Zeta Functions and the Casimir Energy

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Abstract

We use zeta function techniques to give a finite definition for the Casimir energy of an arbitrary ultrastatic spacetime with or without boundaries. We find that the Casimir energy is intimately related to, but not identical to, the one–loop effective energy. We show that in general the Casimir energy depends on a normalization scale. This phenomenon has relevance to applications of the Casimir energy in bag models of QCD.

Within the framework of Kaluza–Klein theories we discuss the one–loop corrections to the induced cosmological and Newton constants in terms of a Casimir like effect. We can calculate the dependence of these constants on the radius of the compact dimensions, without having to resort to detailed calculations.

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1 Introduction

The study of vacuum fluctuations, as embodied in the Casimir effect [1], has been a subject of extensive research [2]. The Casimir energy may be thought of as the energy due to the distortion of the vacuum. This distortion may be caused either by some background field (*e.g.* gravity), or by the presence of boundaries in the space–time manifold (*e.g.* conductors). Early investigations of the effects of a gravitational background were performed by Utiyama and De Witt [3], and work has continued on this important subject [4, 5, 6, 7, 8, 9]. Early work on the effect of boundaries was performed by Casimir [1], and was later extended by Fierz, Boyer, deRaad, and Milton [10, 11, 12, 13]. More recently boundary effects have been central to the calculation of the Casimir energy in bag models of QCD [14, 15, 16].

We feel that interesting things remain to be said. In this paper heat kernel and zeta function techniques will be utilized to investigate these topics [4, 17]. The unified treatment presented here is applicable to a very wide class of models and physical situations.

We start by developing a definition of the Casimir energy which is *finite* and applies to arbitrary static manifolds with or without boundaries

$$E_{\rm Casimir} = \frac{1}{2}\hbar c\mu \cdot PP[\zeta_3(-\tfrac{1}{2}+\epsilon)]. \tag{1.1}$$

Here μ is a normalization scale of dimension $(length)^{-1}$, and the PP symbol indicates that we are to extract the "principal part". This definition yields a finite quantity in both flat and curved space–times, with or without boundaries, for both massive and massless particles. The normalization scale μ appearing in the above is required to keep the zeta function dimensionless for all values of s. The introduction of this scale leads generically to non-trivial scaling behaviour for the Casimir energy. It is pointed out how this definition relates in special cases to well–known results.

Our definition of the Casimir energy allows us to investigate its dependence on the "radius" of the manifold. We find that for massless fields

$$E_{\rm Casimir}(R) = \frac{\hbar c}{R} \cdot \{\epsilon_0 - \epsilon_1 \cdot \ln(\mu R)\}, \tag{1.2}$$

where the μ-independent coefficients ϵ_0 and ϵ_1 are dimensionless numbers depending on the geometry of the manifold. This result has some very interesting consequences when applied to the bag models of hadrons in QCD.

Further, we may relate the Casimir energy to the one–loop effective action (*i.e.* the determinant of a suitable *four* dimensional differential operator). This is done by relating the zeta function of $D_4 = -{\partial_0}^2 + D_3$ to the zeta function of D_3

$$\zeta_4(s) = \frac{\mu c T}{\sqrt{4\pi}} \cdot \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \cdot \zeta_3(s-\tfrac{1}{2}). \tag{1.3}$$

Thus we obtain a non-trivial relationship between the Casimir energy and the one-loop effective energy

$$E_{\rm eff} = E_{\rm Casimir} + \frac{1}{2}\hbar c\mu \ \left[\psi(1) - \psi(-\tfrac{1}{2})\right] \ \frac{C_2}{(4\pi)^2}. \tag{1.4}$$

To help understand the significance of this relationship we include a discussion of the various *different* concepts commonly lumped together as "vacuum energy".

We next apply our analysis to the one–loop corrections to the effective cosmological constant and Newton constant in Kaluza–Klein theories. These one-loop corrections may be interpreted as a Casimir-like effect. We derive the following *finite* expressions for the one-loop four-dimensional effective cosmological and Newton constants.

$$\begin{aligned} \Lambda_{\rm eff} &= \Lambda \cdot {\rm vol}(\Omega) + G^{-1} \cdot \int_\Omega \sqrt{g}\, R_d - \frac{\mu^4}{2(4\pi)^2}\, \left\{ \tfrac{1}{2}\zeta_d'(-2) - \tfrac{3}{4}\zeta_d(-2) \right\}, \\ G_{\rm eff}^{-1} &= G^{-1} \cdot {\rm vol}(\Omega) - k\, \frac{\mu^2}{2(4\pi)^2}\, \left\{ \zeta_d'(-1) - \zeta_d(-1) \right\}. \end{aligned} \tag{1.5}$$

In particular, this allows us to study the dependence of these constants on the "radius" of the compact dimensions, without having to resort to explicit calculations.

2 Zeta functions on manifolds with boundary.

As regularization technique we shall use the zeta-function method due to Dowker and Critchley [4] and Hawking [17]. Its relation to other methods (*e.g.*, dimensional regularization) has been discussed in the literature [4]. In order to make subsequent arguments understandable, we must first briefly review the mathematical machinery of zeta functions. Consider the zeta function associated with a second-order self-adjoint elliptic operator D defined on a compact manifold Ω with boundary $\partial\Omega$

$$\zeta(s) = \mathrm{tr}'\{(\mu^{-2}D)^{-s}\} = {\sum}'(\mu^{-2}\lambda_n)^{-s}, \tag{2.1}$$

where the λ_n are eigenvalues of D; while the prime on tr' and ${\sum}'$ indicates that we should *not* include the zero eigenvalues of D in the sum. We have introduced a "scale" μ, with the dimensions of $(length)^{-1}$, in order to keep the zeta-function dimensionless for all s.

The zeta function is related to the diffusion operator (heat kernel) via a Mellin transform:

$$\begin{aligned}\zeta(s) &= {\sum}' \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1}\; \exp(-\lambda_n \mu^{-2} t) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty dt\; t^{s-1}\; \mathrm{tr}'(e^{-tD\mu^{-2}}).\end{aligned} \tag{2.2}$$

Here t is a dimensionless parameter, not to be confused with physical time (x^0/c). From now on, in the interests of notational simplicity, we ignore zero modes. The trace of the diffusion operator is given by the integral of the diagonal part of the heat kernel over the manifold:

$$\mathrm{tr}(e^{-tD\mu^{-2}}) = \int_\Omega K(t,x,x)\; \sqrt{g}\; d^d x. \tag{2.3}$$

The heat kernel K possesses an asymptotic expansion for small t:

$$K(t,x,x) = \left(\frac{\mu^2}{4\pi t}\right)^{d/2} \cdot \left\{ \sum_0^N a_n(x)\; (\mu^{-2}t)^n + o(t^N) \right\}. \tag{2.4}$$

The sum is over integer values of n. The a_n are functions of the gravitational field, they may be expressed as polynomials in the Riemann tensor, its contractions, and covariant derivatives. (See Appendix A.) The diagonal part of the heat kernel contains exponentially suppressed terms $(e^{-k(x)/t})$ that do not contribute to the asymptotic expansion (2.4). These exponentially suppressed terms do however contribute an explicit boundary term to the trace of the heat kernel

$$\mathrm{tr}(e^{-tD\mu^{-2}}) = \left(\frac{\mu^2}{4\pi t}\right)^{d/2} \cdot \left\{ \sum_0^N \left(\int_\Omega a_n(x)\; (\mu^{-2}t)^n + \int_{\partial\Omega} b_n(y)\; (\mu^{-2}t)^n \right) + o(t^N) \right\}. \tag{2.5}$$

The sum runs over half–integers, (but the a_n vanish for half-odd-integers). The b_n are functions of the second fundamental form of the boundary (extrinsic curvature), the induced geometry on the boundary (intrinsic curvature), and the nature of boundary conditions imposed. These objects are tabulated in many places: *e.g.*, Birrell and Davies [19] and Appendix A of this paper. For future reference we define the dimensionless quantities: $A_n = \mu^{d-2n} \int_\Omega a_n(x)\; \sqrt{g}\; d^d x$, $B_n = \mu^{d-2n} \int_{\partial\Omega} b_n(y)\; \sqrt{\tilde{g}}\, d^{d-1} y$, and $C_n = A_n + B_n$.

In view of the asymptotic expansion (2.5), it is clear that the zeta function $\zeta(s)$ is a meromorphic function of the complex variable s possessing only simple poles whose residues are determined by C_n. Observe that (2.5) implies that $\zeta(s)$ has a pole structure given by

$$\zeta(s) = \frac{1}{\Gamma(s)\ (4\pi)^{d/2}} \cdot \left\{ \sum_0^\infty \frac{C_n}{(s - [\frac{d}{2} - n])} + f(s) \right\}. \tag{2.6}$$

The function $f(s)$ is an entire analytic function of s, but, in general, we have little additional information concerning its behaviour. However, we do know that $\zeta(s)$ is analytic at $s = 0$. It is thus possible to define the determinant of D to be [17]

$$\det{}'(\mu^{-2} D) = \exp\left(- \frac{d}{ds} \zeta(s) \Big|_{s=0} \right). \tag{2.7}$$

Observe that many of the technical details associated with renormalization have been hidden by these zeta function techniques. We shall now utilize this mathematical machinery to define the Casimir energy, and relate $E_{\rm Casimir}$ to the one–loop Effective action $S_{\rm eff} = \frac{1}{2} \ln \det D$.

3 The Casimir energy.

In order to have a well–defined notion of energy, it is useful to work in a static space-time [18], specifically let us take $g_4 = -(dx^0)^2 + g_3$, in which case we decompose the differential operator D_4 as $D_4 = -(\partial_0)^2 + D_3$. The eigen-frequencies associated with D_3 are $\omega_n = \sqrt{\lambda_n(D_3)} \cdot c$. We wish to consider the zero–point energy:

$$E_{\rm Casimir} = \tfrac{1}{2} \sum_n \hbar\omega_n. \tag{3.1}$$

This sum is, of course, divergent. We regularize it by defining

$$E_{\rm reg}(\epsilon) = \tfrac{1}{2}\hbar c\mu \cdot \textstyle\sum_n (\lambda_n \mu^{-2})^{(\frac{1}{2}-\epsilon)} = \tfrac{1}{2}\hbar c\mu \cdot \zeta_3\left(-\tfrac{1}{2}+\epsilon\right). \tag{3.2}$$

Where ζ_3 is the zeta function associated with the three-dimensional operator D_3. A quick glance at the previous section shows that $E_{\rm reg}(\epsilon)$ is a meromorphic function with a pole at $\epsilon = 0$, with residue $-\frac{1}{2}\hbar c\mu\, C_2(g_3)/(4\pi)^2 = -\frac{1}{2}\hbar c\{\int_\Omega a_2 + \int_{\partial\Omega} b_2\}/(4\pi)^2$, where the integral is over three-dimensional space and its two-dimensional boundary. Because of the pole at $\epsilon = 0$, we cannot, in general, remove the regulator; the geometric coefficient C_2 is an obstacle to giving a finite definition for the Casimir energy. Note, however, that in many interesting cases (*e.g.*, flat space with flat boundaries and massless particles) $C_2 = 0$, so that $\lim_{\epsilon\to 0} E_{\rm reg}(\epsilon)$ is finite, and independent of the normalization scale μ.

How is one to understand the unphysical pole and μ dependence of the (zeta-function regulated) Casimir energy? First we note that the Casimir energy in isolation is unphysical. When physicists speak of the Casimir energy they usually are identifying terms in the renormalized total energy which they interpret as arising from boundary or gravitational effects. There is *ipso facto* no pole in the total energy; the pole in equation (3.2) is absorbed into the bare action which must contain a term proportional to C_2. Having seen this we must admit that the way in which the pole is removed is not unique. The possibility of different renormalization schemes means that the Casimir energy has an ambiguity proportional to C_2. Our choice of renormalization scheme is to adopt the minimal subtraction scheme which is equivalent to simply removing the pole from equation (3.2). We define

$$\begin{aligned} E_{\rm Casimir} &\equiv \lim_{\epsilon\to 0} \tfrac{1}{2}\{E_{\rm reg}(+\epsilon) + E_{\rm reg}(-\epsilon)\} \\ &\equiv \frac{1}{2}\hbar c\mu \cdot \lim_{\epsilon\to 0} \tfrac{1}{2}\{\zeta_3(-\tfrac{1}{2}+\epsilon) + \zeta_3(-\tfrac{1}{2}-\epsilon)\} \\ &\equiv \frac{1}{2}\hbar c\mu \cdot PP[\zeta_3(-\tfrac{1}{2}+\epsilon)], \end{aligned} \tag{3.3}$$

where the symbol PP stands for taking the principal part. (This technique yields the "finite part" of any meromorphic function that possesses at worst simple poles.)

The Casimir energy defined in equation (3.3) depends, in general, on the normalization scale. We keep this scale dependence to remind us that the that the renormalization programme, which removes any μ dependence from the total energy, may introduce a second finite ambiguity in the Casimir energy. In section 4 we shall study how the Casimir energy varies with this normalization scale. In section 5 we shall relate the Casimir energy to the one-loop effective energy, which also depends on the normalization scale.

The difference between the two is finite, μ independent, and proportional to the geometric term C_2. In particular, the Casimir and one-loop effective energies agree when C_2 vanishes. The total energy, in the context of bag models, is considered in section 6, and we shall verify that it is independent of μ.

4 The role of the normalization scale

The renormalized Casimir energy defined by equation (3.3) generically will depend on the normalization scale μ. This should not, in fact, be surprising. As we shall soon see, the Casimir energy is intimately related to one–loop physics, and the occurrence of anomalous scale dependence in one–loop field theory calculations is by now a well understood phenomenon [20, 21]. This anomalous scaling behaviour manifests itself in two ways: (i) the Casimir energy may depend on the normalization scale μ; (ii) for conformally coupled fields, the Casimir energy may fail to scale as the inverse of the radius of the system. This effect is related to the existence of the conformal anomaly (trace anomaly). Note however, that the Casimir energy, in isolation, cannot be measured. What is measurable is the total energy which includes (renormalized) zero-loop contributions along with the Casimir energy. If one knew the Lagrangian for the entire system under study (*e.g.*, see the discussion of bag models later in this paper) then one would express the total energy in terms of running coupling constant sand the normalization scale μ. The total energy is independent of μ. If the total Lagrangian is unknown, the Casimir energy still gives the proper geometric dependence for the oder $\hbar$ part of the total energy. In particular, naive scaling behaviour of the total energy is violated. The scale μ should be interpreted as a scale that summarizes the (unknown) physics associated with the boundaries, curvature, and masses; it must be determined experimentally.

Consider the effect of a change in the normalization scale $\mu \to \mu'$. From the definition of the zeta function it is easy to see that this induces a change $\zeta_3(s,\mu') = (\mu'/\mu)^{2s} \cdot \zeta_3(s,\mu)$, so that $E_{\rm reg}(\epsilon,\mu') = (\mu'/\mu)^{2\epsilon} \cdot E_{\rm reg}(\epsilon,\mu)$. Now for any analytic function $f(s)$ it is easy to see that

$$PP[f(s)\zeta(s)] = f(s) \cdot PP[\zeta(s)] + f'(s) \cdot Res[\zeta(s)]. \tag{4.1}$$

This has the immediate consequence that

$$E_{\rm Casimir}(\mu') = E_{\rm Casimir}(\mu) - \hbar c \mu \cdot \frac{C_2(\mu)}{(4\pi)^2} \cdot \ln\left[\frac{\mu'}{\mu}\right]. \tag{4.2}$$

The dependence on the normalization scale is logarithmic, with a coefficient given by the second Seeley-De Witt coefficient. (The combination μC_2 is, despite appearances, independent of the scale μ.) As is to be expected, this dependence on normalization scale leads to a breakdown of scale covariance. (It should be noted that C_2 depends on $\int a_2$, and that a_2 contains a piece proportional to the conformal anomaly [19], in fact $T^\sigma{}_\sigma \propto a_2$, and, for a conformally coupled theory, a_2 *is* the conformal anomaly.)

Now consider the effect of rescaling the metric and masses: $g_3 \to \kappa^2 \cdot g_3$, $m \to \kappa^{-1} \cdot m$. This has a simple effect on the eigenvalues of D_3, namely: $\lambda_n \to \kappa^{-2} \cdot \lambda_n$. So for the zeta function

$$\zeta_3(\kappa^2 g_3; \kappa^{-1} m; s) = \kappa^{2s} \cdot \zeta_3(g_3; m; s). \tag{4.3}$$

Using the properties of the principal part prescription we find

$$E_{\rm Casimir}(\kappa^2 \cdot g_3; \kappa^{-1} \cdot m) = \frac{E_{\rm Casimir}(g_3; m)}{\kappa} - \hbar c \mu \cdot \frac{C_2(g_3; m)}{(4\pi)^2} \cdot \frac{\ln \kappa}{\kappa}. \tag{4.4}$$

This is the generalization, allowing for massive particles, of equation (1.2). It is easy to see that if $\kappa \to \infty$ then $E_{\rm Casimir} \to 0$, thus the approach to massless particles in Minkowski space does in fact lead to zero Casimir energy.

To derive equation (1.2) of the introduction, we note that the radius of the manifold $\kappa^2\ g_3$ is given by $R(\kappa^2 g_3) = \kappa\ R(g_3)$. Then equation (4.4) may be written as

$$E_{\rm Casimir}(R) = \frac{\hbar c}{R} \cdot \{\epsilon_0 - \epsilon_1 \cdot \ln(\mu R)\}, \tag{4.5}$$

where

$$\begin{aligned} \epsilon_1 &= \frac{C_2(g_3, \mu = R(g_3)^{-1})}{(4\pi)^2 \hbar c}, \\ \epsilon_0 &= \left[\frac{E_{\rm Casimir}(g_3,\mu)\cdot R(g_3)}{\hbar c}\right] + [\epsilon_1\ \ln(\mu R(g_3))]. \end{aligned} \tag{4.6}$$

Note that ϵ_0 and ϵ_1 are independent of the normalization scale μ. A little thought will show one that ϵ_1 depends only on the *shape* of the manifold, and are in fact independent of the radius of the manifold. The total energy must contain a term with the same geometric structure as the Casimir energy

$$E_{\rm tot} = \frac{\hbar c}{R}\ \{\epsilon_0(\mu) - \epsilon_1\ \ln(\mu R)\} + \ldots, \tag{4.7}$$

where now $\epsilon_0(\mu)$ depends on μ logarithmically so that $E_{\rm tot}$ is independent of the normalization scale. One might set the scale μ arbitrarily, and determine the "running coupling constant" ϵ_0 as a function of μ. In the context of Casimir energy calculations it is natural to use an alternative procedure: fix $\epsilon_0(\mu)$ to have the value determined by equation (4.6), and determine μ experimentally. (This is completely analogous to the experimental determination of $\Lambda_{\rm QCD}$.)

From (4.5) we see that if $C_2(g_3) > 0$, then the Casimir energy has an absolute minimum at $R_{\rm min} = \mu^{-1} \cdot \exp(1 + |\epsilon_0/\epsilon_1|)$, with $E_{min} = -\hbar c|\epsilon_1|/R_{\rm min}$. If $C_2(g_3) < 0$ then the Casimir energy is unbounded from below, approaching $E \to -\infty$ as $R \to 0$. (There is now an absolute maximum at $R_{max} = \mu^{-1} \cdot \exp(1 + |\epsilon_0/\epsilon_1|)$ and $E_{max} = +\hbar c|\epsilon_1|/R_{\rm max}$. The *sign* of C_2 is thus the determining factor in deciding whether the Casimir effect is repulsive or attractive for small sizes. If $C_2(g_3) = 0$ then an absolute extremum occurs at $R = \infty$ and $E = 0$.

The appearance of the logarithmic dependence on the radius in (1.2), (4.4), and (4.5) is very striking. One may quite justifiably ask, would this term not have been seen in some of the many Casimir energy calculations in the literature? The answer is that in very many situations encountered in the literature C_2 vanishes. Specifically, in flat 3-space, with massless particles, and any collection of infinitely thin boundaries one has $C_2 = 0$ (for either Dirichlet or Neumann boundary conditions). In particular, considering the case of the electromagnetic field, any collection of infinitely thin perfect conductors has $C_2 = 0$. To see this, recall $C_2 = A_2 + B_2$. Now $A_2 = 0$ since we are in flat space. Further $b_2(y)$ contains only *odd* powers of the second fundamental form. Infinitely thin boundaries means that all boundaries consist of two oppositely oriented faces separated by an infinitesimal distance. Thus the second fundamental forms are equal and opposite on the two faces of each boundary, and consequently the net value of b_2 summed over the two faces of each boundary vanishes. Thus $B_2 = 0$, as required.

The case of Robin boundary conditions requires extra care. For Robin boundary conditions $\partial\phi/\partial\eta(y) + \psi(y)\phi(y) = 0$ on the boundary. In this case one still has $C_2 = 0$ for

thin boundaries, provided one makes the additional assumption that $\psi(y+) = -\psi(y-)$. That is, provided ψ is equal and opposite on the two faces of each thin boundary layer.

Some cases where C_2 does not vanish have also been discussed in the literature. These situations have occasioned some rather puzzled comments which we shall discuss more fully below.

5 The one–loop effective action.

We now consider the relationship between the Casimir energy defined by (3.3) and the one–loop effective energy. As in the previous section, we consider an ultrastatic spacetime with $g_4 = -(dx^0)^2 + g_3$. To proceed we Wick rotate to imaginary time so that the Euclidean Laplacian is $D_4 = +\partial_0{}^2 + D_3$. The heat kernel then factorizes, $e^{-D_4\mu^{-2}t} = e^{-\partial_0{}^2\mu^{-2}t} \cdot e^{-D_3\mu^{-2}t}$, so that for the diagonal part of the heat kernel one has:

$$K_4(x,x,t) = \frac{1}{\sqrt{4\pi\mu^{-2}\,t}} \cdot K_3(x,x,t). \tag{5.1}$$

Now, defining $T = \int dx^0/c =$ "age of the universe", and applying the Mellin transform (2.2) one sees

$$\zeta_4(s) = \frac{\mu c T}{\sqrt{4\pi}} \cdot \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \cdot \zeta_3(s-\tfrac{1}{2}). \tag{5.2}$$

Using $E_{\rm eff} \cdot T = S_{\rm eff} = +\frac{1}{2}\ln\det D = -\frac{1}{2}\zeta_4'(0)$, and the known analyticity properties of the zeta function yields:

$$E_{\rm eff} = E_{\rm Casimir} + \tfrac{1}{2}\hbar c\mu \cdot [\psi(1) - \psi(-\tfrac{1}{2})] \cdot \tfrac{C_2}{(4\pi)^2}. \tag{5.3}$$

Where $\psi(s) = d\ln\Gamma(s)/ds$ is the digamma function. The effective energy and Casimir energy differ, but the difference reflects the inherent renormalization-scheme ambiguity introduced in the Casimir energy by removing the pole in equation (3.2). The unambiguous parts of the effective and Casimir energies agree, illustrating a remarkably close connection between zero-point energies and one-loop quantum effects. Note that when $C_2 = 0$, so that the zeta-function regulated Casimir energy is unambiguous and finite, $E_{\rm eff} = E_{\rm Casimir}$.

There are several variations on the concept of "vacuum energy" in common circulation. One of these is the vacuum–expectation–value of the integral of the 00 component of stress energy: $E_{\rm Vacuum} = \int < 0|T_{00}|0 >$. This version of the vacuum energy is, in general, not equal to either one of $E_{\rm Casimir}$ or $E_{\rm eff}$. However, if one were to switch off all interactions, so that $T_{00} \to T_{00}^{\rm Free}$, then an argument, (Presented, *e.g.*, in the review article [2]), shows that under rather general conditions $E_{\rm Casimir} = \int < 0|T_{00}^{\rm Free}|0 >$. Yet another version of vacuum energy is obtained by considering the full effective action in place of the one–loop effective action and its corresponding effective energy $E_{\rm eff}^{\infty} = \Gamma_{\rm eff}/T$. Again this effective energy is quite distinct from the other versions of the vacuum energy discussed above. These at least four subtly different versions of the vacuum energy has unfortunate consequences insofar as many papers in the literature do not take the appropriate care to make these distinctions.

6 Comparison with standard results.

In this section we shall make connections between our formalism and some of the explicit calculations already available in the literature. While agreeing with many of those calculations, we report some subtle differences when considering solid conductors and closely related aspect of bag models.

6.1 Parallel Plates:

Consider a massless scalar field satisfying Dirichlet boundary conditions confined between two parallel plates of surface area S held a distance L apart. The three dimensional heat kernel is easily seen to be $K_3(x,x,t) = K_1(x,x,t)/(4\pi\mu^{-2}t)$, which upon integration over the volume between the plates yields

$$K_3(t) = \frac{\mu^2 S}{4\pi t} \cdot K_1(t). \tag{6.1}$$

But $K_1(t)$ is explicitly known in terms of the eigenvalues of the reduced one dimensional problem $\lambda_n = n^2/L^2$. Evaluation of the three-dimensional zeta function proceeds in a straightforward manner

$$\begin{aligned} \zeta_3(s) &= \frac{\mu^2 S}{\Gamma(s)} \int_0^\infty dt \cdot t^{s-1} \cdot \frac{1}{4\pi t} \cdot \sum_0^\infty \exp(-tn^2/\mu^2 L^2) \\ &= \frac{\mu^2 S}{4\pi} \cdot (\mu L)^{2s-2} \cdot \frac{1}{s-1} \cdot \zeta_R(2s-2). \end{aligned} \tag{6.2}$$

Here ζ_R is the ordinary Riemann zeta function. In taking the limit $s \to -\frac{1}{2}$ one does *not* encounter a pole, so the Casimir energy is simply

$$E_{\text{Casimir}}(L,S) = -\frac{1}{12\pi} \cdot \frac{1}{2} \cdot \frac{\hbar c 2\pi S}{L^3} \cdot \zeta_R(-3). \tag{6.3}$$

It is a standard zeta function result that $z_R(-3) = \frac{1}{120}$, which finally leads to the well-known standard result [2]. This calculation, though trivial, has expressed some important ideas. The absence of a pole in the $s \to -\frac{1}{2}$ limit can be traced back to the fact that the plates are *flat*. Because the plates are flat the second fundamental form vanishes ($\gamma = 0$), consequently $b_2 = 0$, and finally $C_2 = 0$. This has the additional interesting effect that the flat–plate Casimir energy is insensitive to the thickness of the plates.

6.2 Cylindrical Shells and Spherical Shells:

For cylindrical and spherical shells $b_2(outside) = -b_2(inside)$, thus $C_2(net) = 0$, and we may safely use simple dimensional arguments to deduce

$$\begin{aligned} E_{\text{cylinder}} &\propto \frac{L}{R^2}, \\ E_{\text{sphere}} &\propto \frac{1}{R}. \end{aligned} \tag{6.4}$$

Note that these dimensional analysis results are merely *assumed*, not *proved*, in the standard analyses of these problems [11, 12, 13]. It was by no means clear, in the days before conformal anomalies became a well understood part of field theory, that there is anything to prove in deriving (6.4). Fortunately, the naive result works for thin shells, but as we shall soon see, leads to confusion when applied to solid conductors. It should be emphasized that the cancellation of b_2 between the inner and outer faces is the underlying cause of the "delicate cancellations between internal and external modes" noted by many authors [2].

6.3 Solid Cylinders and Solid Spheres:

For solid conductors the "delicate cancellations" alluded to previously no longer occur. Indeed it is easy to see that

$$\begin{aligned} C_2(\mu, L, R)_{\text{solid cylinder}} &\propto \frac{L}{\mu R^2} \\ C_2(\mu, R)_{\text{solid sphere}} &\propto \frac{1}{\mu R} \end{aligned} \tag{6.5}$$

Consequently the Casimir energy possesses a logarithmic dependence on the radius of these systems. The Casimir energy also depends on the normalization scale. In regularization schemes such as proper-time regularization or a mode-sum cut-off the pole associated with C_2 manifests itself as an divergent term that depends logarithmically on the cut–off [8, 22]. Such logarithmic divergences have in fact been encountered in some explicit calculations [15]. Any term of the form $\ln(R\Lambda)$ may be re–cast as $\ln(R\mu) + \ln(\Lambda/\mu)$; the $\ln(\lambda/\mu)$ may then be absorbed into a renormalization of some appropriate piece of the energy, but a term of form $\ln(R\mu)$ always remains in the renormalized energy (with the μ dependence compensated by some other term).

6.4 Membranes:

We now turn to a very different physical system, that of a membrane. Membrane theory, as a generalization of string theory, has enjoyed some recent popularity [23, 24, 25]. Consider a physical field that is constrained to propagate on the surface of a closed static membrane. As far as the Casimir effect is concerned, this is equivalent to considering a 2+1 dimensional spacetime. The analysis of this paper continue to hold, with the sole exception that the pole of the zeta function at $s = -\frac{1}{2}$ is now proportional to $C_{\frac{3}{2}}$. Since $a_{\frac{3}{2}}$ is automatically zero, this means that a closed (*i.e.*, boundary-less) membrane automatically has $C_2 = 0$. Consequently, zeta-function calculations of the Casimir effect on any closed membrane are always *guaranteed* to not encounter a pole. This explains the otherwise quite miraculous cancellation of poles encountered in explicit computations performed by Sawhill [26]. Open membranes, on the other hand, may possess poles in the zeta function as $s \to -\frac{1}{2}$. The residues of such poles are, however, tightly constrained.

These above comments are also relevant to other physical systems: consider any field theory that gives rise to domain walls. It is very easy in such theories to arrange for massless particles to become trapped on the domain wall. This suggests the interesting possibility that for suitable choices of parameters and particle content, one may use the

Casimir energy to stabilize small spherical domain walls against collapse. Preliminary calculations seem encouraging.

At a more general level, the comments of this section imply that the behaviour of the Casimir effect depends crucially on whether the total number of spacetime dimensions is even or odd. This will be discussed more fully when we make some comments on Kaluza–Klein models.

6.5 Bag Models:

Another physical situation where the Casimir effect has been of great importance is in the bag models of QCD [14, 15, 16]. As a first approximation, the idea is to treat quarks and gluons as massless particles confined to the interior of some (3+1)-dimensional bounded region of spacetime called the bag. The free quark-gluon Lagrangian is then augmented by a "bag Lagrangian" responsible for confining the quarks and gluons.

The points we wish to make are twofold. First, generically $C_2 \neq 0$ for these bag models (barring fortuitous cancellations between the effects of quark and gluon boundary conditions). In cut-off regularizations of the mode sum this would correspond to the appearance of a logarithmic divergence, as has indeed been reported by Milton [15]. In our zeta-function approach the Casimir energy of the bag includes a $\ln(\mu R)/R$ term. Since we are working with a model that is supposed to be an approximation to QCD, and since we have argued that the Casimir energy is related to one–loop effects, it is natural for the bag models to expect μ to be related to $\Lambda_{\rm QCD}$ ($\hbar c\mu \approx \Lambda_{\rm QCD}$).

The second point we wish to make concerns the (renormalized) bag energy. The *total* bag energy depends on the zero-loop bag energy, *plus* the Casimir energy (*i.e.*, one–loop physics), *plus* higher loop effects (presumably small). One of the great virtues of the zeta function approach is that it yields an effective way of calculating the Casimir energy without requiring a detailed analysis of the renormalization properties of the bag energy. To extract the structure of the (renormalizable) Bag Lagrangian the proper time cutoff is more appropriate. In the proper time formalism

$$E_{\rm reg}(\epsilon) = \frac{\hbar c\mu}{\sqrt{4\pi}} \int_\epsilon^\infty dt\; t^{-3/2}\; {\rm tr}'(e^{-tD_3\mu^{-2}}). \tag{6.6}$$

The resulting divergences in the Casimir energy are described by

$$E_{\rm reg}(\epsilon) \sim \frac{C_0}{\epsilon^2} + \frac{C_{\frac{1}{2}}}{\epsilon^{3/2}} + \frac{C_1}{\epsilon} + \frac{C_{3/2}}{\epsilon^{1/2}} + C_2 \ln\epsilon + \text{finite pieces}. \tag{6.7}$$

Thus the requirement of renormalizability of the energy implies that the zero-loop bag energy contains (at a minimum) the following terms

$$E_0 = \int_\Omega \sum_0^2 g_n\; a_n + \int_{\partial\Omega} \sum_0^2 h_n\; b_n. \tag{6.8}$$

In flat spacetime this simplifies considerably

$$E_0 = p\cdot V + \sigma\cdot S + \int_{\partial\Omega} \left(h_1\; b_1 + h_{3/2}\; b_{3/2} + h_2 b_2\right). \tag{6.9}$$

Here p is the bag pressure, σ is its surface tension, the parameters h_1, $h_{3/2}$ and h_2 do not appear to have standard names.

If we approximate the bag as spherical, we can easily extract the dependence of these terms on bag radius

$$h_1 \int b_1 = FR, \tag{6.10}$$

$$h_{3/2} \int b_{3/2} = k, \tag{6.11}$$

$$h_2 \int b_2 = h/R. \tag{6.12}$$

Which allows us to write the zero-loop renormalized bag energy as

$$E_0 = p \cdot V + \sigma \cdot S + FR + k + h/R \tag{6.13}$$

It is to be emphasized that these parameters are to be determined by experiment; they cannot be calculated within the confines of the bag model. In principle they would be calculable from the full theory of QCD. Adding the one-loop effects (Casimir energy) and defining $Z = h + \epsilon_0$ finally yields

$$E_{\rm bag} = p \cdot V + \sigma \cdot S + FR + k + Z/R - \epsilon_1 \ln(\mu R)/R. \tag{6.14}$$

The only one of these parameters that is calculable using Casimir energy techniques is ϵ_1. In particular, the parameter Z is not calculable, but rather is to be experimentally determined. The terms involving p and σ are standard. The term involving F has previously been discussed in the work of Milton [15]. The offset term k has (to the best of our knowledge) not previously been discussed. We note in passing that the offset piece k contains a purely topological piece proportional to the Euler characteristic of the bag.

7 Applications to Kaluza–Klein theories.

In this section we seek to extract some information concerning the one-loop contributions to the effective four-dimensional cosmological and Newton constants within the framework of Kaluza–Klein theory. Calculations along these lines have been carried out, for some specific simple choices of the internal geometry, in references [27, 28, 29, 30]. We shall proceed with a bare minimum of assumptions. Consider a $4+d$ dimensional universe with d compactified dimensions, $\mathcal{M}_{4+d} = \mathcal{M}_4 \otimes \Omega$. Assume the theory to possess multidimensional cosmological (Λ) and Newton (G) constants. That is

$$S_{4+d} = \Lambda \cdot \int \sqrt{g_{4+d}}\, d^{4+d}x + G^{-1} \cdot \int R_{4+d} \sqrt{g_{4+d}}\, d^{4+d}x + \cdots \tag{7.1}$$

Using the product decomposition of spacetime one infers $R_{4+d} = R_4 + R_d$, so that for the *tree–level* four dimensional effective Cosmological and Newton constants one deduces:

$$\begin{aligned} \Lambda_{\text{eff}} &= \Lambda \cdot \text{vol}(\Omega) + G^{-1} \cdot \int_\Omega \sqrt{g_d}\, R_d, \\ G^{-1}_{\text{eff}} &= G^{-1} \cdot \text{vol}(\Omega). \end{aligned} \tag{7.2}$$

To evaluate the one–loop contributions to Λ_{eff} and G_{eff} one uses the product decomposition of spacetime to deduce a product decomposition for the diagonal part of the heat kernel

$$K(t) = K_4(t) \cdot K_d(t). \tag{7.3}$$

The asymptotic expansion of the four-dimensional heat kernel may now be used to obtain an expansion for the zeta function

$$\zeta_{4+d}(s) = \sum_0^\infty \frac{C_n(g_4)}{(4\pi)^2} \cdot \frac{\Gamma(s-2+n)}{\Gamma(s)} \cdot \zeta_d(s-2+n). \tag{7.4}$$

This expansion is a formal one in the "size" of the compactified dimensions. To justify the above expansion consider a "long wavelength" approximation implemented by rescaling the external dimensions: $g_{4+d,\kappa} = g_{4,\kappa} \oplus g_d = (\kappa^2 g_4) \oplus g_d$. In this situation the heat kernel enjoys the property that $K_{4+d,\kappa}(t) = K_{4,\kappa}(t) \cdot K_d(t) = K_4(\kappa^{-2}t) \cdot K_d(t)$. Thus the limit $\kappa \to \infty$ allows one to employ the asymptotic expansion of the heat kernel to obtain an asymptotic expansion for the multi-dimensional zeta function

$$\zeta_{4+d,\kappa}(s) = \sum_0^N \frac{C_n(g_4)}{(4\pi)^2}\, \kappa^{4-2n}\, \frac{\Gamma(s-2+n)}{\Gamma(s)}\, \zeta_d(s-2+n) + o(\kappa^{4-2n}). \tag{7.5}$$

By abuse of notation we have rewritten this asymptotic expansion as the physically more reasonable (7.4). Now, recall that $C_0 = \mu^4 \int \sqrt{g_4}\, d^4x$ and $C_1 = k \cdot \int R_4 \sqrt{g_4}\, d^4x$, ($k$ is a constant depending on the statistics and spins of the elementary particles present in the theory). This may be used to extract the one-loop corrections to Λ_{eff} and G_{eff}

$$\begin{aligned} \Lambda_{\text{eff}} &= \Lambda \cdot \text{vol}(\Omega) + G^{-1} \cdot \int_\Omega \sqrt{g} R_d - \frac{\mu^4}{2(4\pi)^2} \left\{ \tfrac{1}{2}\zeta'_d(-2) - \tfrac{3}{4}\zeta_d(-2) \right\}. \\ G^{-1}_{\text{eff}} &= G^{-1} \cdot \text{vol}(\Omega) - k \cdot \frac{\mu^2}{2(4\pi)^2} \left\{ \zeta'_d(-1) - \zeta_d(-1) \right\}. \end{aligned} \tag{7.6}$$

Observe that the zeta functions appearing in the above are guaranteed to be analytic at all non-positive integers, so that these expressions are finite as they stand. Further, the value of the zeta function at non-positive integers is (in principle) known; for example $\zeta_d(-2) = 2C_{2+(d/2)}/(4\pi)^{d/2}$, and $\zeta_d(-1) = -C_{1+(d/2)}/(4\pi)^{d/2}$.

Without evaluating equation (7.6) in full detail, we may profitably inquire as to the dependence of Λ_{eff} and G_{eff} on the "radius" of the internal dimensions. The major point to be made is that the case of an odd number of internal dimensions behaves in a qualitatively different manner form an even number of internal dimensions. Introducing appropriate constants permits us to write

$$\begin{aligned} \Lambda_{\text{eff}} &= a r^d + b r^{d-2} + \{\epsilon_0 - \epsilon_1 \ln(\mu r)\}\, r^{-4}, \\ G_{\text{eff}}^{-1} &= a' r^d + \{\epsilon_0' - \epsilon_1' \ln(\mu r)\}\, r^{-2}. \end{aligned} \tag{7.7}$$

The dimensionless constants ϵ_1 and ϵ_1' are proportional to $\zeta_d(-2)$ and $\zeta_d(-1)$ respectively. In any odd number of dimensions (provided the internal manifold has no boundary) these are guaranteed to vanish. Thus in an odd number of dimensions, Λ_{eff} and G_{eff} have a simple power-law dependence on the radius of the compact dimensions. This breaks down however, for any even number of dimensions where one observes the appearance of logarithmic dependences on the radius. We expect these logarithms to have significant effects, but shall postpone further comments to another paper.

8 Conclusion

The Casimir energy is a very useful concept, it may be viewed as the "zero point energy" of the vacuum, and, from a slightly different viewpoint, is also intimately related to one–loop physics in the form of the one–loop Effective energy. In this paper we have exhibited a unified framework that allows us to regularize and renormalize the zero point mode sum in a way that is extremely general. Our definition yields a well behaved finite quantity in many interesting physical situations: *e.g.* in the presence of a background gravitational field, with massive or massless particles, and in the presence or absence of boundaries of the space–time manifold. It is hoped that with this framework in place, it will be possible to perform extensive explicit calculations.

Note added in proof

After submittal of this paper we were made aware of additional work by the Manchester group [32, 33, 34]. For additional work on the relevance of the Casimir effect to the stability of Kaluza–Klein models see references [35, 36, 37, 38]. In addition we wish to thank Emil Mottola for useful discussions.

Appendix A

The Seeley–de Witt coefficients.

The Seeley–de Witt coefficients $a_n(x)$ are independent of the applied boundary conditions, but the coefficients do depend on the spin of the field in question.

$$a_0(x) = 1. \tag{A.1}$$

$$a_1(x) = k \cdot R. \tag{A.2}$$

$$a_2(x) = {}_A(Weyl)^2 + {}_B[(Ricci)^2 - \tfrac{1}{3}R^2] + {}_C\nabla^2 R + {}_D R^2. \tag{A.3}$$

The boundary coefficients $b_n(y)$ depend on the nature of the boundary conditions imposed. For Dirichlet or Neumann boundary conditions

$$b_0(y) = 0. \tag{A.4}$$

$$b_{1/2}(y) = \mp\frac{\sqrt{\pi}}{2}. \tag{A.5}$$

$$b_1(y) = \frac{1}{3}\mathrm{tr}\gamma. \tag{A.6}$$

$$b_{3/2}(y) = a(\mathrm{tr}\gamma)^2 + b\mathrm{tr}(\gamma^2) + cR \tag{A.7}$$

$$b_2(y) = \tilde{a}(\mathrm{tr}\gamma)^3 + \tilde{b}(\mathrm{tr}\gamma^2)(\mathrm{tr}\gamma) + \tilde{c}(\mathrm{tr}\gamma^3) + \tilde{d}(\mathrm{tr}\gamma)R + \tilde{e}\gamma_{ij}R^{ij} + \tilde{f}\nabla^2(\mathrm{tr}\gamma). \tag{A.8}$$

Where γ is the second fundamental form of $\partial\Omega$, the boundary of Ω. The curvatures appearing in b_n are intrinsic curvatures computed from the induced metric on the boundary. If one adopts Robin boundary conditions $\frac{\partial\phi}{\partial\eta} + \psi(y)\phi(y) = 0$, then additional terms appear in b_n for $n \geq 1$. Since ψ has the same dimensions as γ, these extra terms are of the type exhibited above with $\gamma \mapsto \psi$.

Appendix B

Gamma Function Identities.

We collect some useful Gamma Function identities, see for instance [31]. Take $n \in \{0, 1, 2, \cdots\}$:

$$Res[\Gamma(-n+\epsilon)] = \frac{(-)^n}{n!}. \tag{B.1}$$

$$PP[\Gamma(-n+\epsilon)] = (-)^n \cdot \frac{\psi(n+1)}{\Gamma(n+1)} = \psi(n+1) \cdot Res[\Gamma(-n+\epsilon)]. \tag{B.2}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{B.3}$$

$$\Gamma(-\tfrac{1}{2}) = -\sqrt{4\pi}. \tag{B.4}$$

$$\psi(1) = -\gamma. \tag{B.5}$$

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}. \tag{B.6}$$

$$\psi(\tfrac{1}{2}) = -\gamma - 2\ln 2. \tag{B.7}$$

$$\psi(\tfrac{1}{2} \pm n) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n} \tfrac{1}{(2k-1)}. \tag{B.8}$$

$$\psi(-\tfrac{1}{2}) = -\gamma - 2\ln 2 + 2. \tag{B.9}$$

References

[1] H.G.B. Casimir, Proc. Kon. Nederl. Akad. Wetenschap 51 (1948) 793.

[2] G. Plunien, B. Muller, and W. Greiner, Physics Reports 134 (1986) 87.

[3] R. Utiyama and B.S. De Witt, J. Math. Phys. 3 (1962) 608.

[4] J.S. Dowker and R. Critchley, Phys. Rev. D13 (1976) 3224.

[5] L.H. Ford, Phys. Rev. D11 (1975) 3370, D14 (1976) 3304.

[6] R.T. Hammond and A. McKinley, Phys. Rev. D20 (1979) 3047.

[7] D. Deutsch and P. Candelas, Phys. Rev. D20 (1979) 3063.

[8] B.S. Kay, Phys. Rev D20 (1979) 3052.

[9] J.S. Dowker, Phys. Rev. D36 (1987) 3095.

[10] M. Fierz, Helv. Phys. Acta 33 (1960) 855.

[11] T.H. Boyer, Phys. Rev. 174 (1968) 1764;
Ann. Phys. (N.Y.), 56 (1970) 474.

[12] K.A. Milton, Ann. Phys. (N.Y.) 127 (1980) 49.

[13] L.L.Jr. deRaad and K.A. Milton, Ann. Phys. (N.Y.) 136 (1981) 229.

[14] C.M. Bender and P. Hays, Phys. Rev. D14 (1976) 2622.

[15] K.A. Milton, Phys. Rev. D22 (1980) 1441, 1444; D27 (1983) 439;
Ann. Phys. (N.Y.) 150 (1983) 432.

[16] J. Baacke and Y. Igarashi, Phys. Rev. D27 (1983) 460.

[17] S.W. Hawking, Commun. Math. Phys. 55 (1977) 133.

[18] B.S. De Witt, Physics Reports 19 (1975) 295.

[19] N.D. Birrell and P.C.W. Davies, *"Quantum fields in curved space"*, (Cambridge University Press, Cambridge, 1982).

[20] P. Ramond, *"Field Theory: A Modern Primer"*, (Benjamin/Cummings, Reading, MA, 1981).

[21] C. Itzykson and J. Zuber, *"Quantum Field Theory"*, (McGraw-Hill, New York,1980).

[22] O. Alvarez, Nucl. Phys B216 (1983) 125.

[23] I. Bars, C.N. Pope, and E. Szegin, Phys. Lett. 198B (1987) 455;
I. Bars, USC–87/HEP–06.

[24] K. Kikkawa and M. Yamasaki, Prog. Theor. Phys. 76 (1986) 1379.

[25] K. Fujikawa and J. Kubo, Phys. Lett. B199 (1987) 75.

[26] B. Sawhill, Phys. Lett. B202 (1988) 505.

[27] T. Applequist and A. Chodos, Phys. Rev. 28D (1983) 772.

[28] A. Chodos and E. Meyers, Phys.Rev. 31D (1985) 3064.

[29] G. Gilbert and B. McClain, Nucl. Phys. B244 (1984) 173.

[30] G. Gilbert, B. McClain, and M.A. Rubin, Phys. Lett. 142B (1984) 28.

[31] M. Abramowitz and I.A. Stegun, *"Handbook of Mathematical Functions"*, (Dover, New York, 1970).

[32] Dowker, J.S., and Kennedy, G., J. Phys. A11 (1978) 895.

[33] J.S. Dowker and R. Banach, J. Phys. A11 (1978) 2255.

[34] J.S. Dowker, Class. Quantum Grav. 1 (1984) 359.

[35] C.R. Ordóñez and M. Rubin, Nucl. Phys. B260 (1985) 456.

[36] C.A. Lütken and C.R. Ordóñez, Class. Quantum Grav. 4 (1987) 1525; 1543.

[37] D. Birmingham and S. Sen, Ann. Phys. 172 (1986) 451.

[38] D. Birmingham, R. Kantowski, and K.A. Milton, Oklahoma University preprint 88-0098, 1988.