}(O_1) = \mathrm{fcl}(O_1 \cap T)$. Hence $\mathrm{fcl}(P_1) = \mathrm{fcl}(O_1)$. But, we can now use elementary moves to transform Φ and Ψ into equivalent flowers $\Phi_1 = (P_1^1, P_2^1, \ldots, P_n^1)$ and $\Psi_1 = (O_1^1, O_2^1, \ldots, O_n^1)$, where $O_1^1 = P_1^1$ and if $i \geqslant 2$, then $O_i^1 \cap T = P_i^1 \cap T$. Arguing inductively, assume that we have transformed Φ and Ψ into equivalent flowers $\Phi_k = (P_1^k, P_2^k, \ldots, P_n^k)$ and $\Psi_k = (O_1^k, O_2^k, \ldots, O_n^k)$ for some $k \leqslant n-1$, where $O_i^k = P_i^k$ for $i \leqslant k$, and $O_i^k \cap T = P_i^k \cap T$ otherwise. Then $\mathrm{fcl}(P_{k+1}^k) = \mathrm{fcl}(O_{k+1}^k)$, so that we can use elementary moves to transform Φ_k and Ψ_k into equivalent flowers Φ_{k+1} and Ψ_{k+1} such that if $i \leqslant k+1$, then $P_i^{k+1} = O_i^{k+1}$ and, otherwise, $P_i^{k+1} \cap T = O_i^{k+1} \cap T$. Finally, we have $\Phi_n = \Psi_n$. Now Φ is move-equivalent to Φ_n, and Ψ is move-equivalent to Ψ_n. Since move-equivalence is an equivalence relation, this proves that Φ is move-equivalent to Ψ. $\square$

Finally, we note some corollaries of results in this section. We omit the routine proofs.

Corollary 5.10. *If Φ is a flower, then the order of Φ is the number of petals in any tight flower equivalent to Φ.*

Corollary 5.11. *If Φ and Φ' are equivalent tight flowers of order at least* 2, *then Φ can be transformed to Φ' by a sequence of moves of Type* 0 *and Type* 2a.

Corollary 5.12. *If $\Phi = (P_1, P_2, \ldots, P_n)$ is a tight flower, and $P_1{}'$ is a* 3*-separating set that contains and is equivalent to P_1, then $(P_1{}', P_2 - P_1{}', \ldots, P_n - P_1{}')$ is a tight flower equivalent to Φ. In particular, this holds when $P_1{}' \in \{\mathrm{cl}(P_1), \mathrm{cl}^*(P_1), \mathrm{fcl}(P_1)\}$.*

6. Flower types and equivalence

It would seem clear that equivalent flowers should have the same type. But, for flowers of order less than 3, this is not the case. In this section, we seek to show that flower equivalence preserves type for flowers of order at least three. But, for this to be possible, we need to clarify the status of unresolved flowers. Before doing that, we deal with the Vámos-like case, which is quite special.

Theorem 6.1. *Let Φ be a Vámos-like flower. Then Φ has no loose elements. Hence any flower equivalent to Φ is equal to Φ up to a permutation of the petals.*

Proof. Assume that $\Phi = (P_1, P_2, P_3, P_4)$. Then, by the definition of a Vámos-like flower, $\sqcap(P_i, P_{i+1}) = 1$ for all i. Moreover, we may assume that $\sqcap(P_1, P_3) = 0$, while $\sqcap(P_2, P_4) = 1$. Note that (P_1, P_4, P_3, P_2) is an equivalent Vámos-like flower. In what follows, we take advantage of this symmetry. Assume that Φ has a loose element e. Then, by duality, we may assume that $e \in \mathrm{cl}(P_i) - P_i$ for some petal P_i. Up to symmetry, there are two cases. For the first, assume that $e \in \mathrm{cl}(P_1) - P_1$. As

$\sqcap(P_1, P_3) = 0$, it follows from Lemma 2.5 that $e \notin P_3$. Hence, by symmetry, we may assume that $e \in P_2$. Thus

$$e \in \mathrm{cl}(P_1) \quad \text{and} \quad e \in \mathrm{cl}(P_2). \tag{3}$$

For the second case, we may assume that $e \in \mathrm{cl}(P_2) - P_2$. If $e \in P_1$, then (3) holds, while, if $e \in P_3$, then (3) holds up to symmetry. Say $e \in P_4$. Then $e \in \mathrm{cl}(P_4 \cup P_1)$ and $e \in \mathrm{cl}(P_1 \cup P_2)$. By Lemma 4.12, these two flats form a modular pair whose intersection is spanned by P_1. Hence $e \in \mathrm{cl}(P_1)$. Thus, in all cases, we may assume that (3) holds.

By (3), $e \in \mathrm{cl}(P_1 \cup P_4)$ and $e \in \mathrm{cl}(P_2 \cup P_4)$. While it does not follow from Lemma 4.12 that $\mathrm{cl}(P_1 \cup P_4)$ and $\mathrm{cl}(P_2 \cup P_4)$ are a modular pair of flats, this is still true. To see this, observe that

$$\begin{aligned} r(M) &= r(P_1 \cup P_2) + r(P_3 \cup P_4) - 2 \\ &= r(P_1) + r(P_2) + r(P_3) + r(P_4) - 4. \end{aligned}$$

Also, $r(M) = r(P_1 \cup P_2 \cup P_4) + r(P_3) - 2$. So

$$r(P_1 \cup P_2 \cup P_4) = r(P_1) + r(P_2) + r(P_4) - 2.$$

But

$$r(P_1 \cup P_4) + r(P_2 \cup P_4) = r(P_1) + r(P_2) + 2r(P_4) - 2.$$

Hence $r(P_1 \cup P_4) + r(P_2 \cup P_4) = r(P_1 \cup P_2 \cup P_4) + r(P_4)$, so $\mathrm{cl}(P_1 \cup P_4)$ and $\mathrm{cl}(P_2 \cup P_4)$ are a modular pair of flats whose intersection is spanned by P_4. Thus $e \in \mathrm{cl}(P_4)$.

We now know that $e \in \mathrm{cl}(P_2 \cup P_3)$, and $e \in \mathrm{cl}(P_3 \cup P_4)$, and we can apply Lemma 4.12 to deduce that $e \in \mathrm{cl}(P_3)$. We conclude that $e \in \mathrm{cl}(P_1)$ and $e \in \mathrm{cl}(P_3)$, contradicting the fact that $\sqcap(P_1, P_3) = 0$. $\square$

The last theorem enables us to verify that Vámos-like flowers do not occur in representable matroids.

Corollary 6.2. *If Φ is a Vámos-like flower in a matroid M, then M is not representable over any field.*

Proof. Let $\Phi = (P_1, P_2, P_3, P_4)$ and $\sqcap(P_1, P_3) = 1$. Assume that M is representable over some field F. Then we can view M as a restriction of the vector space $V(r(M), F)$. As $\sqcap(P_1, P_3) = 1$, the subspaces of $V(r(M), F)$ spanned by P_1 and P_3 meet in a rank-one subspace, V_1. By the last theorem, Φ has no loose elements so no element of V_1 is in M. Let M' be the matroid that is obtained by extending M by a non-zero vector e from V_1. Then $(P_1 \cup \{e\}, P_2, P_3, P_4)$ is a Vámos-like flower in M' and e is loose, contradicting the last theorem. $\square$

Lemma 6.3. *Let $\Phi = (P_1, P_2, P_3)$ be an unresolved flower. Assume that Φ has an element $e \in \mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$. Then e is loose and Φ has at most one other loose* *element. Moreover, if* Φ *has a second loose element* f, *then* $f \in \mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$.

Proof. By Lemma 2.5 and Proposition 4.2,

$$\left|\bigcap_{i=1}^{3} \mathrm{cl}(P_i)\right| \leqslant 1 \quad \text{and} \quad \left|\bigcap_{i=1}^{3} \mathrm{cl}^*(P_i)\right| \leqslant 1. \tag{4}$$

6.3.1. *If there is a loose element* g *different from* e, *then there is a loose element* f *that is different from* e *and is in* $\mathrm{cl}^*(P_i) - P_i$ *for some* i.

Subproof. Suppose that 6.3.1 fails. Then the presence of a second loose element means that there is an element z different from e such that, to within relabelling of the petals, P_1, $P_1 \cup \{e\}$, and $P_1 \cup \{e, z\}$ are 3-separating where $\{e, z\} \subseteq P_2 \cup P_3$.

Let $\{j, k\} = \{2, 3\}$ and suppose that $e \in P_j$ and $|P_j| = 2$. Let $P_j - \{e\} = \{x\}$. Then $P_k \cup \{x\}$ is 3-separating, so $x \in \mathrm{cl}^{(*)}(P_k) - P_k$. As $\sqcap(P_j, P_k) = 1$, the unique element of $\mathrm{cl}(P_j) \cap \mathrm{cl}(P_k)$ is e, so $x \notin \mathrm{cl}(P_k)$. Hence $x \in \mathrm{cl}^*(P_k) - P_k$ and 6.3.1 holds. Thus, we may assume that the petal containing e has at least three elements.

Without loss of generality, assume that $z \in P_2$. From the last paragraph, $|P_3 - \{e\}| \geqslant 2$. Moreover, it is easily seen that $e \in \mathrm{cl}(P_3 - \{e\})$. Thus $e \in \mathrm{cl}((P_3 \cup P_2) - \{e, z\})$, so $(P_3 \cup P_2) - \{z\}$ and its complement, $P_1 \cup \{z\}$, are 3-separating. Hence $z \in \mathrm{cl}^{(*)}(P_1)$. It follows, as in the last paragraph, that $z \notin \mathrm{cl}(P_1)$ so $z \in \mathrm{cl}^*(P_1) - P_1$. $\square$

We now show that, when there is a loose element f satisfying the conclusion of 6.3.1, $f \in \mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$. Without loss of generality, $f \in \mathrm{cl}^*(P_1) - P_1$ and $f \in P_2$. We need to show that $f \in \mathrm{cl}^*(P_3)$. Assume this is not the case. Let $M' = M/e$, and, for $i \in \{1, 2, 3\}$, set $P_i' = P_i - \{e\}$. As $f \notin \mathrm{cl}^*(P_3)$, it follows that f is not a coloop of $M|(P_1 \cup P_2)$, so f is not a coloop of $M'|(P_1' \cup P_2')$. As $f \in \mathrm{cl}^*(P_1)$, we see that f is a coloop of $M|(P_2 \cup P_3)$ and hence of $M|P_2$. But $e \in \mathrm{cl}(P_3)$, so f is a coloop of $M|(P_2 \cup P_3 \cup \{e\})$. Thus, f is a coloop of $M'|(P_2' \cup P_3')$ and hence f is a coloop of $M'|P_2'$. But, since $\sqcap(P_1, P_2) = 1$ and $e \in \mathrm{cl}(P_1) \cap \mathrm{cl}(P_2)$, we have $\sqcap_{M'}(P_1', P_2') = 0$. From this, it follows easily that f is a coloop of $M'|(P_1' \cup P_2')$. This contradiction implies that $f \in \mathrm{cl}^*(P_3)$ and the lemma follows by (4). $\square$

Now let $\Phi = (P_1, P_2, P_3)$ be an unresolved flower. If Φ has no loose elements, then it can be viewed equally as well as spike-like or swirl-like. We shall call such a flower *ambiguous*. If Φ has an element e such that either $e \in \mathrm{cl}(P_1) \cap \mathrm{cl}(P_2) \cap \mathrm{cl}(P_3)$ or $e \in \mathrm{cl}^*(P_1) \cap \mathrm{cl}^*(P_2) \cap \mathrm{cl}^*(P_3)$, then Φ is *spike-like*. If Φ has at least one loose element and is not spike-like, then it is *swirl-like*.

Lemma 6.4. *Assume that* Φ *is an anemone with at least four petals.*

(i) *If* $a \in \mathrm{cl}(P_k) - P_k$ *for some petal* P_k, *then* $a \in \mathrm{cl}(P_i)$ *for each petal* P_i *of* Φ.
(ii) *If* $a \in \mathrm{cl}^*(P_k) - P_k$ *for some petal* P_k, *then* $a \in \mathrm{cl}^*(P_i)$ *for each petal* P_i *of* Φ.

Proof. Without loss of generality, we may assume that $a \in \mathrm{cl}(P_1) - P_1$ and that $a \in P_3$. The result will follow if we can show that $a \in \mathrm{cl}(P_2)$. Now $a \in \mathrm{cl}(P_1)$, so $a \in \mathrm{cl}(P_1 \cup P_2)$. Moreover, $a \notin P_n$ as $n \geqslant 4$. It follows immediately from Lemma 5.5 that $a \in \mathrm{cl}(P_2)$ as required. This establishes (i). Part (ii) is the dual of (i). $\square$

Theorem 6.5. *Let Φ be a flower of order at least 3. Then every flower equivalent to Φ has the same type as Φ.*

Proof. Let $\Phi = (P_1, P_2, \ldots, P_n)$ and assume that Φ' is obtained from Φ by performing a single elementary move. We show that Φ and Φ' have the same type. This is clearly the case after a Type-0 move. Say Φ' is obtained by a Type-1 move. By duality, we may assume that $\Phi' = (P_1 \cup \{e\}, P_2 - \{e\}, P_3, \ldots, P_n)$, where $e \in P_2 \cap \mathrm{cl}(P_1)$ and $|P_2| \geqslant 3$. Then $\sqcap(P_1, P_3) = \sqcap(P_1 \cup \{e\}, P_3)$. If Φ has at least four petals, this shows that the local connectivity between non-adjacent petals is the same in both flowers. Also, $\sqcap(P_3, P_4)$ is the local connectivity between adjacent petals in both flowers. By Theorem 6.1, Φ is not a Vámos-like flower. Hence, by Lemma 4.11, Φ and Φ' have the same type.

Assume that Φ has three petals. As $\sqcap(P_1, P_3) = \sqcap(P_1 \cup \{e\}, P_3)$, it follows by Theorem 5.1 and Lemma 5.3 that Φ and Φ' have the same type unless one is spike-like and the other is swirl-like. In this case, since the inverse of a Type-1 move is a Type-1 move, we may assume that Φ is spike-like. But it is easily seen that $e \in \mathrm{cl}(P_2 - \{e\})$ so, by the last lemma, e is in the closure of each petal of Φ'. Hence, by the definition of a spike-like 3-petal flower, Φ' is also spike-like.

Since the inverse of a Type-2 move is a Type-3 move, it only remains to consider Type-2 moves. Say that $\Phi = (P_1, \{e,f\}, P_3, \ldots, P_n)$, and that $\Phi' = (P_1 \cup \{e,f\}, P_3, \ldots, P_n)$ where $e \in \mathrm{cl}^{(*)}(P_1)$, and $f \in \mathrm{cl}^{(*)}(P_1 \cup \{e\})$. Since Φ has order at least 3, we have $n \geqslant 4$. Again it follows easily from Lemma 4.11 that Φ and Φ' have the same type unless $n = 4$ and one of Φ and Φ' is spike-like and the other is swirl-like.

Consider the exceptional case and assume that Φ' is spike-like. Then, by Lemma 6.3, either e or f is in the closure of each petal. Thus, we may assume that $e \in \mathrm{cl}(P_3)$. But then $\sqcap(\{e,f\}, P_3) > 0$, so Φ is not swirl-like and hence must be spike-like. Assume that Φ is spike-like and assume, by taking the dual if necessary, that $e \in \mathrm{cl}(P_1)$. Then, by Lemma 6.4, $e \in \mathrm{cl}(P_4)$ and $e \in \mathrm{cl}(P_3)$. It now follows from the definition of a spike-like 3-petal flower that Φ' is spike-like. $\square$

The last theorem fails for flowers of order 2. For example, consider the cycle matroid of the graph G in Fig. 3. Let (A, B, X, Y, Z) be the partition of $E(G)$ indicated in the diagram. Then $(A \cup Y \cup Z, X, B)$ is a paddle while $(A, Z, Y, B \cup X)$ is a swirl-like flower. Both of these flowers are equivalent to the tight flower $(A, B \cup X \cup Y \cup Z)$. It is not difficult to see how to modify this example to obtain numerous other examples of flowers of order 2 for which the theorem fails.

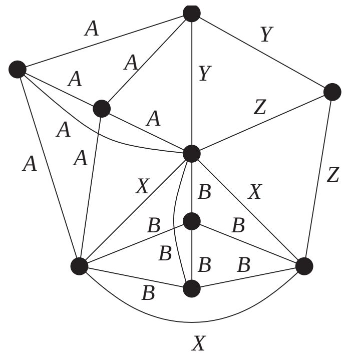

Fig. 3. Theorem 6.5 fails for the cycle matroid of this graph G.

7. More flower structure

In this section, we give a structural description of equivalent flowers. We focus on tight flowers. The extension to general flowers is easy but somewhat messy to describe and we omit it.

Theorem 7.1. *Let M be a 3-connected matroid and let Φ be a tight flower of M of order $n \geqslant 3$ that is a paddle, a copaddle, or is spike-like. Let T and L denote the sets of tight and loose elements of Φ, respectively. For each petal P_i of Φ, let $T_i = P_i \cap T$.*

(i) *If Φ is a paddle, then L is a segment, and $L \subseteq \mathrm{cl}(T_i)$ for each $i \in \{1, 2, \ldots, n\}$;*
(ii) *if Φ is a copaddle, then L is a cosegment, and $L \subseteq \mathrm{cl}^*(T_i)$ for each $i \in \{1, 2, \ldots, n\}$; and*
(iii) *if Φ is spike-like, then $|L| \leqslant 2$. If L contains a single element, then that element is either in the closure of T_i for each i, or is in the coclosure of T_i for each i. If $|L| = 2$, then one member of L is contained in the closure of each T_i, while the other member is contained in the coclosure of each T_i.*

Moreover, up to arbitrary permutations of the petals, the tight flowers equivalent to Φ are precisely the partitions of $E(M)$ of the form

$$(T_1 \cup L_1, T_2 \cup L_2, \ldots, T_n \cup L_n),$$

where $(L_1, L_2, \ldots, L_n)$ is a partition of L.

The next two lemmas build towards the proof of Theorem 7.1.

Lemma 7.2. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower with $n \geqslant 3$.*

(i) *If Φ is a paddle, then each petal of Φ is coclosed.*
(ii) *If Φ is a copaddle, then each petal of Φ is closed.*

Proof. Consider (i). Let Φ be a paddle. We show that P_1 is coclosed. Assume not; say $f \in \mathrm{cl}^*(P_1) - P_1$. Then, as $(P_1, P_2 \cup P_3 \cup \cdots \cup P_n)$ is a 3-separation, it follows from Lemma 3.4(ii) that $\sqcap(P_1, (P_2 \cup P_3 \cup \cdots \cup P_n) - \{f\}) = 1$. Since Φ has at least three petals, $f \notin P_j$ for some $j \in \{2, \ldots, n\}$. Then

$$\sqcap(P_1, P_j) \leqslant \sqcap(P_1, (P_2 \cup P_3 \cup \cdots \cup P_n) - \{f\}) = 1.$$

But, by the definition of a paddle, $\sqcap(P_1, P_j) = 2$. Part (i) follows from this contradiction; (ii) is the dual of (i). $\square$

Lemma 7.3. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight flower with at least 3 petals.*

(i) *If Φ is a paddle, then $\mathrm{fcl}(P_i) = \mathrm{cl}(P_i)$ for each petal P_i of Φ.*
(ii) *If Φ is a copaddle, then $\mathrm{fcl}(P_i) = \mathrm{cl}^*(P_i)$ for each petal P_i of Φ.*

Proof. Say that Φ is a paddle. Consider P_1. By Corollary 5.12, $(\mathrm{cl}(P_1), P_2 - \mathrm{cl}(P_1), \ldots, P_n - \mathrm{cl}(P_1))$ is a paddle equivalent to Φ. But $\mathrm{cl}(P_1)$ is certainly closed and, by Lemma 7.2, it is coclosed. Hence, it is fully closed. This proves (i). Again, (ii) is the dual of (i). $\square$

Proof of Theorem 7.1. Let Φ be a tight flower of order at least 3 that is a paddle, a copaddle, or is spike-like. Assume that Φ is a paddle and that x is loose. We shall show first that $x \in \mathrm{cl}(P_i)$ for all i. By Lemma 7.3, $x \in \mathrm{cl}(P_j)$ for some j and so, by Lemma 6.4, when Φ has at least four petals, $x \in \mathrm{cl}(P_i)$ for all i. Consider the case when Φ has three petals. We may assume that $x \in \mathrm{cl}(P_1) \cap P_2$. Since $\sqcap(P_k, P_{k+1}) = 2$ for all k, an elementary rank argument shows that $\mathrm{cl}(P_1 \cup P_3)$ and $\mathrm{cl}(P_2 \cup P_3)$ are a modular pair of flats whose intersection is spanned by P_3. Thus $x \in \mathrm{cl}(P_3)$ and so $x \in \mathrm{cl}(P_i)$ for all i. We conclude that, when Φ is a paddle, $L \subseteq \mathrm{cl}(P_i)$ for all i. Since $\sqcap(P_1, P_2) = 2$, we have $r(\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2)) \leqslant 2$. Hence L is a segment. Moreover, since $L \subseteq \mathrm{cl}(P_1)$ and $\mathrm{cl}(P_1) \cap P_n \subseteq L$, we deduce that $P_n - \mathrm{cl}(P_1) = T_n$. As $(\mathrm{cl}(P_1), P_2 - \mathrm{cl}(P_1), \ldots, P_n - \mathrm{cl}(P_1))$ is a paddle equivalent to Φ having T_n as a petal, it follows that $L \subseteq \mathrm{cl}(T_n)$. By symmetry, $L \subseteq \mathrm{cl}(T_i)$ for all i. This proves (i); part (ii) follows by duality.

Assume that Φ is spike-like. If Φ has three petals, then (iii) follows by Lemma 6.3. Assume that Φ has at least four petals. By Lemma 6.4, every element of $\bigcup_i (\mathrm{cl}(P_i) - P_i)$ is in $\mathrm{cl}(P_1) \cap \mathrm{cl}(P_2)$. But, by the definition of a spike-like flower, $\sqcap(P_1, P_2) = 1$ so, by Lemma 2.5, $|\bigcup_i (\mathrm{cl}(P_i) - P_i)| \leqslant 1$. Dually, $|\bigcup_i (\mathrm{cl}^*(P_i) - P_i)| \leqslant 1$. Let L' be the union of $\bigcup_i (\mathrm{cl}(P_i) - P_i)$ and $\bigcup_i (\mathrm{cl}^*(P_i) - P_i)$. Then $|L'| \leqslant 2$ and $L' \subseteq L$.

We now show that $L' = L$. By Corollary 5.12 and Theorem 6.5, $\Phi' = (P_1 \cup L', P_2 - L', \ldots, P_n - L')$ is a spike-like flower equivalent to Φ. Say that $P_1 \cup L'$ is not fully closed. Then, up to duality, there is an element $x \in \mathrm{cl}(P_1 \cup L') - (P_1 \cup L')$. Without loss of generality, $x \notin P_2$. As Φ' is an anemone with at least four petals, it follows, from Lemma 6.4, that $x \in \mathrm{cl}(P_2 - L')$, so $x \in \mathrm{cl}(P_2) - P_2$. This contradicts the fact that $\mathrm{cl}(P_2) - P_2 \subseteq L'$. Thus $P_1 \cup L'$ is fully closed. Hence $\mathrm{fcl}(P_1) - P_1 \subseteq \mathrm{fcl}(P_1 \cup L') - P_1 = (P_1 \cup L') - P_1 \subseteq L'$. By symmetry, we deduce that

$L \subseteq L'$ and so $L' = L$. Part (iii) of the theorem now follows routinely and the details are omitted.

To complete the proof, let Φ be a tight flower of order $n \geqslant 3$ that is a paddle, a copaddle, or is spike-like. From above, $\mathrm{cl}(P_1) \cup \mathrm{cl}^*(P_1)$ contains L. Thus, by Corollary 5.12, $(P_1 \cup L, P_2 - L, \ldots, P_n - L)$ is equivalent to Φ. Moreover, for every $i > 1$ and every $l \in L$, we have $l \in \mathrm{cl}^{(*)}(P_i - L)$. Hence we may arbitrarily distribute the members of L amongst the petals of Φ. This shows that every flower of the form described in the statement of the theorem must be equivalent to Φ. Moreover, as this structure is clearly preserved under Type-2a moves, it follows, from Lemma 5.11, that every tight flower equivalent to Φ is of this form. $\square$

We now consider the swirl-like case. If $F = (f_1, f_2, \ldots, f_n)$ is a fan, and $i, j \in \{1, 2, \ldots, n\}$, then we will say that $\{f_1, f_2, \ldots, f_i\}$ is an *initial section* of F, and that $\{f_j, f_{j+1}, \ldots, f_n\}$ is a *terminal section* of F.

Theorem 7.4. *In a matroid M, let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight swirl-like flower of order at least 3 with set T of tight elements and L of loose elements. Let $T_i = P_i \cap T$ for all i. Then there is a partition $(F_1, F_2, \ldots, F_n)$ of L into fans, some of which may be empty, with the following property: a partition $(Q_1, Q_2, \ldots, Q_n)$ of $E(M)$ is a tight swirl-like flower equivalent to Φ if and only if $Q_i = F_{i-1}^- \cup T_i \cup F_i^+$ for all $i \in \{1, 2, \ldots, n\}$, where F_{i-1}^- is a terminal section of F_{i-1}, and F_i^+ is an initial section of F_i.*

The proof of this theorem will use the next three lemmas.

Lemma 7.5. *Let P_i and P_j be petals of a tight swirl-like flower Φ of order at least 3.*

(i) $|\mathrm{cl}(P_i) \cap \mathrm{cl}(P_j)| \leqslant 1$, *and, if P_i and P_j are not consecutive, then* $\mathrm{cl}(P_i) \cap \mathrm{cl}(P_j) = \emptyset$.
(ii) $|\mathrm{cl}^*(P_i) \cap \mathrm{cl}^*(P_j)| \leqslant 1$, *and, if P_i and P_j are not consecutive, then* $\mathrm{cl}^*(P_i) \cap \mathrm{cl}^*(P_j) = \emptyset$.
(iii) *If* $\mathrm{cl}(P_i) \cap P_j \neq \emptyset$, *then* $\mathrm{cl}^*(P_i) \cap P_j = \emptyset$.

Proof. By the definition of a swirl-like flower, $\sqcap(P_i, P_j)$ is 1 if P_i and P_j are consecutive, and is 0 otherwise. Part (i) now follows from Lemma 2.5. Part (ii) is the dual of (i). Consider (iii). Say that $e \in \mathrm{cl}(P_i) \cap P_j$. By (i), we may assume that $(i, j) = (1, 2)$. Assume that there is an element $f \in \mathrm{cl}^*(P_1) \cap P_2$. Then f is a coloop of $M \backslash P_1$, so f is a coloop of $M|P_2$. Now it is easily seen that $e \in \mathrm{cl}(P_2 - \{e\})$ and that f is a coloop of $M|(P_2 - \{e\})$. Thus $e \in \mathrm{cl}(P_2 - \{e, f\})$. By a Type-3 move, transform Φ into the flower $(P_1, \{e, f\}, P_2 - \{e, f\}, P_3, \ldots, P_n\})$. By Theorem 6.5, this flower is swirl-like. Hence $\sqcap(P_1, P_2 - \{e, f\}) = 0$. But $e \in \mathrm{cl}(P_1) \cap (P_2 - \{e, f\})$, so, by Lemma 2.5, $\sqcap(P_1, P_2 - \{e, f\}) > 0$. Part (iii) follows from this contradiction. $\square$

Lemma 7.6. *In a matroid M, let $\Phi = (P_1, P_2, \ldots, P_n)$ be a tight swirl-like flower of order at least 3. Then $\mathrm{fcl}(P_2) \subseteq P_1 \cup P_2 \cup P_3$ and there is a unique ordering*

$(a_1, a_2, \dots, a_l)$ of the elements of $P_3 \cap \mathrm{fcl}(P_2)$ such that, for all $i \in \{1, 2, \dots, l\}$, the set $P_2 \cup \{a_1, a_2, \dots, a_i\}$ is 3-separating.

Proof. Suppose $x \in \mathrm{cl}^{(*)}(P_2) - P_2$. Then, by Lemma 2.5, $x \in P_1$ or $x \in P_3$. We may assume the latter. Then $(P_1, P_2 \cup \{x\}, P_3 - \{x\}, P_4, \dots, P_n)$ is a flower equivalent to Φ. It now follows by an obvious inductive argument that $\mathrm{fcl}(P_2) \subseteq P_1 \cup P_2 \cup P_3$. The elements of $\mathrm{fcl}(P_2) - P_2$ can be ordered $(s_1, s_2, \dots, s_m)$ such that, for all $i \in \{1, 2, \dots, m\}$, the set $P_2 \cup \{s_1, s_2, \dots, s_i\}$ is 3-separating. Now $P_2 \cup P_3$ is 3-separating and Φ is tight so there are at least two elements of M not contained in $P_2 \cup P_3 \cup \{s_1, s_2, \dots, s_i\}$. Thus, by uncrossing, $P_2 \cup (P_3 \cap \{s_1, s_2, \dots, s_i\})$ is 3-separating. It follows that there is an ordering $(a_1, a_2, \dots, a_n)$ of the elements of $P_3 \cap \mathrm{fcl}(P_2)$ such that $P_2 \cup \{a_1, a_2, \dots, a_i\}$ is 3-separating for all $i \in \{1, 2, \dots, l\}$.

We now show that the above ordering is unique. Say that $(b_1, b_2, \dots b_n)$ is another such ordering. Let k be the least integer such that $b_{k+1} \neq a_{k+1}$. Then $(P_1, P_2 \cup \{a_1, a_2, \dots, a_k\}, P_3 - \{a_1, a_2, \dots, a_k\}, P_4, \dots, P_n)$ is a flower equivalent to Φ. But both b_{k+1} and a_{k+1} are in $\mathrm{cl}^{(*)}(P_2 \cup \{a_1, a_2, \dots, a_k\})$, contradicting Lemma 7.5. Thus the ordering is, indeed, unique. $\square$

Lemma 7.7. *Let $\Phi = (P_1, P_2, \dots, P_n)$ be a tight swirl-like flower of order at least 3. Let $(a_1, a_2, \dots, a_l)$ be the ordering of $\mathrm{fcl}(P_2) \cap P_1$ such that, for all $i \in \{1, 2, \dots, l\}$, the set $P_2 \cup \{a_i, a_{i+1}, \dots, a_l\}$ is 3-separating, and let $(a_{l+1}, a_{l+2}, \dots, a_m)$ be the ordering of $\mathrm{fcl}(P_1) \cap P_2$ such that, for all $i \in \{l+1, l+2, \dots, m\}$, the set $P_1 \cup \{a_{l+1}, a_{l+2}, \dots, a_i\}$ is 3-separating. Then $A = (a_1, a_2, \dots, a_m)$ is a fan of M.*

Proof. Consider a triple $\{a_i, a_{i+1}, a_{i+2}\}$ of consecutive elements of A. It is easily seen that $(P_2 - A) \cup \{a_i, a_{i+1}, a_{i+2}, \dots, a_m\}$ and $(P_1 - A) \cup \{a_1, a_2, \dots, a_i, a_{i+1}, a_{i+2}\}$ are both 3-separating. So, by uncrossing, their intersection, $\{a_i, a_{i+1}, a_{i+2}\}$, is 3-separating. Thus, every consecutive triple of elements of A is 3-separating.

If A is not a fan, then, by Lemma 2.2, $|A| \geqslant 4$, and A is either a segment or a cosegment. By duality, we may assume that A and hence $\{a_1, a_2, a_3, a_4\}$ is a segment. But $((P_1 - A) \cup \{a_1, a_2\}, P_2 \cup (A - \{a_1, a_2\}), P_3, \dots, P_n)$ is a swirl-like flower and both a_3 and a_4 are in $\mathrm{cl}((P_1 - A) \cup \{a_1, a_2,\})$ contradicting Lemma 7.5. $\square$

Theorem 7.4 follows straightforwardly from the last three lemmas and we omit the details.

The next corollary will be useful in the proof of the main result of the paper, Theorem 9.1.

Corollary 7.8. *Let $\Phi = (P_1, P_2, \dots, P_n)$ be a tight flower. If $2 \leqslant i \leqslant n-2$ and (X, Y) is a 3-separation that is equivalent to $(P_1 \cup P_2 \cup \dots \cup P_i, P_{i+1} \cup P_{i+2} \cup \dots \cup P_n)$, then there is a tight flower equivalent to Φ that displays (X, Y).*

Proof. We may assume that $\mathrm{fcl}(X) = \mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_i)$ and $\mathrm{fcl}(Y) = \mathrm{fcl}(P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)$. Then all tight elements of $P_1 \cup P_2 \cup \cdots \cup P_i$ and $P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n$ are in X and Y, respectively. We now argue by induction on $|X - (P_1 \cup P_2 \cup \cdots \cup P_i)| + |Y - (P_{i+1} \cup P_{i+2} \cup \cdots \cup P_n)|$. The result is immediate if this sum S is 0. Assume it holds if $S < k$ and let $S = k$. We may assume that $x \in X - (P_1 \cup P_2 \cup \cdots \cup P_i)$. Then $x \in \mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_i) - (P_1 \cup P_2 \cup \cdots \cup P_i)$. Thus, by Lemma 5.9, we may assume that $x \in \mathrm{fcl}(P_i) - P_i$.

If $x \in \mathrm{cl}^{(*)}(P_i)$, then Φ is equivalent to the tight flower Φ' that is obtained by adjoining x to P_i and removing it from its original petal. In this case, the result follows by applying the induction assumption to Φ'.

We may now assume that $x \notin \mathrm{cl}^{(*)}(P_i)$. Then, by Theorem 7.1, Φ is not a paddle, not a copaddle, and is not spike-like. Since Φ has at least four petals, it follows that Φ is swirl-like. Then, by Lemma 7.6, $x \in P_{i+1}$ and there are elements $a_1, a_2, \ldots, a_t$ of $P_{i+1} \cap \mathrm{fcl}(P_i)$ such that $P_i \cup \{a_1, a_2, \ldots, a_j\}$ is 3-separating for all $j \leqslant t$, and $x = a_t$. We may assume that none of $a_1, a_2, \ldots, a_{t-1}$ is in X otherwise we replace x by the first such element. By uncrossing, both $X \cap P_i$ and $(X \cap P_i) \cup \{x\}$ are exactly 3-separating. Hence $x \in \mathrm{cl}^{(*)}(P_i)$; a contradiction. $\square$

8. Maximal flowers

A flower Φ is *maximal* if Φ is equivalent to Φ' whenever $\Phi \preccurlyeq \Phi'$. Let (X, Y) be a 3-separation of M. We say that (X, Y) *conforms* to the flower Φ if either (X, Y) is equivalent to a 3-separation that is displayed by Φ or (X, Y) is equivalent to a 3-separation (X', Y') with the property that either X' or Y' is contained in a petal of Φ.

The goal of this section is to prove the following theorem which is a key result for this paper.

Theorem 8.1. *Let M be a matroid with at least 9 elements and let Φ be a tight maximal flower in M. Then every non-sequential 3-separation of M conforms with Φ.*

The flower Φ is a *refinement* of the flower Φ' if the underlying partition of $E(M)$ for Φ refines that of Φ'. Evidently, if Φ is a refinement of Φ', then $\Phi' \preccurlyeq \Phi$. A partition (X, Y) of $E(M)$ *crosses* the petal P if $P \cap X \neq \emptyset$ and $P \cap Y \neq \emptyset$.

Lemma 8.2. *Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower of a matroid M and let (R, G) be a 3-separation of M such that:*

(i) *neither R nor G is contained in a petal of Φ; and*
(ii) *if (R, G) crosses a petal P, then $|P \cap R|, |P \cap G| \geqslant 2$.*

Then there is a flower that refines Φ and displays (R, G).

Proof. The lemma holds trivially if $n = 1$. If $n = 2$, then, by Lemma 4.11(i), $(P_1 \cap G, P_1 \cap R, P_2 \cap R, P_2 \cap G)$ is the desired flower. Say $n \geqslant 3$. If (R, G) does not cross any petal, then (R, G) is displayed by Φ. Thus assume that (R, G) crosses a petal, say P_1. Set $P_3{}' = P_3 \cup P_4 \cup \cdots \cup P_n$.

8.2.1. *Up to switching* G *and* R, *both* $|P_2 \cap R|$ *and* $|P_3{}' \cap G|$ *exceed* 1.

Subproof. If (R, G) crosses P_2, then, up to switching G and R, we have $|P_3{}' \cap G| \geqslant 2$ and, by (ii), $|P_2 \cap R| \geqslant 2$. We may now assume that (R, G) does not cross P_2. Then, up to switching G and R, we have $P_2 \subseteq R$. But then, by (i), $P_3{}'$ must have at least one green element and, by (ii), it must have at least two such elements, so again $|P_2 \cap R|, |P_3{}' \cap G| \geqslant 2$. $\square$

Assume that labels are chosen so that $|P_2 \cap R|, |P_3{}' \cap G| \geqslant 2$. Then

8.2.2. $\Phi' = (P_1 \cap G, P_1 \cap R, P_2, \ldots, P_n)$ *is a flower.*

Subproof. Evidently, Φ' has at least four petals. So, by Lemma 4.11(i), it suffices to show that the union of all but one consecutive pair of petals is 3-separating. We know that $(P_1 \cap G) \cup (P_1 \cap R) = P_1$ is 3-separating. Thus it suffices to show that $(P_1 \cap R) \cup P_2$ is 3-separating. Since $|P_3{}' \cap G| \geqslant 2$, the set $R \cup (P_1 \cup P_2)$ avoids at least two members of G, so, by uncrossing, $R \cap (P_1 \cup P_2)$ is 3-separating. But $(R \cap (P_1 \cup P_2)) \cap P_2$, which equals $P_2 \cap R$, contains at least two members of R so, by uncrossing, $(R \cap (P_1 \cup P_2)) \cup P_2$, which equals $(P_1 \cap R) \cup P_2$, is 3-separating, as required. $\square$

It now follows from 8.2.2 and an induction on the number of petals crossed by (R, G) that there is a flower that refines Φ and displays (R, G). $\square$

Proof of Theorem 8.1. Let $\Phi = (P_1, P_2, \ldots, P_n)$. Assume that the theorem fails, and that (X, Y) is a non-sequential 3-separation that does not conform with Φ. Let (R, G) be a 3-separation equivalent to (X, Y) with the property that it crosses a minimum number of petals. Since (R, G) is non-sequential, $|R|, |G| \geqslant 4$.

8.1.1. *If* $|R \cap P_i| = 1$, *then* $|G \cap P_i| = 1$.

Subproof. Say $R \cap P_i = \{e\}$ and $|G \cap P_i| \geqslant 2$. Then, by uncrossing, $G \cup P_i$ is 3-separating. But $G \cup P_i = G \cup \{e\}$, so $(R - \{e\}, G \cup \{e\})$ is a 3-separation that is equivalent to (R, G). But $(R - \{e\}, G \cup \{e\})$ crosses fewer petals that (R, G), contradicting the choice of (R, G). $\square$

8.1.2. *There is no petal* P_i *with* $|R \cap P_i| = 1$.

Subproof. Assume that $|R \cap P_1| = 1$, say $R \cap P_1 = \{e\}$. By 8.1.1, $|G \cap P_1| = 1$. Certainly Φ has at least two petals. If Φ has two petals, then Φ displays no non-sequential 3-separation, so Φ is equivalent to the trivial flower and is therefore not tight. We may now assume that Φ has at least three petals. We shall define a partition

(P^+, P^-) of $E(M) - P_1$ into 3-separating sets P^+ and P^- such that

$$P^+ \cup P_1 \text{ is 3-separating; } |P^-| \geqslant 3; \text{ and } |R \cap P^+| \geqslant 2 \text{ or } |G \cap P^+| \geqslant 2. \tag{5}$$

Assume first that Φ has exactly three petals. If $|P_2| = 2$, then Φ displays at most one non-sequential 3-separation, contradicting the fact that Φ is tight. Thus $|P_2|, |P_3| \geqslant 3$. In this case, set $P^+ = P_2$ and $P^- = P_3$. Clearly, (5) holds. Next, assume that Φ has four petals. Then, since $|E(M)| > 8$, one of the petals of Φ has at least 3 elements. This means that we can assume that, amongst 2-element crossed petals, P_1 is chosen so that $|P_2 \cap R| \geqslant 2$ or $|P_2 \cap G| \geqslant 2$. In this case, set $P^+ = P_2$ and $P^- = P_3 \cup P_4$. Again (5) holds. Finally, if Φ has at least five petals, set $P^+ = P_2 \cup P_3$, and $P^- = P_4 \cup P_5 \cup \cdots \cup P_n$. Then (5) holds in this case too and so holds in general.

Next we assert that we may assume, by possibly interchanging R and G, that

$$|P^+ \cap R| \geqslant 2 \quad \text{and} \quad |P^- \cap G| \geqslant 2. \tag{6}$$

By (3), $|P^+ \cap R| \geqslant 2$ or $|P^+ \cap G| \geqslant 2$. If both of the last two inequalities hold, then (6) follows from the fact that both R and G meet $P^+ \cup P^-$ in at least 3 elements. If exactly one of the last two inequalities holds, say $|P^+ \cap R| \geqslant 2$, then $|P^+ \cap G| \leqslant 1$ so $|P^- \cap G| \geqslant 2$ and again (6) holds.

As $(P^+ \cup P_1) \cup R$ avoids $P^- \cap G$, it follows by uncrossing that $(P^+ \cup P_1) \cap R$, which equals $(P^+ \cap R) \cup \{e\}$, is 3-separating. Another uncrossing argument shows that $P^+ \cap R$ is 3-separating and, as this set has at least two elements, we see that $e \in \mathrm{cl}^{(*)}(P^+ \cap R)$ and hence $e \in \mathrm{cl}^{(*)}(P^+)$. But P^+ is a union of at most $n-2$ consecutive petals so, by Lemma 5.9, e is loose in Φ. Thus P_1 contains at most one tight element. But Φ is tight and has at least three petals, so Φ has order at least 3. Hence, by Lemma 5.8, P_1 contains at least two tight elements. The sublemma follows from this contradiction. $\square$

From 8.1.2, we see that (R, G) satisfies the hypotheses of Lemma 8.2. Thus, by that lemma, there is a flower that refines Φ and displays (R, G) contradicting the fact that Φ is maximal. $\square$

The requirement that M has at least 9 elements is essential in the last theorem. For example, let R_8 be the 8-element rank-4 that is represented geometrically by a cube in 3–space (see Fig. 4). Let $\Phi = (\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\})$. Then Φ is a tight maximal flower. However, the non-sequential 3-separation $(\{1,3,5,7\}, \{2,4,6,8\})$ does not conform with Φ. Evidently, we can relax certain circuit-hyperplanes in R_8 to obtain other 8-element matroids for which the theorem fails.

9. Partial 3-trees

Let π be a partition of a finite set E. Let T be a tree such that every member of π labels a vertex of T; some vertices may be unlabelled and no vertex is multiply labelled. We say that T is a *π-labelled tree*; labelled vertices are called *bag vertices* and members of π are called *bags*.

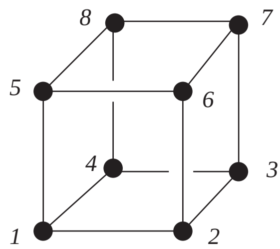

Fig. 4. Theorem 8.1 fails for this matroid, R_8.

Let T' be a subtree of T. The union of those bags that label vertices of T' is the subset of E *displayed* by T'. Let e be an edge of T. The *partition of E displayed by e* is the partition displayed by the components of $T\backslash e$. Let v be a vertex that is not a bag vertex. Then the *partition of E displayed by v* is the partition displayed by the components of $T - v$. The edges incident with v are in natural one-to-one correspondence with the components of $T - v$, and hence with the members of the partition displayed by v. In what follows, if a cyclic ordering $(e_1, e_2, \ldots, e_n)$ is imposed on the edges incident with v, this cyclic ordering is taken to represent the corresponding cyclic ordering on the members of the partition displayed by v.

Let M be a 3-connected matroid with ground set E. An *almost partial 3-tree T* for M is a π-labelled tree, where π is a partition of E such that the following conditions hold:

(i) For each edge e of T, the partition (X, Y) of E displayed by e is 3-separating, and, if e is incident with two bag vertices, then (X, Y) is a non-sequential 3-separation.
(ii) Every non-bag vertex v is labelled either D or A. Moreover, if v is labelled D, then there is a cyclic ordering on the edges incident with v.
(iii) If a vertex v is labelled A, then the partition of E displayed by v is a tight maximal anemone of order at least 3.
(iv) If a vertex v is labelled D, then the partition of E displayed by v, with the cyclic order induced by the cyclic ordering on the edges incident with v, is a tight maximal daisy of order at least 3.

By conditions (iii) and (iv), a vertex v labelled D or A corresponds to a flower of M. The 3-separations displayed by this flower are the 3-separations *displayed by v*. A vertex of a partial 3-tree is referred to as a *daisy vertex* or an *anemone vertex* if it is labelled D or A, respectively. A vertex labelled either D or A is a *flower vertex*. A 3-separation is *displayed by an almost partial* 3-*tree* T if it is displayed by some edge or some flower vertex of T.

A 3-separation (R, G) of M *conforms* with an almost partial 3-tree T if either (R, G) is equivalent to a 3-separation that is displayed by a flower vertex or an edge of T, or (R, G) is equivalent to a 3-separation (R', G') with the property that either R' or G' is contained in a bag of T.

An almost partial 3-tree for M is a *partial* 3-*tree* if

(v) every non-sequential 3-separation of M conforms with T.

We now define a quasi-order on the set of partial 3-trees for M. Let T_1 and T_2 be two partial 3-trees for M. Then $T_1 \preccurlyeq T_2$ if all of the non-sequential 3-separations displayed by T_1 are displayed by T_2. If $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$, then T_1 is *equivalent* to T_2. A partial 3-tree is *maximal* if it is maximal with respect to this quasi order.

The following is the main theorem of the paper.

Theorem 9.1. *Let M be a* 3-*connected matroid with* $|E(M)| \geqslant 9$, *and let T be a maximal partial* 3-*tree for M. Then every non-sequential* 3-*separation of M is equivalent to a* 3-*separation displayed by T.*

Let $\Phi = (P_1, P_2, \ldots, P_n)$ be a flower. We associate with Φ a π-labelled tree T. If $n = 1$, then T consists of a single bag vertex labelled by P_1. If $n = 2$, then T consists of two adjacent bag vertices labelled by P_1 and P_2. Assume that $n \geqslant 3$. Then the vertex set of T is $\{v, v_1, v_2, \ldots, v_n\}$, where v is incident with each v_i and each v_i is labelled by the bag P_i. Finally, label v by A or D according to whether Φ is an anemone or daisy, respectively. In the case that $n = 3$, we are free to label v either A or D. We will often identify Φ with its associated π-labelled tree. Under this identification, we get the following immediate consequence of Theorem 8.1.

Corollary 9.2. *Tight maximal flowers of* 3-*connected matroids are partial* 3-*trees.*

The next result will be useful in the proof of Theorem 9.1.

Lemma 9.3. *If* $(X, E - X)$ *is a non-sequential* 3-*separation of a* 3-*connected matroid M, then there is a tight maximal flower that displays a* 3-*separation equivalent to* $(X, E - X)$.

Proof. Clearly, $(X, E - X)$ is a tight flower Φ_0 that displays $(X, E - X)$. If Φ_0 is not maximal, then there is a maximal flower $\Phi_1 \succcurlyeq \Phi_0$. Since Φ_1 must display some non-sequential 3-separation that is not equivalent to one displayed by Φ_0, we must have that Φ_1 has order at least three. Thus, by Lemma 5.7, Φ_1 is equivalent to a tight maximal flower Φ_2. As $\Phi_2 \succcurlyeq \Phi_0$, there is a 3-separation equivalent to $(X, E - X)$ that is displayed by Φ_2. $\square$

The next lemma contains the core of the proof of Theorem 9.1.

Lemma 9.4. *Let M be a* 3-*connected matroid with* $|E(M)| \geqslant 9$ *and let T be a partial* 3-*tree for M having at least one edge. If M has a non-sequential* 3-*separation* $(W, E - W)$ *that is not equivalent to any* 3-*separation displayed by T, then there is a partial*

3-tree T' such that $T' \succcurlyeq T$ and T' displays some non-sequential 3-separation that is not equivalent to any 3-separation displayed by T.

Proof. By the definition of a partial 3-tree, $(W, E-W)$ conforms with T and so is equivalent to a 3-separation $(X, E-X)$, where X is contained in a bag B of T. Evidently, B is non-sequential. Let u be the vertex of T labelled by B. We distinguish two cases:

(I) u is a leaf of T; and
(II) u is not a leaf of T.

Consider Case I. In that case, $(B, E-B)$ is non-sequential. This follows from the definition of a partial 3-tree when u is adjacent to a bag vertex, and follows from Lemma 5.9 when u is adjacent to a flower vertex.

If B is not fully closed, then we can move the elements of $\mathrm{fcl}(B) - B$ one by one out of their current bags and into the bag B. Each step of this process produces a new partial 3-tree equivalent to T and, at the conclusion of the process, we obtain a partial 3-tree in which u is labelled by $\mathrm{fcl}(B)$. It follows that we may assume that B is fully closed.

Now X is a 3-separating set that is contained in but is not equivalent to B. Let Y be such a set whose full closure is maximal among such sets. By Lemma 9.3, there is a tight maximal flower Φ that displays a 3-separation $(Z, E-Z)$ equivalent to $(Y, E-Y)$. Since B is fully closed, $Z \subseteq B$.

9.4.1. *There is a tight maximal flower equivalent to Φ that has a petal containing $E-B$.*

Subproof. By Theorem 8.1, $(E-B, B)$ conforms with Φ. Thus either

(i) $E-B$ is equivalent to a 3-separating set Q' contained in a petal Q of Φ; or
(ii) $E-B$ is equivalent to a union of petals of Φ.

Consider (i). By Lemma 3.5, $\mathrm{fcl}(Q) - Z$ is equivalent to Q. Also $\mathrm{fcl}(Q) - Z \supseteq E-B$. So, by Corollary 5.12, there is a flower equivalent to Φ that displays Z such that $E-B$ is contained in a petal.

Now consider (ii). Let $\Phi = (Q_1, Q_2, \ldots, Q_n)$. Then we may assume that $E-B$ is equivalent to $Q_1 \cup Q_2 \cup \cdots \cup Q_k$ for some $k \geqslant 2$. As Z is displayed by Φ and Z is not equivalent to B, we must have $n-k \geqslant 2$. By Corollary 7.8 and Lemma 5.8(i), there is a tight flower $\Phi' = (Q_1', Q_2', \ldots, Q_n')$ equivalent to Φ where $(Q_1' \cup Q_2' \cup \cdots \cup Q_k', Q_{k+1}' \cup Q_{k+2}' \cup \cdots \cup Q_n') = (E-B, B)$. As Φ' is tight, $\mathrm{fcl}(Q_1' \cup Q_n')$ contains neither Q_{k+1}' nor Q_k' and so contains neither B nor $E-B$. Similarly, $\mathrm{fcl}(E - (Q_1' \cup Q_n'))$ contains neither B nor $E-B$. Thus every 3-separation equivalent to $(Q_1' \cup Q_n', E - (Q_1' \cup Q_n'))$ crosses both B and $E-B$. Therefore, $(Q_1' \cup Q_n', E - (Q_1' \cup Q_n'))$ does not conform with T contradicting the fact that T is a partial 3-tree. $\square$

By 9.4.1, we may assume that $\Phi = (P_1, P_2, \ldots, P_n)$ where $E - B \subseteq P_n$ and Z is some union of consecutive petals from $\{P_1, P_2, \ldots, P_{n-1}\}$. If $n = 2$, then $Z = P_1$ and we modify T to produce T' by adding a new vertex z adjacent to u, relabelling u by $B - Z$, and labelling z by Z. If $n \geqslant 3$, we construct T' from T as follows: first adjoin a new flower vertex v adjacent to u labelling v either A or D depending upon whether Φ is an anemone or a daisy, respectively; then adjoin bag vertices $v_1, v_2, \ldots, v_{n-1}$ adjacent to v labelling these by $P_1, P_2, \ldots, P_{n-1}$; finally, relabel the vertex u by $B - (P_1 \cup \cdots \cup P_{n-1})$. To verify that T' is a partial 3-tree, it suffices to consider the non-sequential 3-separations with $R \subseteq B$. By Theorem 8.1, such a 3-separation conforms with Φ and hence with T' unless (R, G) is equivalent to (R', G') where R' or G' is contained in P_n. Consider the exceptional case. If $R' \subseteq P_n$, then $R' \subseteq E - (P_1 \cup P_2 \cup \cdots \cup P_{n-1})$. But $\mathrm{fcl}(R') = \mathrm{fcl}(R) \subseteq B$, so $R' \subseteq B$. Hence $R' \subseteq B - (P_1 \cup P_2 \cup \cdots \cup P_{n-1})$. As the last set labels a bag of T', it follows, in this case, that (R, G) conforms with T'. We may now assume that $G' \subseteq P_n$. Then $R' \supseteq P_1 \cup P_2 \cup \cdots \cup P_{n-1}$. Moreover, we may assume that $B \supsetneqq \mathrm{fcl}(R') \supsetneqq \mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_{n-1})$ otherwise (R, G) is equivalent to a 3-separation displayed by T'. But $\mathrm{fcl}(Y) = \mathrm{fcl}(Z) \subseteq \mathrm{fcl}(P_1 \cup P_2 \cup \cdots \cup P_{n-1})$. Thus R' contradicts the choice of Y and we conclude that T' is a partial 3-tree. Clearly, $T' \succcurlyeq T$. Moreover, $(P_1, E - P_1)$ is a non-sequential 3-separation for which there is no equivalent 3-separation displayed by T. Hence the lemma holds in Case I.

Consider Case II. Choose a 3-separating set Z of M that is maximal with the property that $X \subseteq Z \subseteq B$. Let T' be the tree that is obtained from T by adjoining a new leaf v adjacent to u such that v is a bag vertex labelled by Z, and u is relabelled by $B - Z$. It is easily verified that T' satisfies the first four properties of a partial 3-tree. Assume that it does not satisfy (v). Then there is a non-sequential 3-separation $(Y, E - Y)$ that does not conform with T'. Since T is a partial 3-tree and T' only differs from T by adding v and changing the bag B, we may assume, by possibly replacing $(Y, E - Y)$ by an equivalent 3-separation, that $Y \subseteq B$ and that both $Y \cap Z$ and $Y \cap (B - Z)$ are non-empty. Assume that $|Y \cap Z| = 1$, say $Y \cap Z = \{z\}$. Since $Z \supseteq X$ and $(X, E - X)$ is non-sequential, we have $|Z - \{z\}| \geqslant 2$. But $Z - \{z\} = E - (Y \cup (E - Z))$, and so, by uncrossing, $Y \cap (E - Z)$, which equals $Y - \{z\}$, is 3-separating. Thus Y is equivalent to $Y - \{z\}$. But, as $Y - \{z\} \subseteq B - Z$, we see that $(Y - \{z\}, E - (Y - \{z\}))$ conforms with T'. Hence $(Y, E - Y)$ conforms with T'; a contradiction. Thus we may assume that $|Y \cap Z| \geq 2$. Therefore, by uncrossing, $Y \cup Z$ is 3-separating, contradicting the maximality of Z. Hence T' is indeed a partial 3-tree.

Clearly, $T \preccurlyeq T'$ and $(Z, E - Z)$ is a non-sequential 3-separation. Thus the lemma holds or Z is equivalent to a 3-separating set displayed by T. Since X is not equivalent to such a 3-separating set, the sets X and Z are not equivalent. Now we may assume that $(X, E - X)$ is not equivalent to any 3-separation displayed by T' otherwise the lemma holds. Since X is contained in the bag Z of T' and this bag is a leaf bag, it follows from Case I that there is a partial 3-tree $T'' \succcurlyeq T'$ such that T'' displays some non-sequential 3-separation that is not equivalent to any 3-separation displayed by T' and hence is not equivalent to any 3-separation displayed by T. $\square$

Proof of Theorem 9.1. Let E be the ground set of M. If M has no non-sequential 3-separations, then T consists of a single bag vertex labelled by E, and T satisfies the theorem. If M has a non-sequential 3-separation (R, G), then, by Lemma 9.3, there is a tight maximal flower displaying a 3-separation equivalent to (R, G) and so, by Corollary 9.2, there is a partial 3-tree T displaying a 3-separation equivalent to (R, G). Thus we may assume that T has at least one edge. Then the theorem holds, otherwise, by Lemma 9.4, we obtain the contradiction that T is not maximal. $\square$

References

[1] W.H. Cunningham, J. Edmonds, A combinatorial decomposition theory, Canad. J. Math. 32 (1980) 734–765.

[2] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected binary matroids, J. Combin. Theory Ser. B 66 (1996) 334–360.

[3] G. Ding, B. Oporowski, J. Oxley, D. Vertigan, Unavoidable minors of large 3-connected matroids, J. Combin. Theory Ser. B 71 (1997) 244–293.

[4] J. Geelen, J. Oxley, D. Vertigan, G. Whittle, Totally free expansions of matroids, J. Combin. Theory Ser. B 84 (2002) 130–179.

[5] J. Geelen, G. Whittle, Matroid 4-connectivity: a deletion–contraction theorem, J. Combin. Theory Ser. B 83 (2001) 15–37.

[6] R. Hall, J. Oxley, C. Semple, G. Whittle, On matroids of branch-width three, J. Combin. Theory Ser. B 86 (2002) 148–171.

[7] J. Kahn, On the uniqueness of matroid representation over $GF(4)$, Bull. London Math. Soc. 20 (1988) 5–10.

[8] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.

[9] J. Oxley, D. Vertigan, G. Whittle, On inequivalent representations of matroids, J. Combin. Theory Ser. B 67 (1996) 325–343.

[10] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.

[11] H. Whitney, On the abstract theory of linear dependence, Amer. J. Math. 57 (1935) 509–533.

[12] Z. Wu, On extremal connectivity properties of unavoidable matroids, J. Combin. Theory Ser. B 75 (1999) 19–45.