Journal of Combinatorial Theory, Series B **71**, 215–230 (1997)
Article No. TB971789

A 2-Isomorphism Theorem for Hypergraphs

Dirk Vertigan*

Mathematics Department, Louisiana State University, Baton Rouge, Louisiana

and

Geoff Whittle†

Mathematics Department, Victoria University, Wellington, New Zealand

Received October 23, 1996

One can associate a polymatroid with a hypergraph that naturally generalises the cycle matroid of a graph. Whitney's 2-isomorphism theorem characterises when two graphs have isomorphic cycle matroids. In this paper Whitney's theorem is generalised to hypergraphs and polymatroids by characterising when two hypergraphs have isomorphic associated polymatroids. © 1997 Academic Press

1. INTRODUCTION

Without doubt, one of the most important invariants of a graph is its cycle matroid. Since Whitney's 2-isomorphism Theorem [7] characterises exactly when two graphs have isomorphic cycle matroids it is one of the cornerstones of this area of mathematics.

Now polymatroids generalise matroids in much the same way that hypergraphs generalise graphs; in the latter case one lifts the restriction that edges meet at most two vertices, while in the former one lifts the restriction that singletons have rank at most one. Moreover, given a hypergraph H, one can associate with it a polymatroid χ_H just as one can associate a matroid with a graph. Indeed, if H is a graph, then χ_H is the usual cycle matroid of this graph.

It is also the case that the polymatroid χ_H carries essentially the same information as that carried by the cycle matroid of a graph (see for example [1, 3, 8]). It follows that χ_H is as important an invariant of the hypergraph H as is the cycle matroid of a graph. Given this, it is natural to generalize Whitney's theorem to polymatroids and hypergraphs, and the main theorem of this paper (Theorem 3.1) does just that by

* E-mail: vertigan@math.lsu.edu.
† E-mail: whittle@kauri.vuw.ac.nz.

0095-8956/97 $25.00
Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.

characterising exactly when two hypergraphs have isomorphic associated polymatroids.

There are several proofs of Whitney's theorem in the literature [5–7] (see also [2, Chapter 5]), but none of these generalise easily to hypergraphs. Specific difficulties occur in the hypergraph case that do not arise for graphs. Thus the proof given here is quite independent of known proofs for graphs. Moreover, when specialized to graphs, our proof gives yet another proof of Whitney's theorem for graphs.

Theorem 3.1 also generalises a result of Swaminathan and Wagner [4]. They characterise when two graphs have the same sets of vertex sets of cycles. This characterisation follows from Theorem 3.1 by considering hypergraphs obtained from graphs by interchanging vertices and edges. It should be noted that while both Whitney's Theorem and that of Swaminathan and Wagner are stated in terms of cycles, this does not seem to be possible for Theorem 3.1.

The remainder of this introduction is devoted to giving a geometric description of χ_H that may aid the reader's intuition in reading this paper. Formal definitions are given in Section 2.

Let G be a graph. Represent the vertices of G as the points of a simplex V in some projective space P. (Any projective space of sufficiently high dimension will do.) Apart from loops, we can regard the edges of G as being lines of P. Now let F be a hyperplane of P that does not contain any of the points of V. (It is not hard to show that such a hyperplane can always be found.) Any edge of G meets F in a single point. The set of such points with rank induced by P forms a matroid. It is straightforwardly shown that this matroid is the cycle matroid of G.

Loosely speaking, one can regard polymatroids as axiomatising the notion of configuration of subspaces in the same way that one can regard matroids as abstracting point configurations. If we perform the above construction for hypergraphs instead of just graphs we obtain such a configuration of subspaces. Specifically let H be a hypergraph. Again the vertices of H can be represented by a simplex V in a projective space P. A hyperedge of H of size k spans a rank-k subspace of P, and such a subspace meets a hyperplane F in a rank-$(k-1)$ subspace. The collection of such subspaces with rank function induced by rank in P forms a polymatroid. This polymatroid is χ_H. Note that while there are other polymatroids that can be associated with a hypergraph it is only χ_H that is considered in this paper.

2. PRELIMINARIES

In this section we introduce basic definitions and notational conventions for hypergraphs and polymatroids that will be used throughout the paper. Other notions will be introduced as they are needed.

Hypergraphs. A *hypergraph* H is a triple (V, E, I) where V and E are finite sets called *vertices* and *edges* respectively, and $I \subseteq V \times E$ is the *incidence relation* of H. In this paper every edge is incident with at least one vertex.

For a subset A of edges, $\bar{A}$ denotes the set of vertices incident with at least one edge in A. Also $H \mid A$ denotes the restriction of H to A, that is $H \mid A$ denotes the hypergraph $(\bar{A}, A, I \cap (\bar{A} \times A))$.

For a subset W of vertices $\bar{W}$ denotes the maximal set F of edges such that $\bar{F}$ is contained in W. Also $H \mid W$ denotes the *restriction* of H to W, that is $H \mid W$ denotes the hypergraph $(W, \bar{W}, I \cap (W \times \bar{W}))$.

The notation $\bar{A}$ and $\bar{W}$ may be ambiguous if more than one hypergraph is being considered. In most cases the intention is clear from the context. In other cases ambiguity is removed by explicitly specifying the appropriate hypergraph H, using the notation $V_H(A)$ and $E_H(W)$.

A subset W of vertices is a *separator* of H if, for any edge e, either $\bar{e}$ is contained in W or $\bar{e}$ is disjoint from W. A *component* of H is a minimal non-empty separator. A non-trivial hypergraph is *connected* if it has exactly one component. Two edges e and f are *parallel* if $\bar{e} = \bar{f}$. An edge e is a *loop* if $|\bar{e}| = 1$. A hypergraph H is *simple* if it has no loops or parallel edges.

Polymatroids. Let E be a finite set and consider an integer valued set function $\rho: 2^E \to Z$. The function ρ is *normalised* if $\rho(\emptyset) = 0$, is *increasing* if $\rho(A) \leqslant \rho(B)$ whenever $A \subseteq B \subseteq E$, and is *submodular* if $\rho(A \cup B) + \rho(A \cap B) \leqslant \rho(A) + \rho(B)$ for all subsets A and B of E. If ρ is normalised, increasing and submodular, then ρ is a *polymatroid.*

For a subset A of the edges of the hypergraph $H = (V, E, I)$, let $k(H \mid A)$ denote the number of components of $H \mid A$. The *hypergraphic polymatroid* χ_H of H is defined, for all subsets A of E, by

$$\chi_H(A) = |\bar{A}| - k(H \mid A).$$

It is well known that χ_H is a polymatroid. Indeed it is straightforward to verify that χ_H is just the polymatroid described geometrically in the Introduction (see [8]). Evidently, if H is a graph, then χ_H is the rank function of the cycle matroid of the graph.

Rank-equivalent hypergraphs. Two hypergraphs H and I are *rank-equivalent* if $\chi_H = \chi_I$, that is, H and I have the same edge set E and $\chi_H(A) = \chi_I(A)$ for every $A \subseteq E$. It is clear that neither vertex labelling nor the presence of an arbitrary number of isolated vertices affects rank-equivalence. Therefore we shall be casual about this and generally will regard two hypergraphs as being equal if they are equal up to vertex labelling and isolated vertices.

However, relabelling edges of a hypergraph H gives a hypergraph H' having the property that $\chi_{H'}$ is isomorphic, but not necessarily equal to χ_H.

Evidently, two hypergraphs have isomorphic hypergraphic polymatroids if and only if, up to relabelling of the edges, they are rank-equivalent. Thus the main task of this paper is to characterise when two hypergraphs are rank-equivalent.

Twisting, splitting, and joining. A *twisting partition* of $H=(V, E, I)$ is a partition $\{U, W, u, w\}$ of V such that for every edge e of H, either $\bar{e} \subseteq U \cup \{u, w\}$ or $\bar{e} \subseteq W \cup \{u, w\}$ or $\{u, w\} \subseteq \bar{e}$. Associated with a twisting partition we define a *twisting* of H as follows: for each edge e of E with $\bar{e} \subseteq W \cup \{u, w\}$ and $|\bar{e} \cap \{u, w\}| = 1$, change $\bar{e}$ to $\bar{e} \Delta \{u, w\}$ (where Δ denotes symmetric difference). Evidently, if H is a graph, the notion of twisting defined here reduces to the familiar one. Observe that the possible presence of edges meeting both U and W is a feature which does not occur in the graph case.

If H' is obtained from H by twisting using the twisting partition $\{U, W, u, w\}$, then H' is said to be obtained by twisting *around* $\{u, w\}$. Note that there may be more than one possible twisting around $\{u, w\}$—consider, for instance, the graph $K_{2,n}$.

Now consider splitting and joining. A vertex v of H is a *cut vertex* if the edges of H can be partitioned into subsets A and B such that $\bar{A} \cap \bar{B} = \{v\}$. We can define a *splitting* of H at a cut vertex just as for graphs. Also, if v_1 and v_2 are vertices in different components of H the operation can be reversed to obtain a hypergraph in which v_1 and v_2 are *joined* into a single cut vertex.

Given that we are relaxed about the presence or absence of isolated vertices it is worth noting that splitting and joining are special cases of twisting. To split H at the cut vertex $\{v\}$ proceed as follows: add a new isolated vertex w and perform a twisting on the twisting partition $\{\bar{A} - v, \bar{B} - v, v, w\}$. Joining is just the reverse of this.

Note that it can happen that a twisting does not change the hypergraph; we call such twistings *trivial*.

2-*isomorphism.* Hypergraphs H and I are 2-*isomorphic* if H can be transformed into I by a sequence of twistings, splittings and joinings. (If we are to be pedantic, we can also include the operations of vertex relabelling and addition and removal of isolated vertices.)

3. THE MAIN RESULT

We are now in a position to state the main result of the paper.

THEOREM 3.1. *Let H and I be hypergraphs. Then H and I are rank-equivalent if and only if they are 2-isomorphic.*

Before proceeding to the proof we note the following immediate corollary of Theorem 3.1. The hypergraph H is *rank-unique* if the only hypergraph that is rank-equivalent to H is H itself (up to vertex labelling and isolated vertices of course).

COROLLARY 3.2. *A hypergraph is rank-unique if and only if it has no non-trivial twisting partitions.*

Another corollary follows from comments in Section 2.

COROLLARY 3.3. *Let H and I be hypergraphs. Then χ_H is isomorphic to χ_I if and only if I is isomorphic to a hypergraph that is 2-isomorphic to H.*

Note that Corollary 3.3 is strictly weaker than Theorem 3.1 since there are hypergraphs H and I which are isomorphic and rank-equivalent, but are not equal.

In one direction Theorem 3.1 is straightforward.

LEMMA 3.4. *If H and I are 2-isomorphic, then they are rank-equivalent.*

Proof. It clearly suffices to consider the case when I is obtained from H via a single twist. Assume then, that I is obtained from H by twisting on the twisting partition $\{U, W, u, w\}$. Let A be a subset of edges. If $\bar{A} \cap \{u, w\}$ has the same cardinality in H as in I, then it is clear that $\chi_H(A) = \chi_I(A)$. If this is not the case, then we may assume without loss of generality, that $\bar{A}$ contains more elements of $\{u, w\}$ in I than in H. There are two cases, the only one that could cause difficulty is when $\bar{A}$ contains one element of $\{u, w\}$ in H and $\bar{A}$ contains both. In this case it is readily checked that $k(I \mid A) = k(H \mid A) + 1$ and it follows that $\chi_H(A) = \chi_I(A)$. ∎

The rest of the paper is devoted to the converse of Lemma 3.4. We begin by examining connectivity in more detail. A hypergraph is 2-*connected* if it is connected and has no cut vertices. A separator of a polymatroid ρ on E is a subset A of E with the property that $\rho(A) + \rho(E - A) = \rho(E)$. A *component* of ρ is a minimal non-empty separator. It is easily seen that unions of components are separators. The polymatroid ρ is *connected* if its only separators are E and $\emptyset$. The following proposition generalises a well-known fact for graphs and matroids; we omit the routine proof.

PROPOSITION 3.5. *Let H be a hypergraph. Then χ_H is connected if and only if H is 2-connected.*

A simple argument that is essentially the same as that for the graph case shows that Theorem 3.1 will follow if the following lemma holds.

LEMMA 3.6. *If H and I are rank-equivalent hypergraphs and H is simple and 2-connected, then H and I are 2-isomorphic.*

The task then is to prove Lemma 3.6. First an example to illustrate just how different things are for hypergraphs and polymatroids than for graphs and matroids.

Let H be a hypergraph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{a, b, c, d\}$ where $\bar{a} = \{1, 2\}$, $\bar{b} = \{1, 3, 4\}$, $\bar{c} = \{2, 3, 5\}$ and $\bar{d} = \{4, 5\}$. It is easily seen that H is 2-connected and rank-unique. However deleting edges incident with 3, namely b and c, leaves a hypergraph that is not connected. In the graph case this is not possible for 2-connected graphs, let alone rank-unique graphs.

A *complete* hypergraph is a hypergraph (V, E, I), $|V| \geqslant 2$ having the property that for every subset W of V having at least two elements, there is a unique edge $\bar{e}$ such that $\bar{e} = W$. The following lemma is evident.

LEMMA 3.7. *Complete hypergraphs are rank-unique.*

The next lemma follows immediately from Proposition 3.5.

LEMMA 3.8. *If H and I are rank-equivalent hypergraphs and A is a set of edges for which $H \mid A$ is 2-connected, then $I \mid A$ is 2-connected.*

A subset A of the polymatroid ρ on E is *spanning* if $\rho(A) = \rho(E)$.

LEMMA 3.9. *Let H and I be 2-isomorphic hypergraphs and let F be a subset of edges. If F is spanning in χ_H and $H \mid F$ is 2-connected, then $H \mid F$ and $I \mid F$ are 2-isomorphic.*

Proof. We can assume that neither H nor I have isolated vertices. Since F is spanning in χ_H, $\bar{F} = V(H)$. A twisting partition of H is certainly a twisting partition of $H \mid F$. By choosing the twists corresponding to the sequence of twisting partitions that transforms H to I we transform $H \mid F$ to $I \mid F$. ∎

Note that if $V_H(F) = V(H)$ and $V_I(F) = V(I)$, then the condition that F be spanning or even 2-connected can be dropped so long as we regard splittings and joinings as twistings.

We can now outline the structure of the proof. We show that two rank-equivalent hypergraphs H and I can be transformed to rank-equivalent (and therefore equal by Lemma 3.7) complete hypergraphs via a sequence of operations. The operations consist of twistings, and the simultaneous addition of an edge to both hypergraphs in such a way that rank-equivalence is preserved. By Lemma 3.9, the twistings correspond to a well-defined sequence of twistings that transforms H to I.

We need two types of lemma. One type to tell us when edges can be added, and the other to tell us when twistings can be performed. We first give some more basic lemmas and definitions.

4. SOME BASICS

There are various natural ways one could define "paths" and "circuits" for hypergraphs. The following definitions are appropriate for our purposes.

A *walk from v_0 to v_n* in a hypergraph is a sequence $v_0, e_0, v_1, e_1, ..., e_{n-1}, v_n$ of vertices and edges such that, for $1 \leqslant i \leqslant n-1$, e_i is incident with v_i and v_{i+1}. An *inpath from vertex v_0 to vertex v_n* is such a sequence where, if $|i-j|>1$ then $\overline{e_i} \cap \overline{e_j} = \emptyset$. An *incircuit* is such a sequence where $n \geqslant 2$, $v_0 = v_n$, and $\overline{e_i} \cap \overline{e_j} = \emptyset$ unless $|i-j| \in \{1, n-1\}$. We may abuse notation by identifying an incircuit or inpath with its set of edges. Note that an incircuit or an inpath may properly contain a two-edge incircuit (where the two edges are mutually incident with more than one vertex).

The notion of contraction for graphs and matroids easily extends to hypergraphs and polymatroids. Let e be an edge of H. Then the *contraction* of e from H, denoted H/e, is obtained by coalescing $\bar{e}$ into a single vertex. An edge is incident with this new vertex if and only if it is incident with at least one vertex in $\bar{e}$. If e is an element of the polymatroid ρ, then the *contraction* of e from ρ is the set function on $E-e$ defined, for all $A \subseteq E-e$, by $\rho/e(A) = \rho(A \cup e) - \rho(e)$. It is routinely checked that $\chi_{H/e} = (\chi_H)/e$.

The set J is *independent* in the polymatroid ρ if each element of J is a component of $\rho \mid J$. Equivalently, J is independent if $\rho(J) = \sum_{i \in J} \rho(i)$. A set F of edges of a hypergraph is a *hyperforest* if F is independent in χ_H. Equivalently F is a hyperforest if and only if it does not contain the edge set of an incircuit.

The following lemma lists several properties of a hypergraph H that are determined by its rank function χ_H.

LEMMA 4.1. *Let ρ be a polymatroid. The following hold for any hypergraph H with $\rho = \chi_H$ and for any edges e, f, g and any subset of edges A.*

(i) $|\bar{e}| = \rho(e) + 1$.

(ii) *If $\rho(e) + \rho(f) - \rho(\{e, f\}) = k \geqslant 1$, then $|\bar{e} \cap \bar{f}| = k+1$ and $H \mid \{e, f\}$ is rank unique.*

(iii) *If $\rho(e) + \rho(f) - \rho(\{e, f\}) = 0$, then $|\bar{e} \cap \bar{f}| \in \{0, 1\}$ and $H \mid \{e, f\}$ is not rank unique.*

(iv) *If* $|\bar{e} \cap \bar{f}| \geqslant 2$ *and* $|(\bar{e} \cup \bar{f}) \cap \bar{g}| \geqslant 2$ *then* $H \mid \{e, f, g\}$ *is rank unique.*

(v) $\rho(f) = \rho(\{e, f\})$ *if and only if* $\bar{e} \subseteq \bar{f}$, *unless e is a loop.*

(vi) $\rho(f) = \rho(\{e, f\}) = \rho(e)$ *if and only if* $\bar{e} = \bar{f}$, *unless e or f is a loop.*

(vii) *If* $\rho(e) + \rho(A) - \rho(A \cup e) > 0$ *and* $e \notin A$, *then* $H \mid (A \cup e)$ *has an incircuit that contains e, or equivalently,* $H \mid A$ *contains an inpath between two vertices in* $\bar{e}$.

Proof. Those items that are not immediately self evident, are routine to verify. ∎

LEMMA 4.2. *If H and I are rank-equivalent and e is an edge, then* $H \mid \bar{e}$ *and* $I \mid \bar{e}$ *are rank-equivalent.*

Proof. It suffices to show that $H \mid \bar{e}$ and $I \mid \bar{e}$ have the same edge set. But this follows from Lemma 4.1(v). ∎

5. EDGE ADDING LEMMAS

Let H be a hypergraph and let I be a hypergraph that is rank-equivalent to H. Consider an extension of H by the edge e. We wish to know when it is also possible to extend I by e and preserve rank-equivalence. Usually this will depend on the choice of I. But for certain extensions we are guaranteed that we can simultaneously extend I. These extensions turn out to be of considerable interest. We now make this notion precise.

The hypergraph H' is an *extension* of H if $H = H' \mid E(H)$. The extension is *proper* if H' is simple. (Of course, if H' is simple, then so is H.) If H and I are rank-equivalent, then (H', I') is a *coherent extension* of (H, I) if H' and I' are respectively extensions of H and I, and H' and I' are rank-equivalent. Also H' is a *coherent extension* of H if, for every I that is rank-equivalent to H, there exists I' such that (H', I') is a coherent extension of (H, I). The coherent extension is *non-trivial*, if H' is simple and at least one edge is added. We wish to characterize some situations when coherent extensions can be guaranteed. We begin by noting yet another elementary lemma.

LEMMA 5.1. *Let e be an edge of the hypergraph H, and A be a subset of edges of H with the property that* $\chi_H(e) = \chi_H(A) = \chi_H(A \cup e)$. *Suppose* $H \mid (A \cup e)$ *has no loops. Then* $H \mid A$ *is connected and* $\bar{A} = \bar{e}$. *Moreover, for any* $B \subseteq E(H)$, $\chi_H(B \cup A) = \chi_H(B \cup e)$.

The next lemma describes a coherent extension of a pair (H, I).

LEMMA 5.2. *Let H and I be rank-equivalent hypergraphs and let H' and I' be extensions of H and I by the edge e. Assume that there exists a set A of edges that is connected in both H and I and has the property that $\bar{e} = \bar{A}$ in both H' and I'. Then H' and I' are also rank-equivalent.*

Proof. Say that H and I have edge set E. We need to show that $\chi_{H'}(B) = \chi_{I'}(B)$ for all subsets B of $E \cup e$. If e is not in B this follows from the fact that H and I are rank-equivalent. Say that e is in B. Then, using Lemma 5.1 and the fact that H and I are rank-equivalent we see that

$$\begin{aligned}\chi_{H'}(B) &= \chi_{H'}((B-e) \cup A)\\ &= \chi_H((B-e) \cup A)\\ &= \chi_I((B-e) \cup A)\\ &= \chi_{I'}((B-e) \cup A)\\ &= \chi_{I'}(B). \quad \blacksquare\end{aligned}$$

The next lemma describes some coherent extensions of a hypergraph H.

LEMMA 5.3. *Let H be a hypergraph with edge set E, and let H' be a proper extension of H by the edge e. Then H' is a coherent extension of H if any of the following conditions are satisfied.*

(i) *There is a subset A of E with the property that $H \mid A$ is 2-connected and $\bar{e} = \bar{A}$.*

(ii) *There is a subset A of E, and an edge $f \in E - A$ such that $H/f \mid A$ is 2-connected and $\bar{e} = \bar{A}$.*

(iii) *There are edges f and g such that $|\bar{f} \cap \bar{g}| \geqslant 2$ and $\bar{e} = \bar{f} \cap \bar{g}$.*

Proof. Let I be a hypergraph that is rank-equivalent to H. We extend I by e in the same way as described for H.

Let A be as in (i). By Lemma 3.8 $I \mid A$ is 2-connected, that is, A is connected in both H and I. Thus (i) follows by Lemma 5.2.

Let A and f be as in (ii). Assume that A is not connected in H, so that A can be partitioned into non-empty subsets A_1 and A_2 such that $V_H(A_1) \cap V_H(A_2) = \varnothing$ in H. Then, $V_{H/f}(A_1)$ and $V_{H/f}(A_2)$ can have at most one vertex in common, namely the new vertex obtained by coalescing the vertices in $V_H(f)$. This contradicts the fact that $H/f \mid A$ is 2-connected. It follows that A is connected in H. The same argument shows that A is also connected in I, and (ii) now follows by Lemma 5.2.

Let e, f, g be as in (iii). By Lemma 4.1(i, ii), $\chi_H(e) = \chi_I(e)$. Suppose there is a set $A \subseteq E$ such that $\chi_H(A \cup e) \neq \chi_I(A \cup e)$ and suppose that $|E|$ is as small as possible. Deleting any edge not in $A \cup \{e, f, g\}$ would give a smaller counterexample. Also, contracting any edge in A that preserves the property that $\bar{e} = \bar{f} \cap \bar{g}$ in both hypergraphs, would give a smaller counterexample. If follows from the minimality of $|E|$ that every edge $h \in A$ has, in at least one of H and I, vertices in common with both f and g, but not with e. By Lemma 4.1(iv), g has this property in both hypergraphs. Thus $\bar{A} \cap \bar{e} = \emptyset$ in both hypergraphs. But then $\chi_H(A \cup e) = \chi_H(A) + \chi_H(e) = \chi_I(A) + \chi_I(e) = \chi_I(A \cup e)$, contradicting the existence of a counterexample. ∎

The hypergraph H is *closed under coherent extensions* if it has no nontrivial coherent extensions. By Lemma 3.9, there is no loss of generality in only considering such hypergraphs. (There may be coherent extensions other than those obtained by repeated applications of Lemma 5.3, but we do not use them.) The next corollary mentions some useful structure implied by closure under coherent extensions.

COROLLARY 5.4. *Suppose H is simple and closed under coherent extensions. Let C be the edge set of an incircuit of H, and let $f \in C$. In each of the following cases there is an edge e such that*

(i) $\bar{e} = \bar{C}$

(ii) $\bar{e} = \overline{C - f}$

(iii) $\bar{e} = \overline{C - f} \cap \bar{f}$

Moreover, if D is the edge set of an incircuit of H with $D \subset C$ (so that $|D| = 2$) then there is an edge e such that

(iv) $\bar{e} = \overline{C - D}$

(v) $\bar{e} = \overline{C - D} \cap \bar{D}$

Proof. Parts (i), (ii) and (iii) follow from Lemma 5.3, noting that $H \mid C$, $(H \mid C)/f$ are 2-connected. Consider parts (iv) and (v). First note that by (i), there is an edge d, with $\bar{d} = \bar{D}$. The result now follows by (ii) and (iii). ∎

The next lemma describes a coherent extension of a pair (H, I).

LEMMA 5.5. *let H and I be rank-equivalent hypergraphs and assume that H is simple and closed under coherent extensions. If the edge f is such that $H \mid V_H(f) = I \mid V_I(f)$, then H and I can be extended to rank-equivalent hypergraphs H' and I' where $H' \mid \bar{f}$ and $I' \mid \bar{f}$ are equal complete hypergraphs.*

Proof. Let H, I and f be as described in the lemma. We may relabel the vertices of I so that the vertex labellings of $H \mid \bar{f}$ and $I \mid \bar{f}$ agree. For $W \subseteq \bar{f}$, with $|W| \geqslant 2$, let H_W and I_W denote extensions of H and I by adding an edge g with $\bar{g} = W$. It is sufficient to show that H_W and I_W are rank-equivalent. Suppose not. Choose a minimal W such that H_W and I_W are not rank-equivalent. Then $|W| = 2$, say $W = \{u, w\}$, since if all the 2-vertex edges can be added, then so can the others by Lemma 5.3(i). Without loss of generality there exists a set F of edges such that $\chi_{H_W}(F \cup g) - \chi_{H_W}(F) = 0$ and $\chi_{I_W}(F \cup g) - \chi_{I_W}(F) = 1$. Choose F so that $|F|$ is minimised, and subject to that, $|\bar{F}|$ is minimised. By Lemma 4.1(vii), F contains an inpath from u to w in H, but not in I. By minimality, F must be that inpath and by Corollary 5.4, F contains no incircuit. Clearly $H \,|\, (F \cup f)$ is 2-connected, so that for each edge $h \in F$ there is an incircuit C containing h, as well as f. By Corollary 5.4(iii), there is an edge with vertex set $\overline{C - f} \cap \bar{f}$, and by minimality h must be that edge. It follows that $\bar{h} \subseteq \bar{f}$ for every $h \in f$. But then $H \,|\, F$ is a restriction of $H \,|\, \bar{f} = I \,|\, \bar{f}$, so that F is also an inpath from u to w in I; a contradiction. ▮

6. LOCAL TWISTING LEMMAS

For distinct vertices u and w of the hypergraph H, set $V_{uw} = V - \{u, w\}$, and $E_{uw} = \{e \in E(H) \,|\, \{u, w\} \not\subseteq \bar{e}\}$. Define an equivalence relation t on $V_{uw} \cup E_{uw}$ to be the least equivalence relation having the property that, for $z \in V_{uw}$ and $e \in E_{uw}$, if $z \in \bar{e}$, then z t e. The t -equivalence classes are the *parts* of H with respect to $\{u, w\}$. If the pair $\{u, w\}$ is clear from the context we will simply refer to the parts of H. Let P be a part of H. Edges in P meet some subset X of $\{u, w\}$. In this case we say that P is an X-*part*. We also say that P is a 0-, 1- or 2-*part* depending on whether $|X|$ is 0, 1, or 2 respectively.

What are the possible twistings around $\{u, w\}$? Such twistings are given by partitions of the form $\{U, W, u, w\}$, where, for any part P, the vertices in P are contained in either U or W. Such a twisting is non-trivial provided that both U and W contain the vertices of a 1- or 2-part. Thus non-trivial twistings around $\{u, w\}$ are possible provided that there are at least two parts other than 0-parts.

Note also that in a 2-connected graph only 2-parts occur. But 1-parts can occur in 2-connected hypergraphs. This represents a new complication that arises in generalising from graphs to hypergraphs.

As an example, let H be a hypergraph with $V = \{1, 2, 3, 4\}$ and $E = \{a, b, c\}$ where $\bar{a} = \{1, 2\}$, $\bar{b} = \{3, 4\}$, $\bar{c} = \{1, 2, 3, 4\}$. The parts of H with respect to $\{2, 3\}$ are $\{1, a\}$ and $\{4, b\}$. These are both 1-parts, even though H is 2-connected. It is clear that it is possible to perform a non-trivial twisting around $\{2, 3\}$.

A $\{u, w\}$-*walk from* v_0 *to* v_n in a hypergraph is a walk $v_0, e_0, v_1, e_1, ..., e_{n-1}, v_n$ such that, $\{v_0, v_1, ..., v_n\} \subseteq V_{uw}$ and $\{e_0, e_1, ..., e_{n-1}\} \subseteq E_{uw}$. Note that edges in this walk may be incident with one of u and w, but not both. It is easily seen that vertices v_0 and v_n are in the same part of H with respect to $\{u, w\}$ if and only if there is a $\{u, w\}$-walk from v_0 to v_n. The *length* of a walk is the number of edges in it.

It had been hoped that for an edge e of a hypergraph H that is closed under coherent extensions, any twisting partition of $H \mid \bar{e}$ could have been extended to a twisting partition of H. However there is a very specific exception as described in the case below.

LEMMA 6.1. *Suppose H is simple and 2-connected with edge set E and vertex set V. Assume that H is closed under coherent extensions. Suppose that $e \in E$, and that $\{U, W, u, w\}$ is a twisting partition of $H \mid \bar{e}$. Then, either (1) or (2) below holds, but not both.*

(1) *There exists a twisting partition $\{U', W', u, w\}$ of H with $U \subseteq U'$ and $W \subseteq W'$.*

(2) *There is a u-part of $H \mid \bar{e}$ with vertex set $R_0 \subseteq U$, and there is a w-part of $H \mid \bar{e}$ with vertex set $R_2 \subseteq W$ and there are edges $f, g \in E_{uw}$ and there are vertices $v_0 \in R_0$ and $v_2 \in R_2$ such that $\{u, v_0\} \subseteq \bar{f} \cap \bar{e} \subseteq R_0 \cup \{u\}$, $\{w, v_2\} \subseteq \bar{g} \cap \bar{e} \subseteq R_2 \cup \{w\}$, $\bar{f} \cap \bar{g} \neq \emptyset$, and $\bar{e} \cap \bar{f} \cap \bar{g} = \emptyset$, or the same situation with the roles of u and w reversed holds.*

Proof. It is clear that if (2) holds, then (1) does not hold. Assume that (1) does not hold. Then there exists, in H, a $\{u, w\}$-walk, $P = v_0, e_0, v_1, e_1, ..., e_{n-1}, v_n$, from some $v_0 \in U$ to some $v_n \in V$. Choose such a walk of minimum length. Let R_0 and R_2 be the vertex sets of parts P_0 and P_2, respectively, of $H \mid \bar{e}$ such that $v_0 \in R_0$ and $v_n \in R_2$.

The remainder of the proof is devoted to showing that P has only two edges f and g as in (2). From the minimality of P we immediately obtain

6.1.1. (i) *If e_i and e_j are edges in P with $|i-j| \geqslant 2$, then* $\overline{e_i} \cap \overline{e_j} \in \{\emptyset, \{u\}, \{w\}\}$.

(ii) $\overline{e_i} \cap U \neq \emptyset$ *if and only if* $i = 0$.

(iii) $\overline{e_i} \cap W \neq \emptyset$ *if and only if* $i = n-1$.

(iv) *If $|i-j| \geqslant 2$, then there is no edge $h \in E_{uw}$ incident with both v_i and v_j.*

For $1 \leqslant i < j \leqslant n-1$, let $P[i, j]$ denote the subwalk $v_i, e_i, ..., e_{j-1}, v_j$. (Note that e_{j-1} is the "last" edge in $P[i, j]$.) Most of the proof will use the following type of argument. When $n > 2$ we shall find a subwalk $P[i, j]$, with $|i-j|\geqslant 2$, that can be replaced by a shorter subwalk v_i, h, v_j, where $\{v_i, v_j\}\subseteq \bar{h}$ and $h\in E_{uw}$. In other words we shall obtain a contradiction to 6.1.1(iv). The existence of such an edge h will be implied by closure under coherent extensions.

6.1.2. *u and w are each incident with at most one edge in P.*

Proof. Suppose the lemma is false. Without loss of generality there exist integers i and j, with $1\leqslant i<j\leqslant n-1$, such that u is incident with both e_i and e_j and u is not incident with any edge in $P[i+1, j]$, and w is incident with at most one edge in $P[i+1, j]$. The edges of $P[i, j+1]$ form an incircuit C. If $w\notin \bar{C}$ then by Corollary 5.4(i) there is an edge h with $\bar{h}=\bar{C}$, which contradicts 6.1.1(iv). Otherwise there exists k such that $i<k<j$ and $w\in\overline{e_k}$. Now by Corollary 5.4(ii) there is an edge h with $\bar{h}=\overline{C-e_k}$, again contradicting 6.1.1(iv). ∎

6.1.3. (i) $\overline{e_0}\cap\{u, w, v_n\}\neq\emptyset$.

(ii) $\overline{e_{n-1}}\cap\{u, w, v_0\}\neq\emptyset$.

(iii) $n>1$.

(iv) *u and w are each incident with exactly one edge in P.*

Proof. Let k be the minimum integer with the property that $\overline{e_k}\cap\{u, w, v_n\}\neq\emptyset$. Suppose (i) is false so that $k>0$. Then $P[0, k+1]$ together with e forms an incircuit C. By Corollary 5.4(i) there is an edge h with $\bar{h}=\bar{C}$, which contradicts 6.1.1(iv). Item (ii) follows similarly.

Suppose (iii) is false, so that $n=1$ and $\{v_0, v_n\}\subseteq\overline{e_0}$. By Lemma 5.3(iii), there is an edge h with $\bar{h}=\overline{e_0}\cap\bar{e}$. Now h is an edge of $H\,|\,\bar{e}$ with $\{v_0, v_n\}\subseteq\bar{h}$ and $\{u, w\}\nsubseteq\bar{h}$. This contradicts the fact that v_0 and v_n are in different parts of $H\,|\,\bar{e}$. Thus (iii) follows. Combining 6.1.2 and 6.1.3(i)(ii)(iii) gives (iv). ∎

By swapping the roles of u and w if necessary we assume without loss of generality that $u\in\overline{e_0}$ and $w\in\overline{e_{n-1}}$.

6.1.4. *P has length* $n=2$.

Proof. Since $\{u, v_0\}\subseteq\bar{e}\cap\overline{e_0}$, it follows that $H\,|\,\{e, e_0\}$ is an incircuit D say. Using facts so far established, P together with edge e forms an incircuit C say. Suppose $n>2$. By Corollary 5.4 there is an edge h with $\bar{h}=\overline{C-D}$. Now h is incident with v_1, v_n and w, but not with u, contradicting 6.1.1(iv). ∎

We now know that P has two edges e_0 and e_1. Set $f=e_0$ and $g=e_1$, so that $P=v_0, f, v_1, g, v_2$. Evidently $\bar{f}\cap\bar{g}\neq\emptyset$. By Lemma 5.3, there are edges h_u and h_w in H with $\overline{h_u} = \bar{e} \cap \bar{f}$ and $\overline{h_w} = \bar{e} \cap \bar{g}$. We now show that $\bar{f} \cap \bar{g} \cap \bar{e} = \overline{h_u} \cap \overline{h_w} = \emptyset$. Assume not. Say $z \in \overline{h_u} \cap \overline{h_w}$. Then v_0, h_u, z, h_w, v_2 is a $\{u, w\}$-walk in $H \mid \bar{e}$. It follows that v_0 and v_2 are in the same part of $H \mid \bar{e}$. This contradicts the fact that $v_0 \in R_0$ and $v_2 \in R_2$.

Clearly h_u is an edge of part P_0 of $H \mid \bar{e}$ and $u \in \overline{h_u} = \bar{f} \cap \bar{e} \subseteq R_0 \cup u$. Also h_w is an edge of part P_2 of $H \mid \bar{e}$ and $w \in \overline{h_w} = \bar{g} \cap \bar{e} \subseteq R_2 \cup w$. Hence P_0 is either a u-part or a uw-part and P_2 is either a w-part or a uw-part. It remains to show that P_0 is a u-part and P_2 is a w-part.

Assume that P_0 is a uw-part. Then P could be extended to a walk $P' = w, d_0, w_0, ..., w_{q-1}, d_q, w_q, f, v_1, g, w,$ such that all edges and vertices in P' except f, v_1, g and w are in part P_0, and such that d_0 and g are the only edges incident with w. By arguments similar to those above it can be inferred that there exists an edge h such that $\bar{h} \subseteq \bar{e}$ and $w, v_0, v_2 \in \bar{h}$ and $u \notin \bar{h}$. Thus v_0 and v_2 are in the same part of $H \mid \bar{e}$; a contradiction. Hence P_0, and similarly P_2 are not uw-parts. Therefore P_0 is a u-part and P_2 is a w-part. ∎

An edge e in a simple hypergraph H is called *minimal incomplete* if $H \mid \bar{e}$ is not complete but for every edge f with $\bar{f} \subset \bar{e}$, the hypergraph $H \mid \bar{f}$ is complete. The hypergraph H is *near-complete* if it has a minimal incomplete edge e with $\bar{e} = V(H)$. Clearly, for any simple hypergraph H, the edge e is minimal incomplete if and only if $H \mid \bar{e}$ is near-complete.

For $n \geqslant 4$, let C_n denote the graph which is a circuit on n edges. Let K_4^- denote the graph obtained by deleting an edge from K_4. Let the hypergraphs HC_n and HK_4^- be obtained, respectively, from C_n and K_4^- by performing all possible coherent extensions, or equivalently (in these cases) adding every hyperedge h such that $\bar{h}$ is the vertex set of a circuit. For a vertex set W with $|W| \geqslant 3$ and a hyperforest with edge set A and $\bar{A} \subseteq W$, let $HF(W, A)$ denote the hypergraph with vertex set W and an edge e with $\bar{e} = W$ and every edge f such that $|\bar{f}| \geqslant 2$ and $\bar{f} \subseteq \bar{g}$ for some $g \in A$. Observe that these hypergraphs are all near-complete (except $HF(W, A)$ in the case that $|A| = 1$ and $\bar{A} = W$).

LEMMA 6.2. *Assume that H is 2-connected, simple and is closed under coherent extensions. Let e be a minimal incomplete edge. Then $H \mid \bar{e}$ is isomorphic to one of the following hypergraphs*

(i) *HC_n for some $n \geqslant 4$, or*

(ii) *HK_4^-, or*

(iii) *$HF(W, A)$ for some vertex set W with $|W| \geqslant 3$ and a hyperforest with edge set A and $\bar{A} \subseteq W$, (unless $|A| = 1$ and $\bar{A} = W$).*

Proof. Let G be the graph formed by the 2-edges of $H \mid \bar{e}$. Suppose G has, as an induced subgraph, K_4^- or C_n for some $n \geqslant 4$. By minimality,

G must be that subgraph, and by closure, $H\,|\,\bar{e}$ must be as in (i) or (ii). Otherwise every block of G is a clique, and by minimality and closure, $H\,|\,\bar{e}$ must be as in (iii). ∎

The following lemma is routine to verify.

LEMMA 6.3. *Suppose H is 2-connected, simple and is closed under coherent extensions and I is rank-equivalent to H. Let e be a minimal incomplete edge of H. Then $H\,|\,\bar{e}$ and $I\,|\,\bar{e}$ are 2-isomorphic.*

LEMMA 6.4. *Suppose H is 2-connected, simple, closed under coherent extensions. Suppose I is rank-equivalent to H. Let e be a minimal incomplete edge of H.*

(a) *Every twisting partition of $H\,|\,\bar{e}$ extends to a twisting partition of H unless $H\,|\,\bar{e}$ consists of e, edges h_1 and h_2 such that $|\overline{h_1}|, |\overline{h_2}| \geqslant 2$, $\overline{h_1} \cup \overline{h_2} = \bar{e}$ and $\overline{h_1} \cap \overline{h_2} = \emptyset$, and every edge h such that $\bar{h} \geqslant 2$ and $\bar{h} \subseteq \overline{h_1}$ or $\bar{h} \subseteq \overline{h_2}$, and there are edges f and g in H such that $\bar{f} \cap \bar{e} = h_1$, $\bar{g} \cap \bar{e} = h_2$, and $\bar{f} \cap \bar{g} \neq \emptyset$, in which case no twisting partition of $H\,|\,\bar{e}$ extends to a twisting partition of H.*

(b) *H is 2-isomorphic to a hypergraph H_1 such that $H_1\,|\,\bar{e} = I\,|\,\bar{e}$.*

Proof. Suppose that $\{U, W, u, w\}$ is a twisting partition of $H\,|\,\bar{e}$ that cannot be extended to H. Then, by Lemma 6.1, there exist edges f and g of H satisfying condition (2) of Lemma 6.1. Let the other symbols in Lemma 6.1(2) also be as in that statement. Now, $\{v_0, u\} \subseteq \bar{e} \cap \bar{f}$ and $\{v_2, w\} \subseteq \bar{e} \cap \bar{g}$ and $\{e, f, g\}$ is an incircuit, so H has edges h_1, h_2 and h_3 where $\overline{h_1} = \bar{e} \cap \bar{f}$, $\overline{h_2} = \bar{e} \cap \bar{g}$, and $\overline{h_3} = \bar{e} \cap (\bar{f} \cup \bar{g}) = \overline{h_1} \cup \overline{h_2}$. Clearly, $|\overline{h_1}| \geqslant 2$, $|\overline{h_2}| \geqslant 2$, $\overline{h_1} \cap \overline{h_2} = \bar{e} \cap \bar{f} \cap \bar{g} = \emptyset$, $\overline{h_1} \subseteq U \cap \{u\}$, $\overline{h_2} \subseteq W \cup \{w\}$, $v_0 \in \overline{h_1}$ and $v_2 \in \overline{h_2}$. Since $\overline{h_1}$ and $\overline{h_2}$ are proper subsets of $\bar{e}$, both $H\,|\,\overline{h_1}$ and $H\,|\,\overline{h_2}$ are complete and either $H\,|\,\overline{h_3}$ is complete or $\overline{h_3} = \bar{e}$. But if $H\,|\,\overline{h_3}$ is complete, the edge $\{v_0, v_2\}$ contradicts the fact that $\{U, W, u, w\}$ is a twisting partition. Hence $\bar{e} = \overline{h_3} = \overline{h_1} \cup \overline{h_2}$, while $\overline{h_1} \cap \overline{h_2} = \emptyset$. Hence $H\,|\,\bar{e}$ is of the form described in (a).

Let $B = \{h \in E \,|\, \bar{h} \subseteq \bar{e}$ or $h = f$ or $h = g\}$. It is easily shown that $H\,|\,B$ is rank-unique and that no twisting partition of $H\,|\,\bar{e}$ extends to a twisting partition of H.

By Lemma 6.3 there is a sequence of twistings (each associated with a twisting partition) from $H\,|\,\bar{e}$ to $I\,|\,\bar{e}$. If we can extend every twisting partition in the sequence to the whole hypergraph, then we are done. If not then we can conclude that H has as a restriction, the rank unique hypergraph $H\,|\,B$ described above. But then $H\,|\,B = I\,|\,B$, so that $H\,|\,\bar{e} = I\,|\,\bar{e}$. In this case we can simply set $H_1 = H$. ∎

7. THE PROOF OF THE THEOREM

We are now in a position to complete the proof of Lemma 3.6, which will complete the proof of the theorem.

Proof. Let H be a 2-connected simple hypergraph and let I be a hypergraph that is rank-equivalent to H. Say $E(H)=E(I)=E$. Suppose that the lemma is false, so that I is not 2-isomorphic to H. Choose a counterexample (H, I) such that $|V(H)|=|V(I)|=n$ is minimum and, subject to that, $|E|=m$ is maximum. (Note that m is bounded above by 2^n-n-1, the number of edges in a n-vertex complete hypergraph.) Clearly H is simple, closed under coherent extensions and is not complete. Therefore H has a minimal incomplete edge e. By Lemma 6.4 H is 2-isomorphic to a hypergraph H_1 such that $H_1 \mid \bar{e}=I \mid \bar{e}$. By Lemma 5.5 there is a non-trivial coherent extension of (H_1, I) to (H', I') such that $H' \mid \bar{e}=I' \mid \bar{e}$ is complete. Since $E(H')>m$, then by our maximality assumption, H' is 2-isomorphic to I'. But then $H_1=H' \mid E$ is 2-isomorphic to $I=I' \mid E$, by Lemma 3.9. It follows that H is 2-isomorphic to I. ∎

REFERENCES

[1] T. Helgason, Aspects of the theory of hypermatroids, *in* "Hypergraph Seminar" (C. Berge and D. K. Ray-Chaudhuri, Eds.), pp. 191–221, Lecture Notes in Math., Vol. 411, Springer-Verlag, Berlin, 1974.

[2] J. G. Oxley, "Matroid Theory," Oxford Univ. Press, New York, 1992.

[3] J. Oxley and G. Whittle, A characterization of Tutte invariants of 2-polymatroids, *J. Combin. Theory Ser. B* **59** (1993), 210–244.

[4] R. Swaminathan and D. Wagner, Vertex 2-isomorphism, *J. Graph Theory* **16** (1992), 335–347.

[5] K. Treumper, On Whitney's 2-isomorphism theorem for graphs, *J. Graph Theory* **4** (1980), 43–49.

[6] D. K. Wagner, On theorems of Whitney and Tutte, *Discrete Math.* **57** (1985), 147–154.

[7] H. Whitney, 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 150–168.

[8] G. Whittle, A geometric theory of hypergraph colouring, *Aeq. Math.* **43** (1992), 45–58.