Victoria University of Wellington
Te Whare Wānanga o te Ūpoko o te Ika a Māui

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Bounding the Bogoliubov coefficients.

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Abstract:

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We have all seen the WKB estimate for barrier penetration probability.

Unfortunately, the WKB estimate is an example of an uncontrolled approximation, and you do not know if it is high or low.

In this talk I will explain some rigorous bounds that you can place on barrier penetration probabilities, or equivalently on the Bogoliubov coefficients associated with a time-dependent potential.

This is an example of finding new physics in an old and apparently well-understood area.

Background:

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Bounding the Bogoliubov coefficients.
(with Petarpa Boonserm)
e-Print: arXiv:0801.0610 [quant-ph]
Annals of Physics (2008) in press.

Some general bounds for 1-D scattering.
e-Print arXiv: quant-ph/9901030
Physical Review A59 (1999) 427--438

Bounding the greybody factors for Schwarzschild black holes.
- (with Petarpa Boonserm)
- e-Print arXiv: quant-ph/9901030

Scattering:

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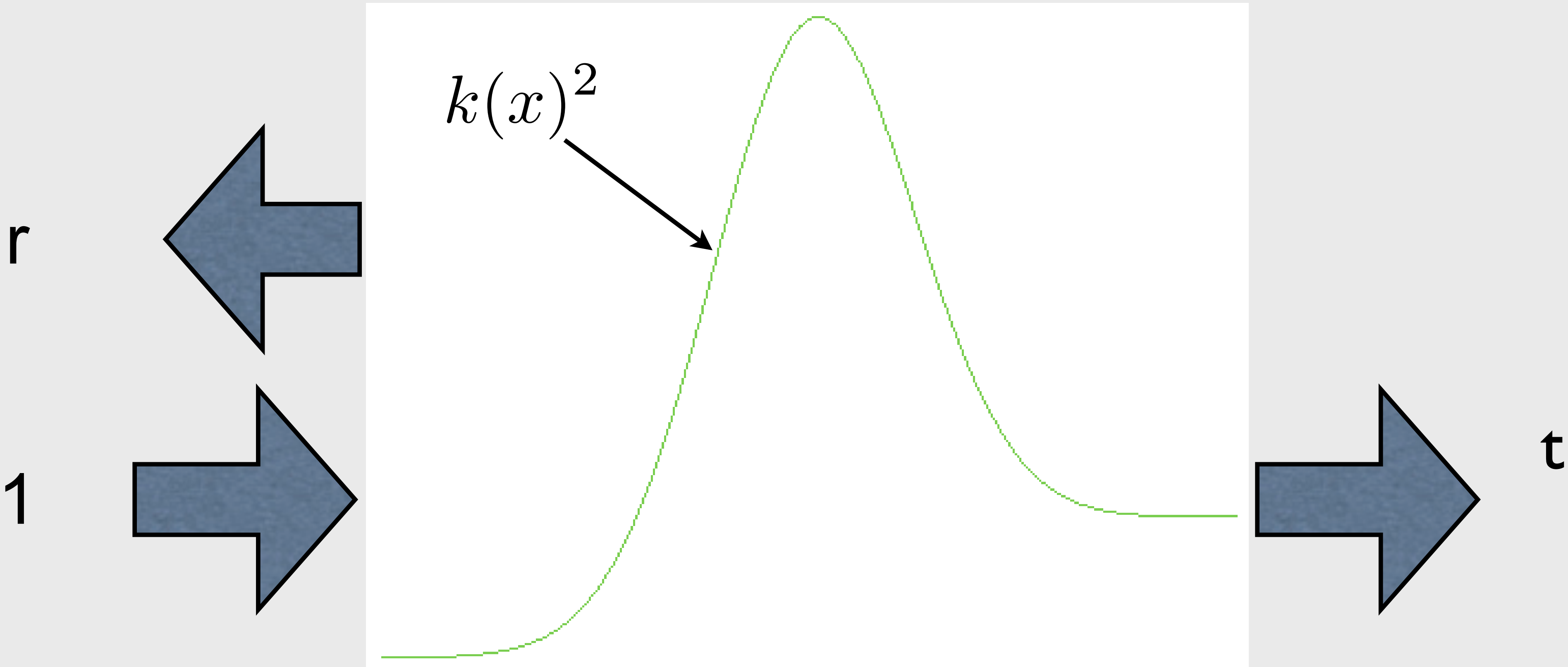
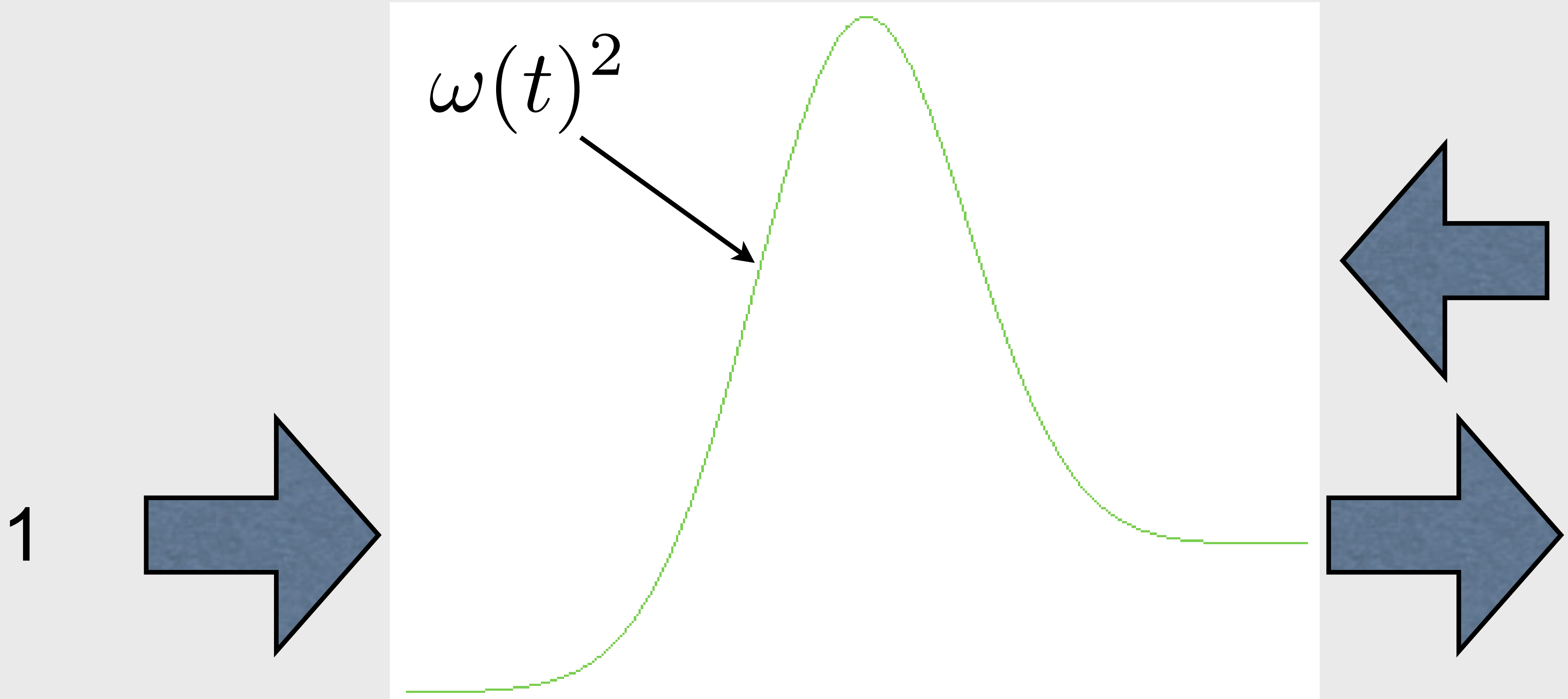


$$\phi''(x) + k^2(x)\,\phi(x) = 0;$$

$$1 = |t|^2 + |r|^2$$

Parametric oscillator:



$$\ddot{\phi}(t) + \omega^2(t)\,\phi(t) = 0.$$

$$|\alpha|^2 - |\beta|^2 = 1$$

Equivalences:

$$|\alpha| \leftrightarrow \frac{1}{|t|} \qquad |\beta| \leftrightarrow \frac{|r|}{|t|}$$

The two problems are formally completely equivalent, and one can treat them completely interchangeably:

$$|\alpha| = \cosh Q \qquad |\beta| = \sinh Q \qquad |t| = \operatorname{sech} Q \qquad |r| = \tanh Q$$

Put a bound on the functional "Q"...

$$Q = Q[k(x)] = Q[\omega(t)]$$

Equivalences:

$$T = |t|^2 = \operatorname{sech}^2 Q \qquad N = |\beta|^2 = \sinh^2 Q$$

$$R = |r|^2 = \tanh^2 Q$$

$$T \leftrightarrow \frac{1}{1+N}$$

Simple case:

(spatial)

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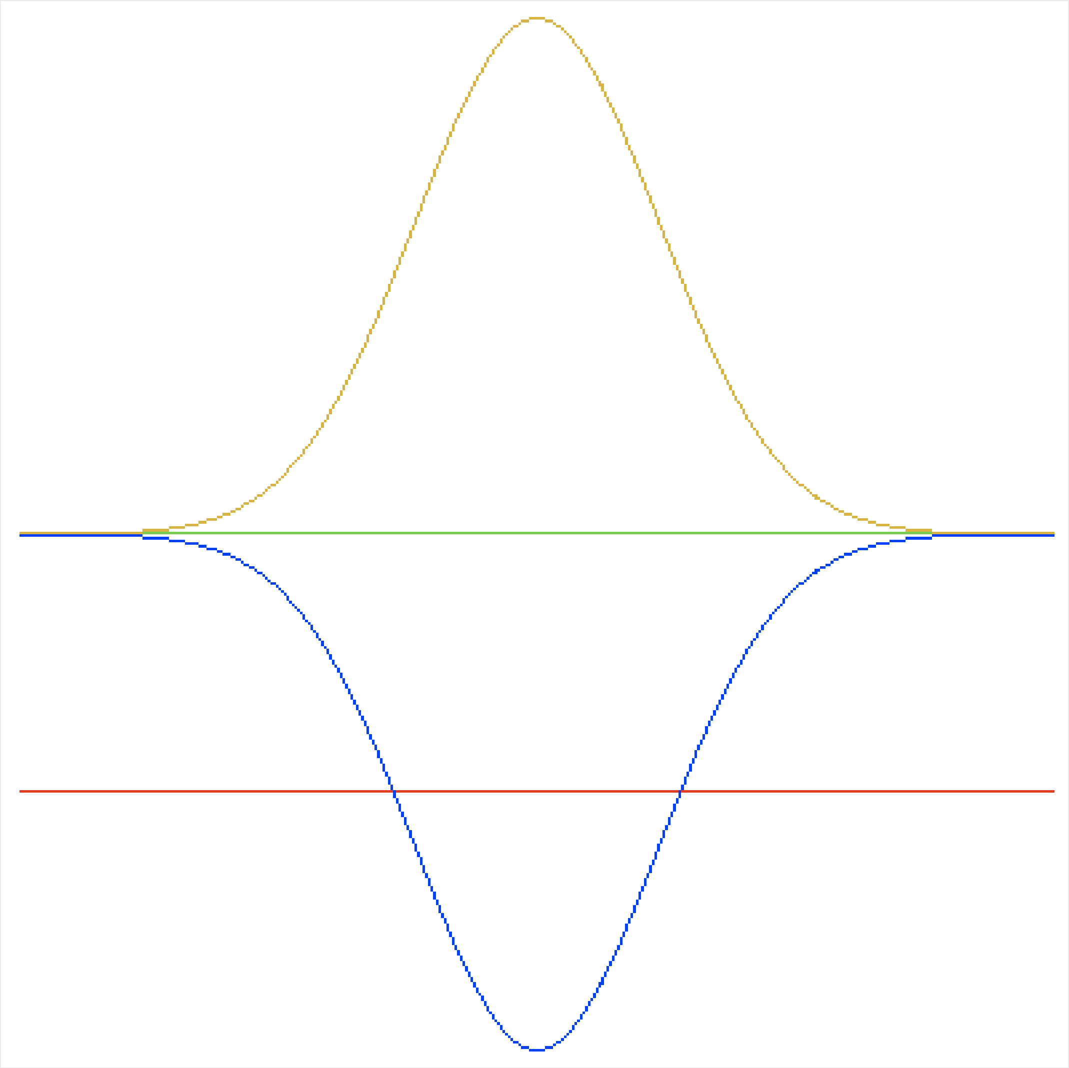
$$k_\infty = k(+\infty) = k(-\infty)$$

$$T = |t|^2 = \text{sech}^2 Q$$

$$R = |r|^2 = \tanh^2 Q$$

Theorem:

$$Q \leq \oint \left| \frac{k_\infty^2 - k(x)^2}{2k_\infty} \right| \mathrm{d}x$$

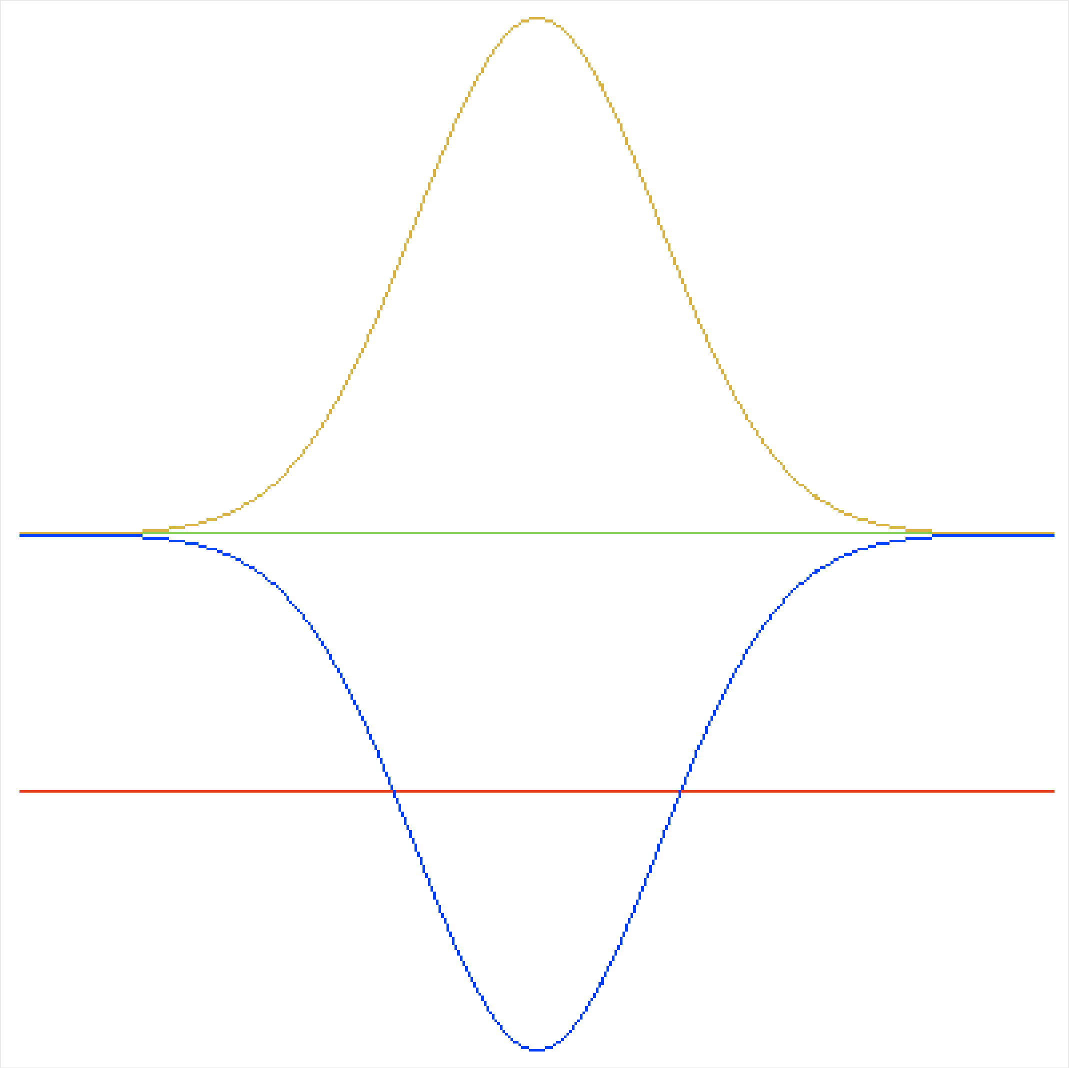


Proof: Let's delay that for now....

Simple case:
(temporal)

$$\omega_\infty = \omega(+\infty) = \omega(-\infty)$$

$$N = |\beta|^2 = \sinh^2 Q$$



Theorem:

$$Q \leq \oint \left| \frac{\omega_\infty^2 - \omega(t)^2}{2\omega_\infty} \right| \mathrm{d}t$$

Proof: Let’s delay that for now....

These bounds

$$Q \leq \oint \left| \frac{k_\infty^2 - k(x)^2}{2k_\infty} \right| \mathrm{d}x \qquad\qquad Q \leq \oint \left| \frac{\omega_\infty^2 - \omega(t)^2}{2\omega_\infty} \right| \mathrm{d}t$$

are limited to situations where the asymptotic limits of the potential are equal, but are otherwise very general...

They work perfectly well in the "classically forbidden" region:

$$k(x)^2 < 0 \qquad\qquad \omega(t)^2 < 0$$

Comments:

Two significantly different proofs have been developed (arXiv: 0810.0610 [quant-ph], and quant-ph/9901030).

Many consistency checks against known exact solutions are presented in quant-ph/9901030.

I have not seen anything like these bounds anywhere else.

These are not JWKB-like bounds.
JWKB theory would lead to integrals such as:

$$\int_{\mathrm{forbidden}} |k(x)|\,\mathrm{d}x \qquad\qquad \int_{\mathrm{forbidden}} |\omega(t)|\,\mathrm{d}t$$

Reprise:

$$T = |t|^2 = \operatorname{sech}^2 Q \geq \operatorname{sech}^2 \oint \left| \frac{k_\infty^2 - k(x)^2}{2k_\infty} \right| \mathrm{d}x$$

$$R = |r|^2 = \tanh^2 Q \leq \tanh^2 \oint \left| \frac{k_\infty^2 - k(x)^2}{2k_\infty} \right| \mathrm{d}x$$

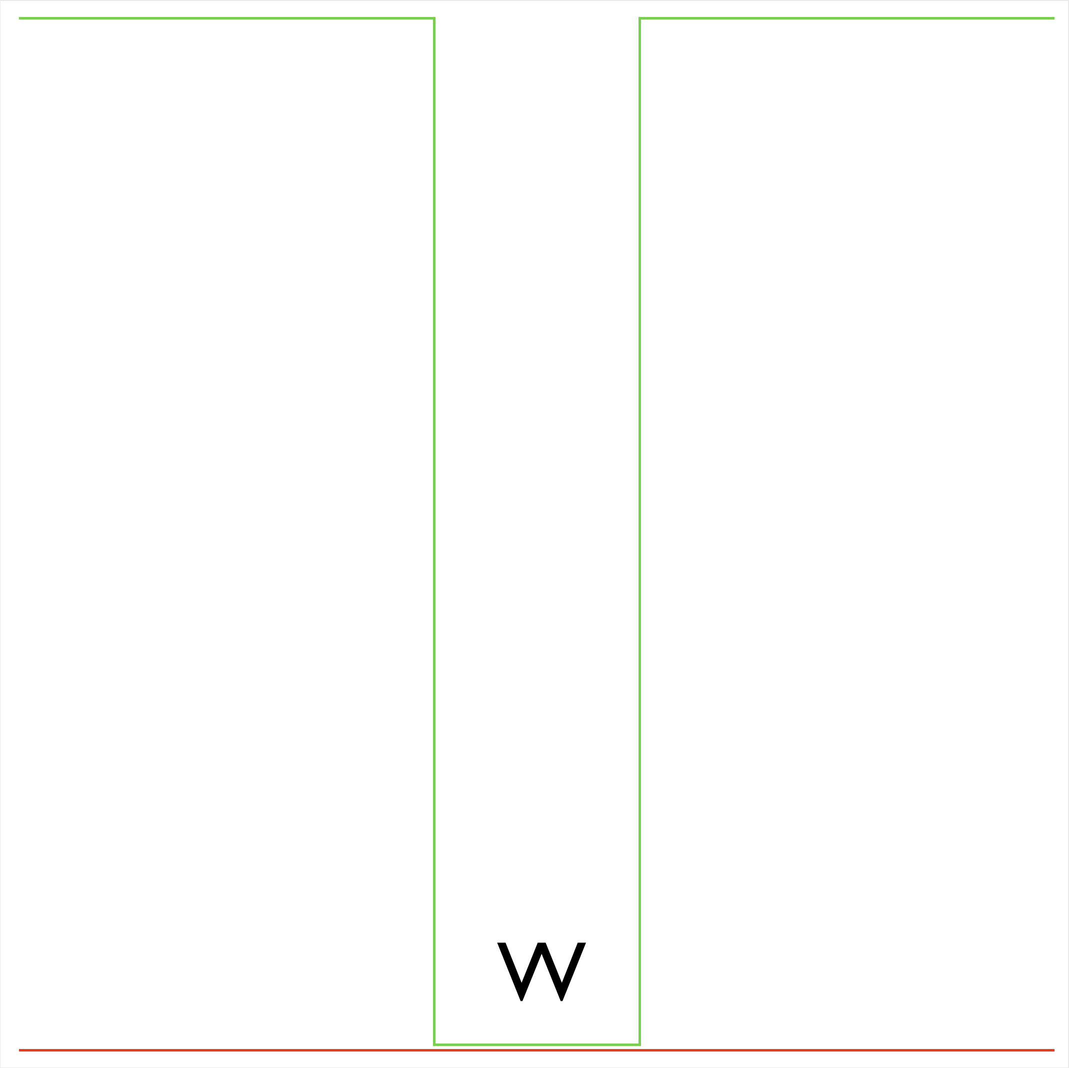
$$N = |\beta|^2 = \sinh^2 Q \leq \sinh^2 \oint \left| \frac{\omega_\infty^2 - \omega(t)^2}{2\omega_\infty} \right| \mathrm{d}t$$

There are situations where these bounds are arbitrarily close to saturation...

$$T_{\text{exact}} = \frac{1}{1 + k_\infty^2 W^2/4}$$

$$T_{\text{bound}} = \text{sech}^2(k_\infty W/2)$$

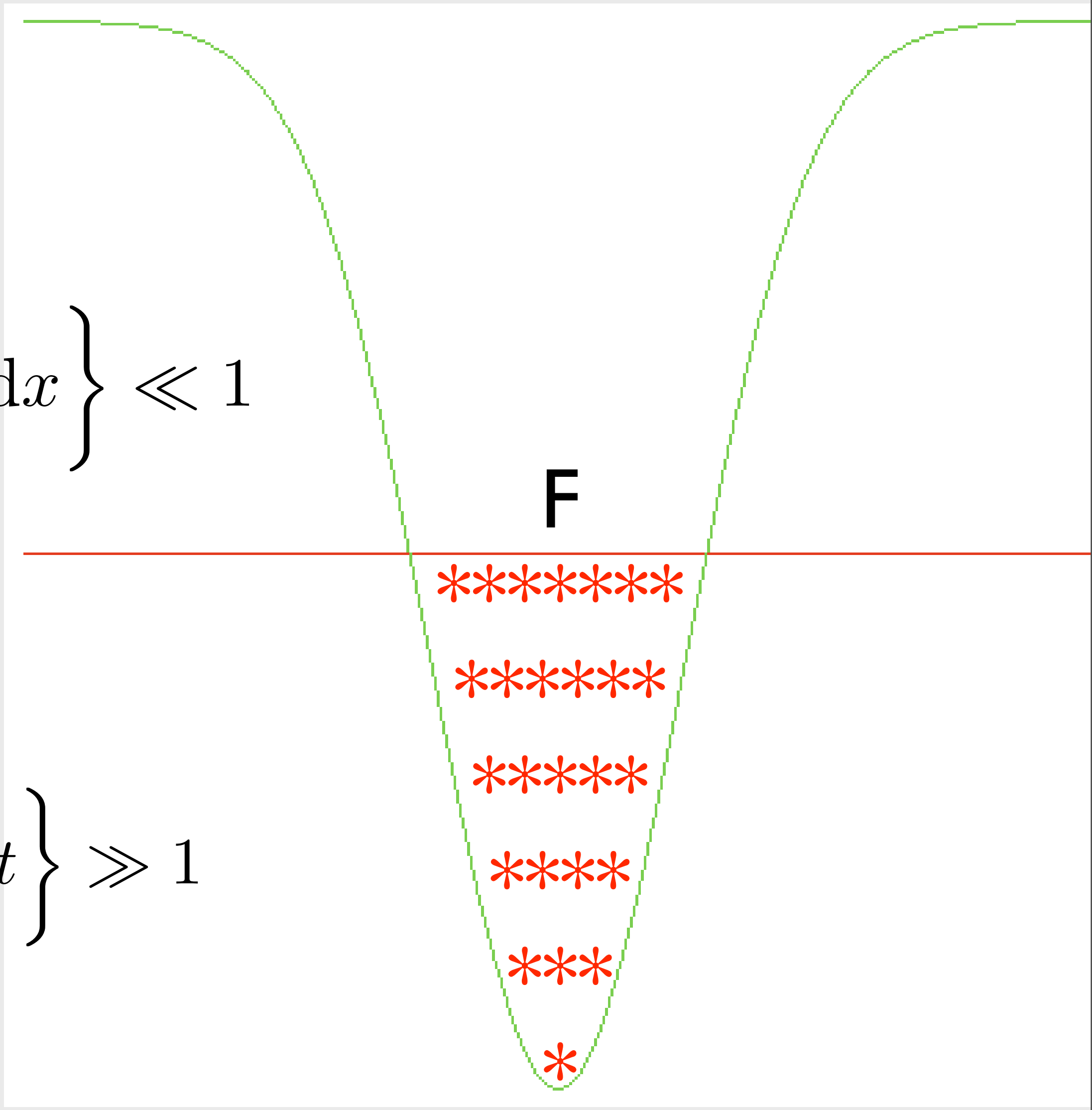
Arbitrarily close for narrow wells (W ==> 0).



Compare JWKB:

$$T \approx \exp\left\{-2\int_{\text{forbidden}} |k(x)|\,\mathrm{d}x\right\} \ll 1$$

$$N \approx \exp\left\{+2\int_{\text{forbidden}} |\omega(t)|\,\mathrm{d}t\right\} \gg 1$$



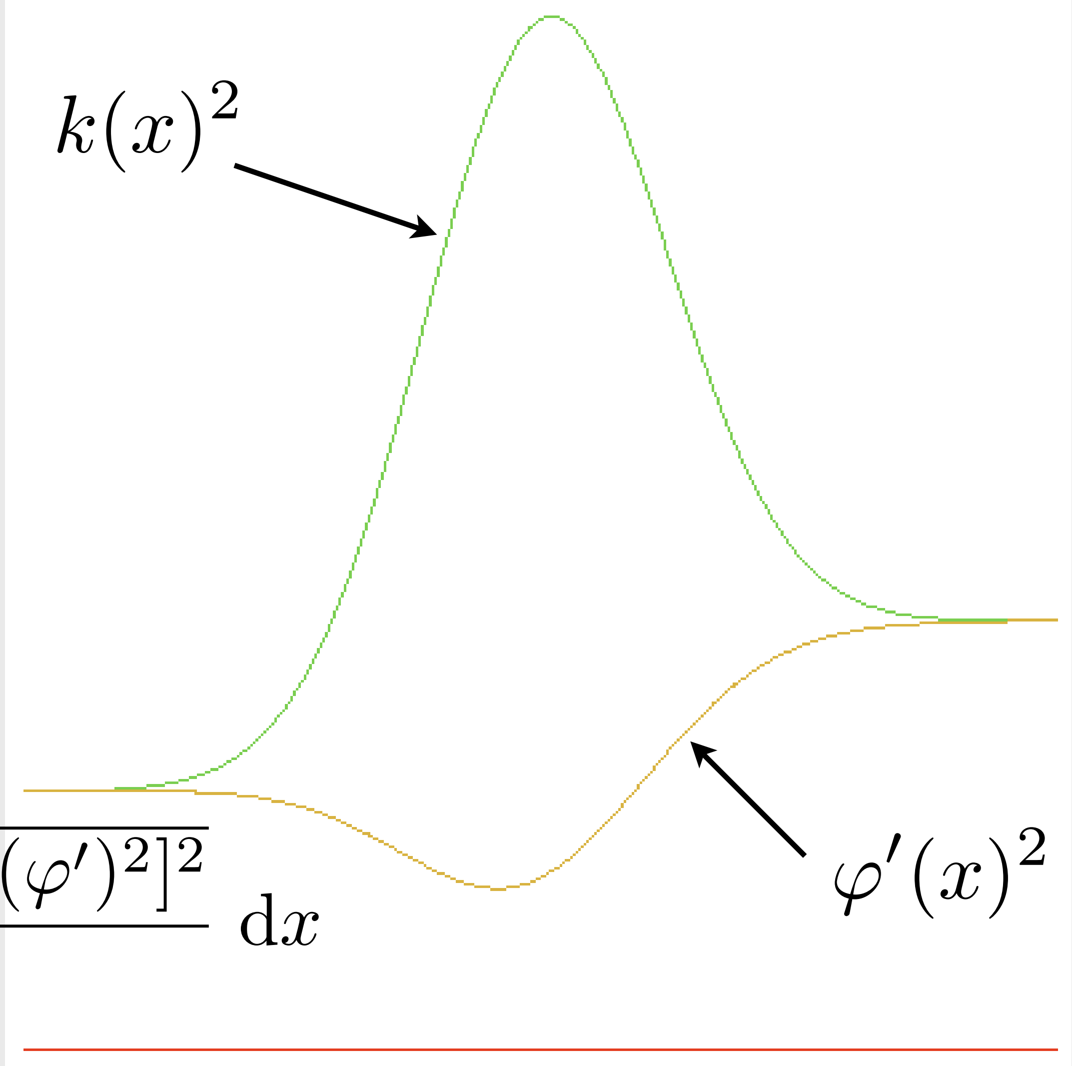
More general bound: (spatial)

Need $\varphi'(x)^2 > 0$.

Permit $k_{-\infty} \neq k_{+\infty}$.

$\varphi'(x)$ essentially arbitrary.

$$T = |t|^2 = \text{sech}^2 Q$$



Theorem:

$$Q \leq \oint \frac{\sqrt{(\varphi'')^2 + [k^2 - (\varphi')^2]^2}}{2|\varphi'|} \, dx$$

Generalizes previous bound.

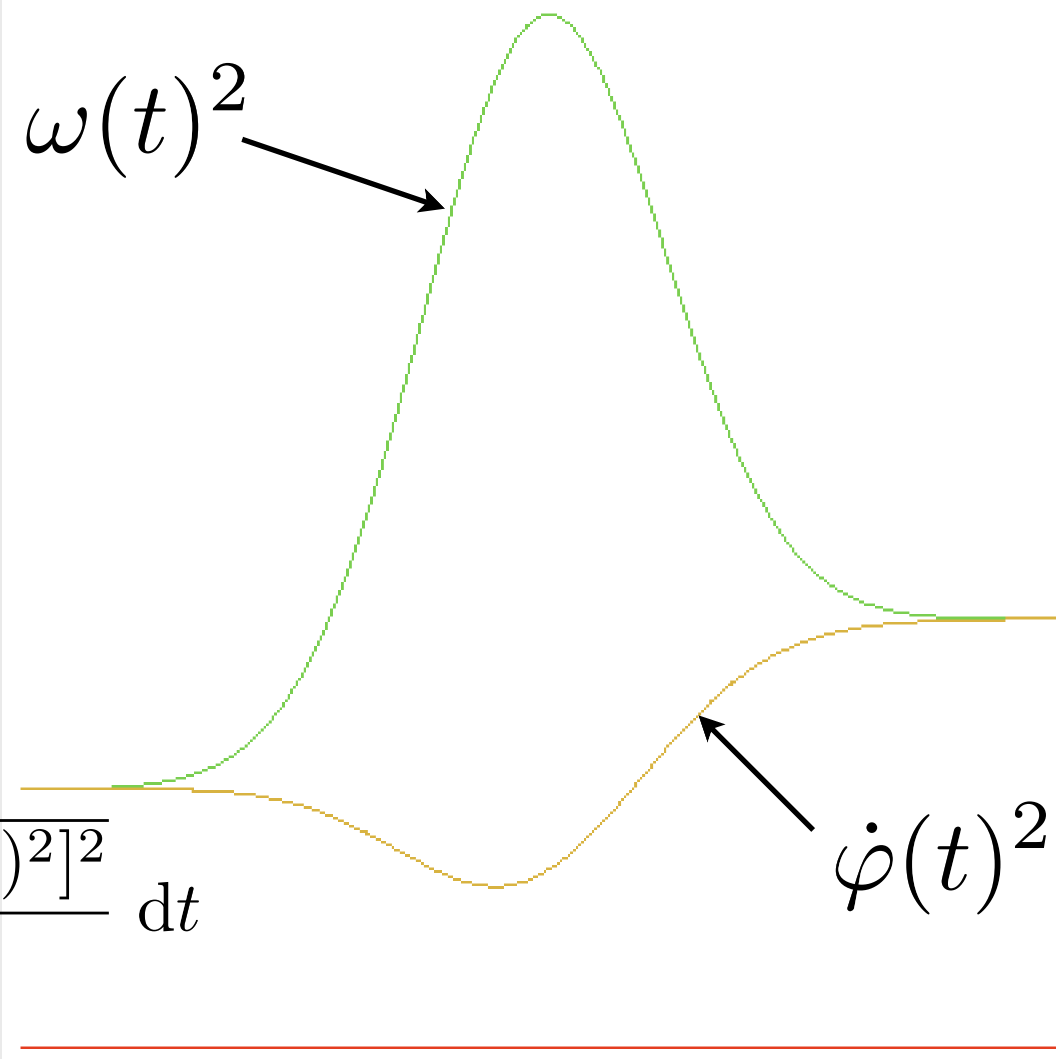
More general bound: (temporal)

Need $\dot{\varphi}(t)^2 > 0$.

Permit $\omega_{-\infty} \neq \omega_{+\infty}$.

$\dot{\varphi}(t)$ essentially arbitrary.

$N = |\beta|^2 = \sinh^2 Q$



Theorem:

$$Q \leq \oint \frac{\sqrt{(\ddot{\varphi})^2 + [\omega^2 - (\dot{\varphi})^2]^2}}{2|\dot{\varphi}|}\, \mathrm{d}t$$

Generalizes previous bound.

Reprise:

$$T \geq \operatorname{sech}^2 \left\{ \oint \frac{\sqrt{(\varphi'')^2 + [k^2 - (\varphi')^2]^2}}{2|\varphi'|} \, \mathrm{d}x \right\}$$

$$R \leq \tanh^2 \left\{ \oint \frac{\sqrt{(\varphi'')^2 + [k^2 - (\varphi')^2]^2}}{2|\varphi'|} \, \mathrm{d}x \right\}$$

$$N \leq \sinh^2 \left\{ \oint \frac{\sqrt{(\ddot{\varphi})^2 + [\omega^2 - (\dot{\varphi})^2]^2}}{2|\dot{\varphi}|} \, \mathrm{d}t \right\}$$

Special case of this more general bound:

$$\varphi'(x) = k(x) \qquad \dot{\varphi}(t) = \omega(t)$$

(Thus no “classically forbidden” region allowed.)

$$Q \leq \frac{1}{2} \oint \frac{|k'|}{|k|}\, \mathrm{d}x \qquad Q \leq \frac{1}{2} \oint \frac{|\dot{\omega}|}{|\omega|}\, \mathrm{d}t$$

$$T \geq \operatorname{sech}^2 \left\{ \frac{1}{2} \oint \frac{|k'|}{|k|}\, \mathrm{d}x \right\} \qquad N \leq \sinh^2 \left\{ \frac{1}{2} \oint \frac{|\dot{\omega}|}{|\omega|}\, \mathrm{d}t \right\}$$

Application:

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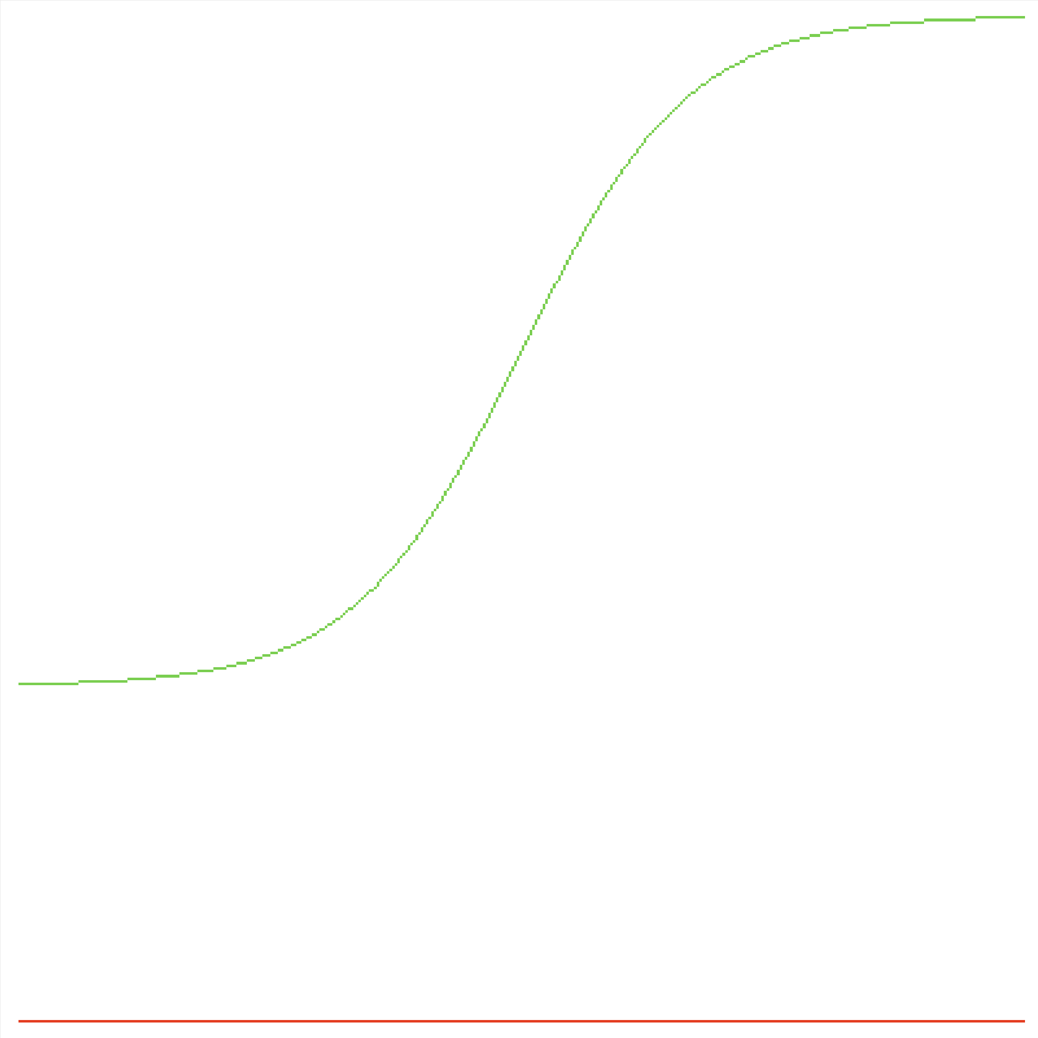
$$T \geq \operatorname{sech}^2\left\{\frac{1}{2}\ln\left[\frac{k_{+\infty}}{k_{-\infty}}\right]\right\}$$

$$N \leq \sinh^2\left\{\frac{1}{2}\ln\left[\frac{\omega_{+\infty}}{\omega_{-\infty}}\right]\right\}$$

Rewrite as:

$$T \geq \left(\frac{2}{\sqrt{k_{+\infty}/k_{-\infty}} + \sqrt{k_{-\infty}/k_{+\infty}}}\right)^2$$

$$N \leq \left(\frac{\sqrt{\omega_{+\infty}/\omega_{-\infty}} - \sqrt{\omega_{-\infty}/\omega_{+\infty}}}{2}\right)^2$$



Application:

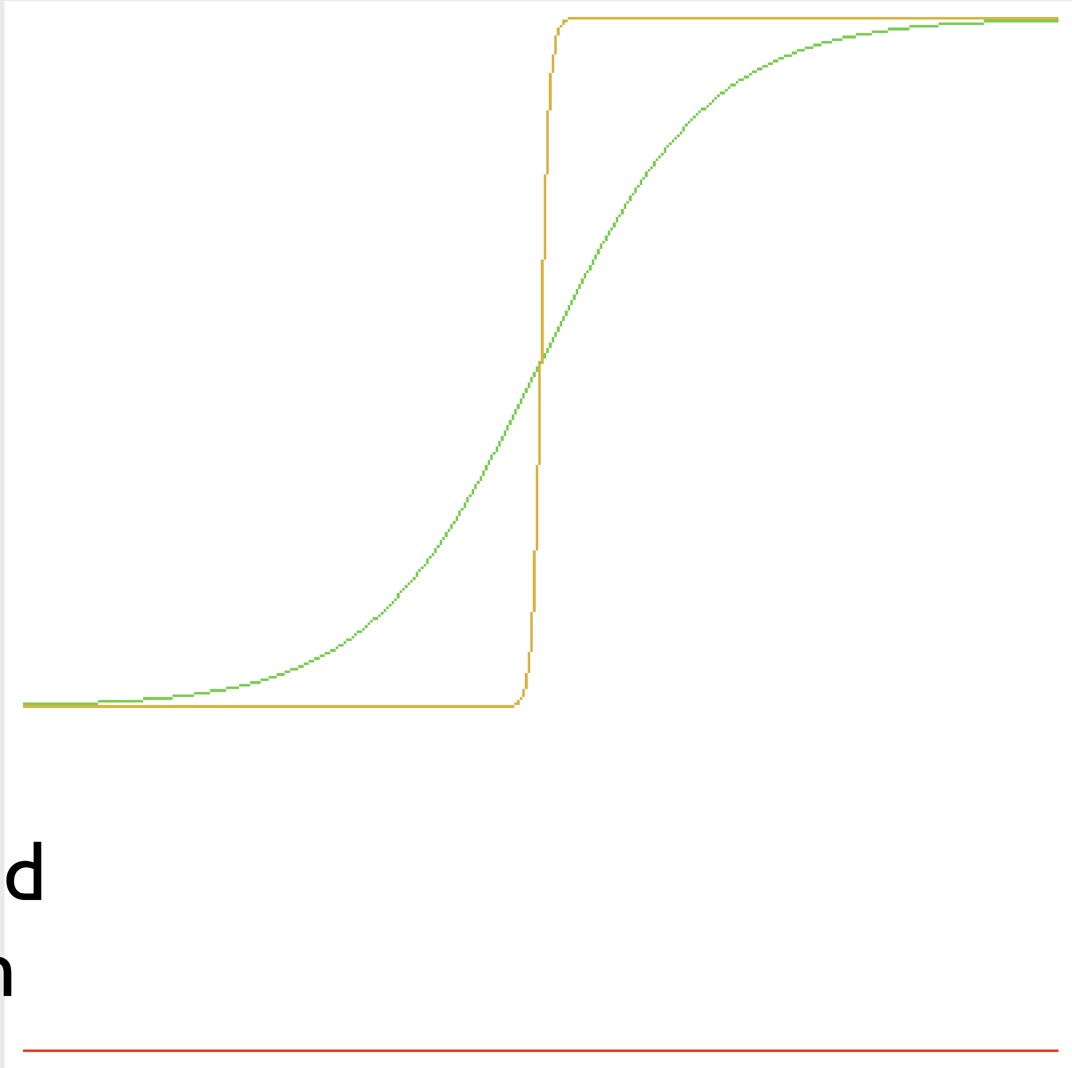
Rewrite as:

$$T \geq \frac{4k_{+\infty}k_{-\infty}}{(k_{+\infty} + k_{-\infty})^2}$$

$$N \leq \frac{(\omega_{+\infty} - \omega_{-\infty})^2}{4\omega_{+\infty}\omega_{-\infty}}$$

These bounds are saturated by a "sudden" step...

Sharp corners are guaranteed to be the best for reflection and particle production...



Discussion:

These are some very unexpected results...

How on earth did I (discover/ derive/ prove) these bounds?

Ultimately it was a side-effect of trying to understand the Birrell and Davies discussion of the adiabatic vacuum and cosmological particle production...

(Though the connection is somewhat tenuous.)

(And I'm still not sure I fully understand Birrell and Davies; there's got to be more you can say...)

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Strategy:

Pick a convenient basis:

$$\frac{\exp(+i\varphi)}{\sqrt{\varphi'}} \qquad \frac{\exp(-i\varphi)}{\sqrt{\varphi'}}.$$

Define position-dependent Bogoliubov coefficients:

$$\psi(x) = a(x)\frac{\exp(+i\varphi)}{\sqrt{\varphi'}} + b(x)\frac{\exp(-i\varphi)}{\sqrt{\varphi'}}.$$

Restrict with an "auxiliary condition":

$$\frac{d}{dx}\left(\frac{a}{\sqrt{\varphi'}}\right)e^{+i\varphi} + \frac{d}{dx}\left(\frac{b}{\sqrt{\varphi'}}\right)e^{-i\varphi} = 0.$$

The first derivative is simple...

$$\frac{d\psi}{dx} = i\sqrt{\varphi'}\left\{a(x)\exp(+i\varphi) - b(x)\exp(-i\varphi)\right\}.$$

As long as phi(x) is real (some generalizations possible):

$$\mathcal{J} = \mathrm{Im}\left\{\psi^* \frac{d\psi}{dx}\right\} = \left\{|a|^2 - |b|^2\right\}.$$

So then:

$$|a|^2 - |b|^2 = 1.$$

Strategy:

Rewrite the Schrodinger equation in terms of a(x) and b(x)...

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(i\frac{\varphi'}{\sqrt{\varphi'}}\left\{ a e^{+i\varphi} - b e^{-i\varphi}\right\}\right)$$

$$= \frac{(i\varphi')^2}{\sqrt{\varphi'}}\left\{ a e^{+i\varphi} + b e^{-i\varphi}\right\}$$

$$+ i\varphi'\left\{ \frac{d}{dx}\left(\frac{a}{\sqrt{\varphi'}}\right) e^{+i\varphi} - \frac{d}{dx}\left(\frac{b}{\sqrt{\varphi'}}\right) e^{-i\varphi}\right\}$$

$$+ i\frac{\varphi''}{\sqrt{\varphi'}}\left\{ a e^{+i\varphi} - b e^{-i\varphi}\right\}$$

Two distinct forms...

$$= -\frac{\varphi'^2}{\sqrt{\varphi'}}\left\{ a e^{+i\varphi} + b e^{-i\varphi}\right\} \qquad ***$$

$$+ \frac{2i\varphi'}{\sqrt{\varphi'}}\frac{da}{dx}e^{+i\varphi} - i\frac{\varphi''}{\sqrt{\varphi'}} b e^{-i\varphi}$$

$$= -\frac{\varphi'^2}{\sqrt{\varphi'}}\left\{ a e^{+i\varphi} + b e^{-i\varphi}\right\} \qquad ***$$

$$- \frac{2i\varphi'}{\sqrt{\varphi'}}\frac{db}{dx}e^{-i\varphi} + i\frac{\varphi''}{\sqrt{\varphi'}} a e^{+i\varphi}.$$

Strategy:

The Schrodinger equation is equivalent to:

$$\frac{da}{dx} = +\frac{1}{2\varphi'}\left\{\varphi''\, b\ \exp(-2i\varphi) + i\left[k^2(x) - (\varphi')^2\right]\left(a + b\exp(-2i\varphi)\right)\right\},$$

$$\frac{db}{dx} = +\frac{1}{2\varphi'}\left\{\varphi''\, a\ \exp(+2i\varphi) - i\left[k^2(x) - (\varphi')^2\right]\left(b + a\exp(+2i\varphi)\right)\right\}.$$

Strategy:

Calculate:

$$\frac{d|a|}{dx} = \frac{1}{2|a|}\left(a^* \frac{da}{dx} + a \frac{da^*}{dx}\right)$$

Obtain:

$$\frac{d|a|}{dx} = \frac{1}{2|a|}\frac{1}{2\varphi'}\Big(\varphi'' \left[a^* b \exp(-2i\varphi) + a b^* \exp(+2i\varphi)\right]$$
$$+ i[k^2 - (\varphi')^2]\left[a^* b \exp(-2i\varphi) - a b^* \exp(+2i\varphi)\right]\Big)$$

Strategy:

Implying:

$$\frac{d|a|}{dx} = \frac{1}{\cancel{2}|a|}\,\frac{1}{2\varphi'}\,\mathrm{Re}\left([\varphi'' + i[k^2 - (\varphi')^2]] \qquad [a^* b \exp(-2i\varphi)] \right).$$

Use: $\mathrm{Re}(X) \leq |X|$

Then:

$$\frac{d|a|}{dx} \leq \frac{|\varphi'' + i[k^2 - (\varphi')^2]|}{2|\varphi'|}\;|b|$$

$$\frac{d|a|}{dx} \leq \frac{\sqrt{(\varphi'')^2 + [k^2 - (\varphi')^2]^2}}{2|\varphi'|}\;|b|.$$

Integrate:

$$\vartheta[\varphi(x), k(x)] \equiv \frac{\sqrt{(\varphi'')^2 + [k^2(x) - (\varphi')^2]^2}}{2|\varphi'|},$$ (definition)

$$\frac{d|a|}{dx} \le \vartheta \sqrt{|a|^2 - 1}.$$

$$\left\{\cosh^{-1}|a|\right\}\Big|_{x_i}^{x_f} \le \int_{x_i}^{x_f} \vartheta \; dx.$$

$$\cosh^{-1}|\alpha| \le \int_{-\infty}^{+\infty} \vartheta \; dx.$$

Conclude:

$$|\alpha| \leq \cosh\left(\int_{-\infty}^{+\infty} \vartheta \; dx\right).$$

$$|\beta| \leq \sinh\left(\int_{-\infty}^{+\infty} \vartheta \; dx\right).$$

$$T \geq \operatorname{sech}^2\left(\int_{-\infty}^{+\infty} \vartheta \; dx\right),$$

$$R \leq \tanh^2\left(\int_{-\infty}^{+\infty} \vartheta \; dx\right).$$

This is one proof, (quant-ph/9901030).

Distinct proof in arXiv:0801.0610 [quant-ph]

I am certain that significant generalizations are still possible...

Another application:

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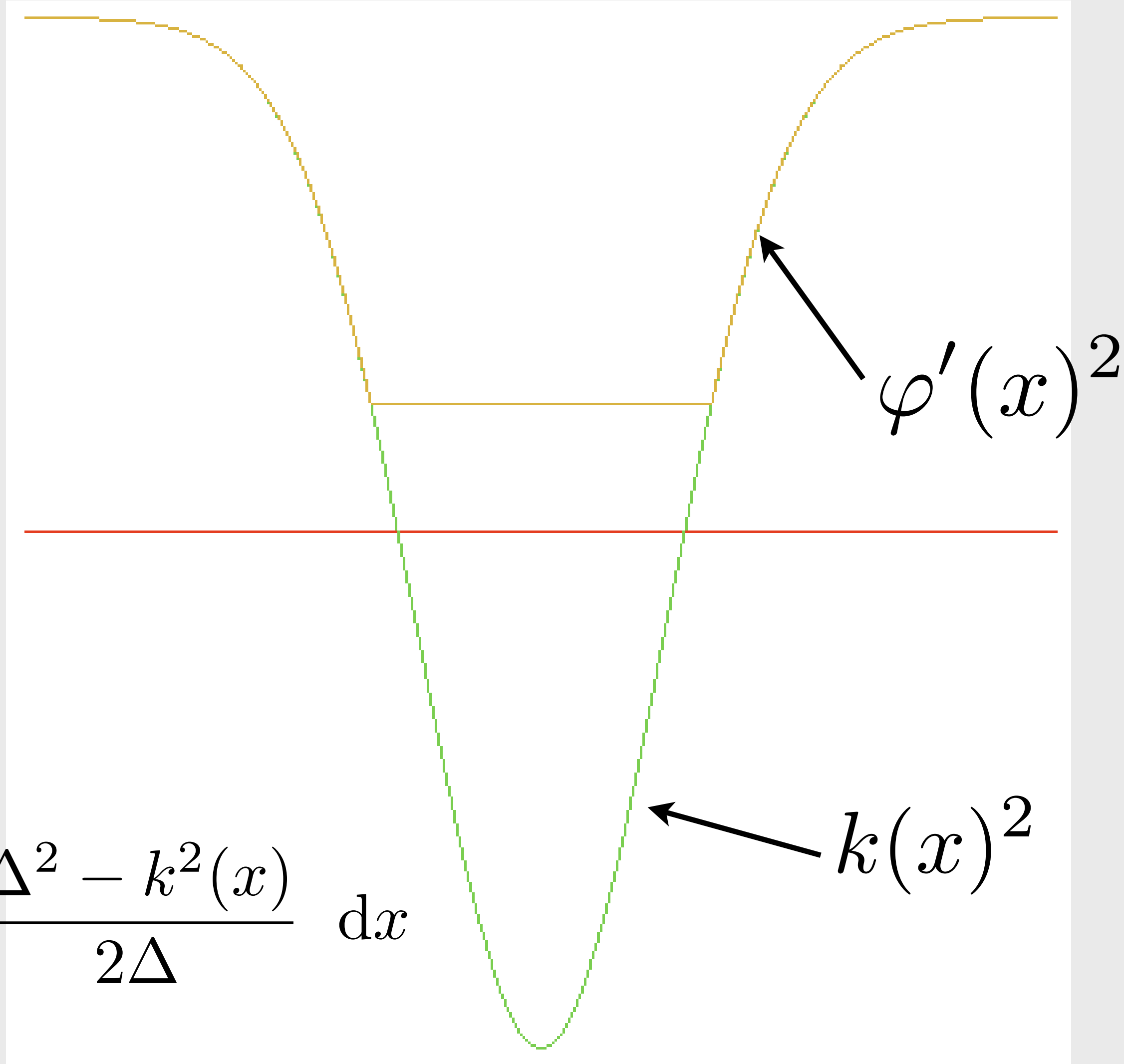
$$\forall \Delta > 0$$

Let phi'(x) track k(x) until one gets close to the classically forbidden region, then hold phi'(x) constant...



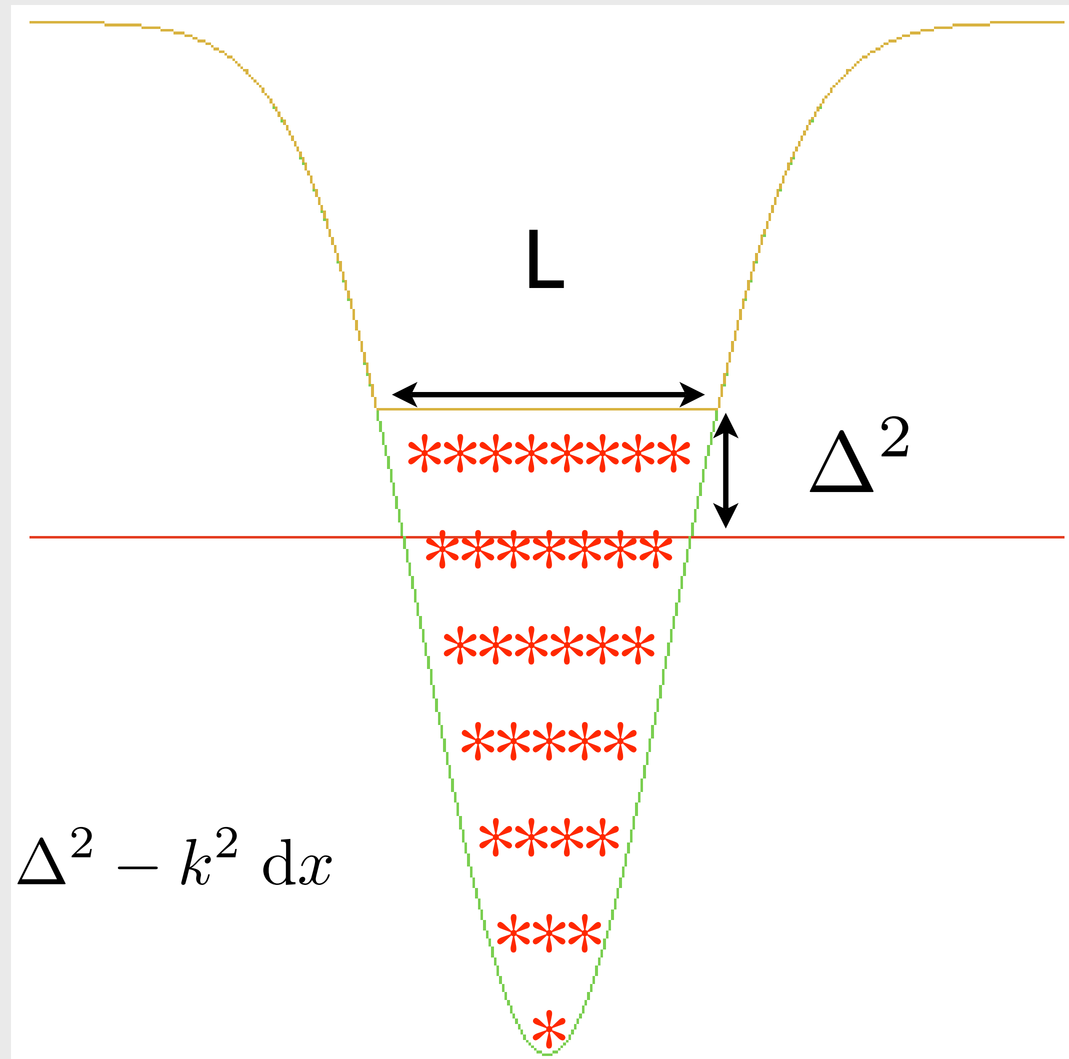
$$Q \leq \frac{1}{2} \ln \left[\frac{k_{+\infty} k_{-\infty}}{\Delta^2} \right] + \int_{\Delta^2 - k^2 > 0} \frac{\Delta^2 - k^2(x)}{2\Delta} \, \mathrm{d}x$$

Extremize to find the "best" value for Delta...

$$L_\Delta = \int_{\Delta^2 - k^2 > 0} 1 \; \mathrm{d}x$$

Horrible nonlinear equation for Delta...

$$\Delta L_\Delta + (\Delta L_\Delta)^2 = L_\Delta \int_{\Delta^2 - k^2 > 0} \Delta^2 - k^2 \, \mathrm{d}x$$



Another application:

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$$\Delta = \frac{\displaystyle\int_{\Delta^2 - k^2 > 0} \Delta^2 - k^2 \, \mathrm{d}x}{1 + \Delta \, L_\Delta}$$

$$\Delta \leq \int_{k^2 < 0} |k^2(x)| \, \mathrm{d}x$$

$$\Delta \geq \int_{k^2 < 0} |k^2(x)| \, \mathrm{d}x - \Delta^2 (L_\Delta - L_0)$$

Another application:

For shallow wells:

$$L_\Delta \int_{\Delta^2 - k^2 > 0} \Delta^2 - k^2 \, \mathrm{d}x \ll 1$$

We can then approximate:

$$\Delta \lesssim \int_{k^2 < 0} |k^2(x)| \, \mathrm{d}x$$

[Remember: bound holds for *all* Delta.]

Another application:

So we certainly can certainly assert:

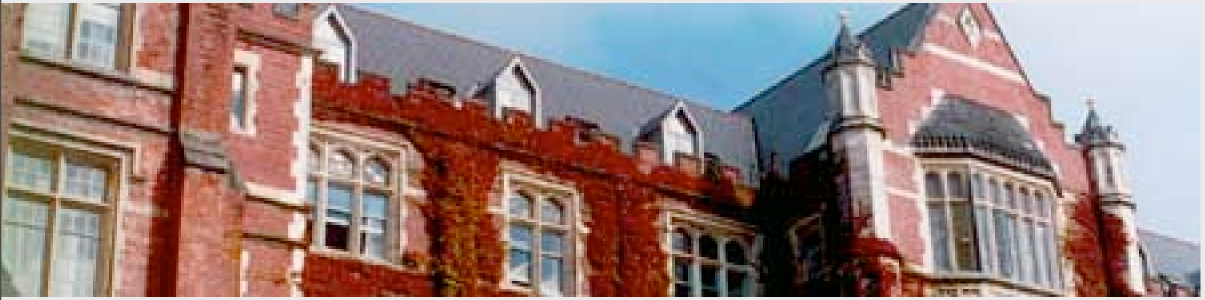
Choose: $$\Delta = \int_{k^2<0} |k^2(x)| \,\mathrm{d}x$$

Deduce:

$$Q \leq \frac{1}{2} \ln \left[\frac{k_{+\infty} k_{-\infty}}{\Delta^2} \right] + \int_{\Delta^2 - k^2 > 0} \frac{\Delta^2 - k^2(x)}{2\Delta} \,\mathrm{d}x$$

But this bound might not be "optimal".

(And finding a general "optimal" bound is exceedingly difficult.)



Comparison with JWKB:

If there is no forbidden region then:

$$T_{\mathrm{JWKB}} = 1$$

But for (almost all) potentials without a forbidden region there will still be *some* scattering, so:

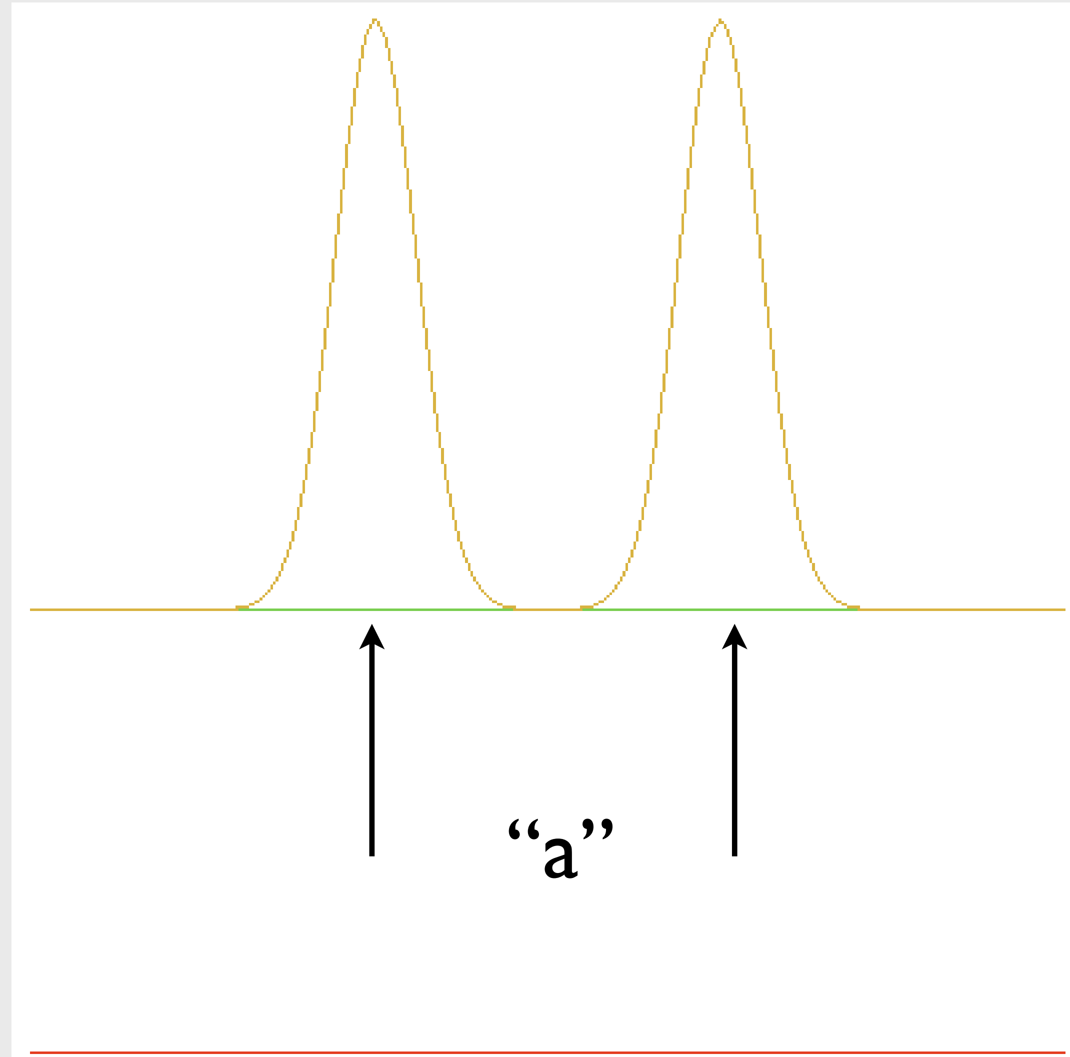
$$T_{\mathrm{exact}} < 1$$

So there definitely are situations where JWKB *over-estimates* the transmission probability.

Comparison with JWKB:

There are also situations where JWKB definitely *under-estimates* the transmission probability.

Consider a "repeated" potential, with the two repeats separated by distance "a".



Resonant tunnelling:

For such a "repeated" potential, one can encounter "resonant tunnelling".

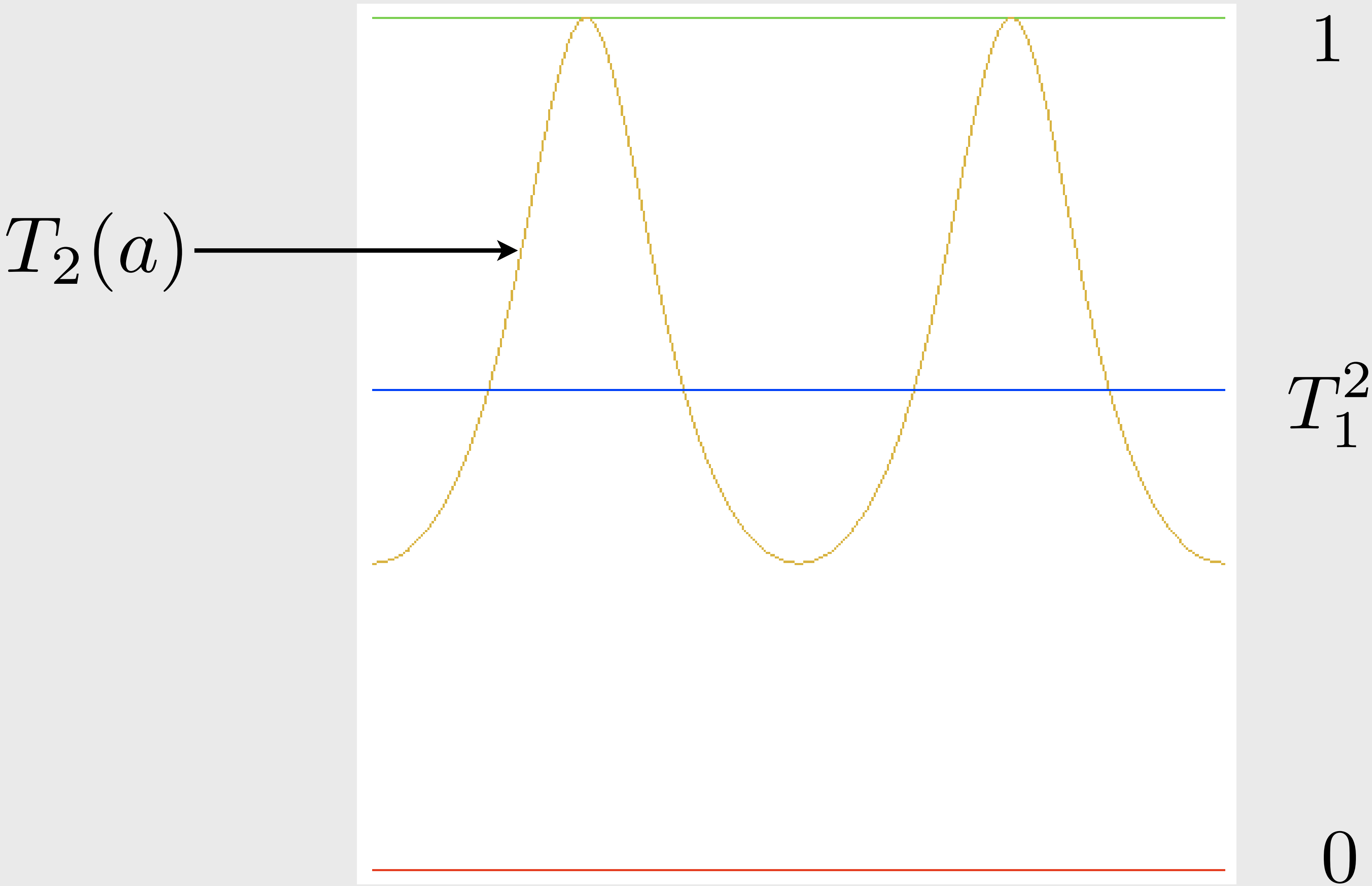
It is an "easy" exercise to see that:

$$T_2(a) = \frac{T_1^2}{1 + 2R_1 \cos[2k_\infty a - \phi(t)] + R_1^2}$$

Thus, as the separation "a" is adjusted the exact transmission probability oscillates between:

$$T_2(a) \in \left[\frac{T_1^2}{(1+R_1)^2},\ 1 \right]$$

Resonant tunnelling:



Resonant tunnelling:

Proof: Describe the effect of each individual potential by a "transfer matrix" [Jones matrix]:

$$
\begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1^* & \alpha_1^* \end{bmatrix} \qquad\qquad \begin{bmatrix} \alpha_1 & \beta_1 e^{2ika} \\ \beta_1^* e^{-2ika} & \alpha_1^* \end{bmatrix}
$$

As long as the 2 potentials do not overlap:

$$
\begin{bmatrix} \alpha_2(a) & \beta_2(a) \\ \beta_2^*(a) & \alpha_2^*(a) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1^* & \alpha_1^* \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 e^{2ika} \\ \beta_1^* e^{-2ika} & \alpha_1^* \end{bmatrix}
$$

Resonant tunnelling:

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Matrix multiply:

$$\alpha_2(a) = \alpha_1^2 + |\beta_1|^2 e^{2ika}$$

Evaluate:

$$T_2(a) = \frac{1}{|\alpha_2(a)|^2} = \frac{1}{\left| |\alpha_1^2| e^{2i\phi_\alpha} + |\beta_1|^2 e^{2ika} \right|^2}$$

$$T_2(a) = \frac{T_1^2}{1 + 2R_1 \cos[2k_\infty a - \phi(t)] + R_1^2}$$

Comparison with JWKB:

But for all potentials with a forbidden region:

$$T_{1,\mathrm{JWKB}} < 1 \qquad T_{2,\mathrm{JWKB}} = T_{1,\mathrm{JWKB}}^2 < 1$$

But we have just seen that there are some choices of the separation "a" for which

$$T_{2,\mathrm{exact}} = 1$$

That is, there are some potentials for which the JWKB estimate is an *under-estimate* of the true transmission probability.

This is what I mean by an "uncontrolled approximation".

JWKB is sometimes high, sometimes low, and there is no (known) a-priori estimate of just how good/ bad JWKB might be in a specific situation.

Given this, the "a priori" bounds I discussed earlier seem (at present) to be the best one can do.

Is there room for improvement?

Reprise:

$$T \geq \operatorname{sech}^2 \left\{ \oint \frac{\sqrt{(\varphi'')^2 + [k^2 - (\varphi')^2]^2}}{2|\varphi'|} \, \mathrm{d}x \right\}$$

$$R \leq \tanh^2 \left\{ \oint \frac{\sqrt{(\varphi'')^2 + [k^2 - (\varphi')^2]^2}}{2|\varphi'|} \, \mathrm{d}x \right\}$$

$$N \leq \sinh^2 \left\{ \oint \frac{\sqrt{(\ddot{\varphi})^2 + [\omega^2 - (\dot{\varphi})^2]^2}}{2|\dot{\varphi}|} \, \mathrm{d}t \right\}$$

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--- Probably there are "optimal" bounds still waiting to be discovered...

--- Definitely there are "more general" bounds in the process of being discovered...

--- Just because it's quantum mechanics, it does not mean that everything has already been done...

--- Watch this space....

Application to black holes:

Regge--Wheeler equation:

$$\frac{d^2\psi}{dr_*^2} = [\omega^2 - V(r)]\psi,$$

Tortoise coordinate:

$$\frac{dr}{dr_*} = 1 - \frac{2m}{r},$$

Regge--Wheeler potential:

$$V(r) = \left(1 - \frac{2m}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2m(1-s^2)}{r^3}\right].$$

Use this to bound greybody factors:

$$T \geq \operatorname{sech}^2 \left\{ \int_{-\infty}^{\infty} \vartheta \, \mathrm{d}r_* \right\}.$$

$$\vartheta = \frac{\sqrt{(h')^2 + [\omega^2 - V - h^2]^2}}{2h}.$$

Omega is now frequency at spatial infinity.

If we set $h = \omega$ then

$$T \geq \operatorname{sech}^2 \left\{ \frac{1}{2\omega} \int_{-\infty}^{\infty} V(r_*) \, \mathrm{d}r_* \right\},$$

Bounding greybody factors:

$$T \geq \operatorname{sech}^2 \left\{ \frac{1}{2\omega} \int_{2m}^{\infty} \left[\frac{\ell(\ell+1)}{r^2} + \frac{2m(1-s^2)}{r^3} \right] \mathrm{d}r \right\}$$

$$T \geq \operatorname{sech}^2 \left\{ \frac{2\ell(\ell+1) + (1-s^2)}{8\omega m} \right\}.$$

$$T \geq \operatorname{sech}^2 \left\{ \frac{(\ell+1)^2 + (\ell^2 - s^2)}{8\omega m} \right\}.$$

Explicit bound for all L and S...

Bounding greybody factors:

If we now set $h = \sqrt{\omega^2 - V}$,

$$T \geq \operatorname{sech}^2 \left\{ \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{h'}{h} \right| \, \mathrm{d}r_* \right\}.$$

$$\begin{aligned} T &\geq \operatorname{sech}^2 \left\{ \ln \left(\frac{h_{\text{peak}}}{h_\infty} \right) \right\} \\ &= \operatorname{sech}^2 \left\{ \ln \left(\frac{\sqrt{\omega^2 - V_{\text{peak}}}}{\omega} \right) \right\}, \end{aligned}$$

Find V_{peak}!

$$T \geq \frac{4\omega^2(\omega^2 - V_{\text{peak}})}{(2\omega^2 - V_{\text{peak}})^2} = 1 - \frac{V_{\text{peak}}^2}{(2\omega^2 - V_{\text{peak}})^2}.$$

Bounding greybody factors:

$$T_{s=1} \geq \frac{108\omega^2 m^2[27\omega^2 m^2 - \ell(\ell+1)]}{[54\omega^2 m^2 - \ell(\ell+1)]^2}.$$

$$T_{s=0,\ell=0} \geq \frac{4096\omega^2 m^2[1024\omega^2 m^2 - 27]}{[2048\omega^2 m^2 - 27]^2}.$$

$$T_{s=0,\ell\geq 1} > \frac{108\omega^2 m^2[27\omega^2 m^2 - (\ell^2+\ell+1)]}{[54\omega^2 m^2 - (\ell^2+\ell+1)]^2},$$

$$T_{s>1} > \frac{108\omega^2 m^2[27\omega^2 m^2 - \ell(\ell+1)]}{[54\omega^2 m^2 - \ell(\ell+1)]^2},$$

Bounding greybody factors:

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I am certain that significant generalizations are still possible...

--- Watch this space....