Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Physics Letters A ●●● (●●●●) ●●●–●●●
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Physical wavelets: Lorentz covariant, singularity-free, finite energy, zero action, localized solutions to the wave equation

Matt Visser

School of Mathematical and Computing Sciences, Victoria University of Wellington, Wellington, New Zealand

Received 10 June 2003; received in revised form 23 June 2003; accepted 27 June 2003

Communicated by P.R. Holland

Abstract

I exhibit a particularly simple "physical wavelet"—it is a Lorentz covariant classical field configuration that lives in physical Minkowski space. The field is everywhere finite and nonsingular, and has quadratic falloff in both space and time. The total energy is finite, the total action is zero, and the field configuration solves the wave equation.

Keywords: Wavelet; Localized wave; Soliton; Instanton; Sphaleron

1. Motivation

While the particle physics community has for some time made extensive use of extended field configurations such as solitons, instantons, and sphalerons, no direct use has yet been made of the quite extensive literature on "localized wave" configurations developed by the engineering, optics, and mathematics communities. (For selected references see [1–6].) These localized waves are classical solutions of the wave equation that are partially localized in space or time, this localization generally coming at a cost such as infinite total energy and/or instability (leading to dispersion or diffraction). The catalogue of known localized waves is large and growing,[1] but most of the known examples are not in a form that would be easy to apply to particle physics problems.

In this Letter I will exhibit a particularly simple "physical wavelet" that is more promising from a particle physics standpoint. It satisfies the properties that:

- It is a localized wave that solves the wave equation.
- It is a Lorentz covariant classical field configuration that lives in physical Minkowski space.
- The field is everywhere finite and nonsingular, and has quadratic falloff in both space and time.
- The total energy is finite, depending on the peak field and the width of the pulse.

E-mail address: matt.visser@vuw.ac.nz (M. Visser).

[1] Field configurations similar to the one considered in this Letter may variously be encountered under names such as: "localized waves", "focus wave modes", "pulses", "X-waves", "limited diffraction beams", "wavelets", and "physical wavelets".

0375-9601/$ – see front matter © 2003 Published by Elsevier B.V.
doi:10.1016/S0375-9601(03)01051-X

S0375-9601(03)01051-X/SCO AID:12659 Vol.●●●(●●●)
ELSGMLTM(PLA):m3 v 1.164 Prn:18/07/2003; 12:09 pla12659 P.2 (1-6)
by:Vita p. 2

- The total action is zero.

These physical wavelets can be constructed for both complex and real scalar fields. Extending these idea to the Maxwell and Yang–Mills fields is straightforward.

2. Complex scalar field

Let $\eta_{ab} = \text{diag}[+1, -1, -1, -1]$ be the Minkowski metric [particle physics signature], let x_0 be an arbitrary 4-vector, and let ζ^a be arbitrary timelike 4-vector, then

$$\phi(x) = -\frac{\phi_0(\eta_{ab}\zeta^a\zeta^b)}{\eta_{ab}[x^a - x_0^a - i\zeta^a][x^b - x_0^b - i\zeta^b]} \tag{1}$$

is a Lorentz covariant, finite energy, zero-action solution of the d'Alembertian wave equation $\Delta\phi = 0$. The "center" of the pulse is at x_0 and its "width" is $a = \sqrt{\eta_{ab}\zeta^a\zeta^b}$. The field is everywhere finite and in fact $|\phi(x)| \leqslant |\phi_0|$. To see this, use the fact that ζ is timelike. Then, using the manifest Lorentz covariance of the field configuration, we can without loss of generality first translate $x_0 \to 0$, and then go into the zero-momentum frame where

$$\zeta^a = (a, 0, 0, 0). \tag{2}$$

Then the field configuration is

$$\phi(x) = -\frac{\phi_0 a^2}{[t - ia]^2 - x^2 - y^2 - z^2} = \frac{\phi_0 a^2}{r^2 - t^2 + a^2 + 2iat}. \tag{3}$$

Once written in this form it is a straightforward exercise to verify that the wave equation is satisfied. To see that the field is everywhere bounded note

$$\begin{aligned} |\phi|^2 &= \frac{|\phi_0|^2 a^4}{(r^2 - t^2 + a^2)^2 + 4a^2t^2} = \frac{|\phi_0|^2 a^4}{(r^2 + t^2 + a^2)^2 - 4r^2t^2} \\ &\leqslant \frac{|\phi_0|^2 a^4}{(r^2 + t^2 + a^2)^2 - (r^2 + t^2)^2} = \frac{|\phi_0|^2 a^4}{a^4 + 2a^2(r^2 + t^2)} \leqslant |\phi_0|^2. \end{aligned} \tag{4}$$

From the penultimate inequality we also derive

$$|\phi|^2 \leqslant \frac{1}{2}|\phi_0|^2 \frac{a^2}{r^2 + t^2}, \tag{5}$$

demonstrating the promised quadratic falloff in both space and time. Indeed for fixed t the magnitude of the field is maximized when $r^2 = \max\{t^2 - a^2, 0\}$, showing that the configuration disperses to spatial infinity at both $t \to \pm\infty$.

To calculate the 4-momentum, we remind the reader that the stress–energy tensor for a massless complex scalar is

$$T_{ab} = \frac{1}{2}\left[\phi_a^*\phi_b + \phi_a\phi_b^*\right] - \frac{1}{2}\eta_{ab}|\nabla\phi|^2. \tag{6}$$

Then

$$\nabla_a T^{ab} \equiv \frac{1}{2}[\Delta\phi^*\nabla_b\phi + \Delta\phi\nabla_b\phi^*], \tag{7}$$

which vanishes by the equations of motion. But this means that

$$P^\mu = \oint T^{ab}\,\mathrm{d}\Sigma_b \tag{8}$$

S0375-9601(03)01051-X/SCO AID:12659 Vol.●●●(●●●) ELSGMLTM(PLA):m3 v 1.164 Prn:18/07/2003; 12:09 pla12659 P.3 (1-6) by:Vita p. 3

is a conserved quantity, the 4-momentum of the configuration, which is independent of the particular spacelike hypersurface Σ chosen to do the integration. By simple dimensional analysis $P^a = C|\phi_0|^2\zeta^a$, where C is a dimensionless number to be calculated. (Note that ζ^a has the dimensions of a position vector—a distance.) The energy density is

$$\rho = \frac{1}{2}\left[|\partial_t\phi|^2 + |\partial_r\phi|^2\right], \tag{9}$$

and in the zero-momentum frame is easily calculated to be

$$\rho = \frac{2a^4|\phi_0|^2(r^2+t^2+a^2)}{(r^2-t^2+a^2+2iat)^2(r^2-t^2+a^2-2iat)^2}. \tag{10}$$

For arbitrary t this integrates to

$$\mathcal{E} = \oint \mathrm{d}^3r\,\rho = \int_0^\infty 4\pi r^2\rho = \frac{1}{2}\pi^2|\phi_0|^2 a. \tag{11}$$

(This is independent of t as it should be.) This is the invariant mass of the field configuration. By spherical symmetry, the total momentum is zero. Thus, for any timelike ζ^a

$$P^a = \frac{1}{2}\pi^2|\phi_0|^2\zeta^a. \tag{12}$$

Furthermore the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left[|\partial_t\phi|^2 - |\partial_r\phi|^2\right], \tag{13}$$

which evaluates (in the zero-momentum frame) to

$$\mathcal{L} = \frac{2a^4|\phi_0|^2(t^2+a^2-r^2)}{(r^2-t^2+a^2+2iat)^2(r^2-t^2+a^2-2iat)^2}. \tag{14}$$

It is easy to check that

$$\oint \mathrm{d}^4x\,\mathcal{L} = 0, \tag{15}$$

so that the configuration is zero action.

In summary, what we have is a Lorentz covariant, singularity-free, finite energy, zero action, exact localized solution to the d'Alembertian equation. In many ways this configuration has more right to be called an "instanton" than do the instantons of QFT; those instantons live in Euclidean signature. This field configuration lives in real physical time.

Now the fact that there are finite energy solutions to the wave equation is not a surprise; that these finite energy solutions can coalesce, bounce, and disperse without producing field singularities is more interesting. One way of guessing that the field configuration in Eq. (1) is worth investigating is the following: it is easy to convince oneself that in 4 Euclidean dimensions the solution to Laplace's equation with a delta function source at the origin (the Green function) is

$$\phi(x) \propto \frac{1}{x^2+y^2+z^2+t^2}. \tag{16}$$

Thus in $(3+1)$ Lorentzian dimensions the (singular) solution to the wave equation with a delta function source at the origin is

$$\phi(x) \propto \frac{1}{x^2+y^2+z^2-t^2}. \tag{17}$$

S0375-9601(03)01051-X/SCO AID:12659 Vol.●●●(●●●)
ELSGMLTM(PLA):m3 v 1.164 Prn:18/07/2003; 12:09 pla12659 P.4 (1-6)
by:Vita p. 4

If the source is now moved to a real position x_0^a we have

$$\phi(x) \propto \frac{1}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-(t-t_0)^2} \tag{18}$$

which is still a singular field configuration. Finally, move the source away from physical Minkowski space to the complex position $x_0^a - i\zeta^a$, then

$$\phi(x) \propto \frac{1}{(x-x_0+i\zeta^1)^2+(y-y_0+i\zeta^2)^2+(z-z_0+i\zeta^3)^2-(t-t_0+i\zeta^0)^2}, \tag{19}$$

which is essentially Eq. (1) above. This style of approach has been particularly advocated by Kaiser [5]. This approach is also reminiscent of the "complex tubes" used for other purposes in axiomatic quantum field theory [7]. Further afield, in spirit (though not in detail) it has limited parallels with Feynman's $i\epsilon$ prescription for quantum Green functions. As we have just seen, if ζ is timelike the resulting field configuration is singularity free. However, for null and spacelike ζ^a, while the field is still a solution of the wave equation, the field is not bounded. Because of the singularities the energy and action integrals then diverge. Details are deferred for now and will be presented in Sections 4 and 5.

One should also note that in the optics and engineering literature the most commonly used notations are not manifestly Lorentz covariant. Thus it is common to see expressions such as (see, for instance, [6])

$$\phi \propto \frac{1}{x^2+y^2+[b_1-i(z+t)][b_2+i(z-t)]} \tag{20}$$

whose Lorentz transformation properties are less than obvious—in fact this field configuration is equivalent to Eq. (1) with the identification

$$\zeta^a = -\left(\frac{b_1+b_2}{2}, 0, 0, \frac{b_1-b_2}{2}\right), \qquad \|\zeta\| = b_1 b_2. \tag{21}$$

3. Real scalar field

By taking real and imaginary parts of the complex solution above we can write down two solutions for the real scalar field:

$$\phi_1 = \frac{\phi_0(\eta_{ab}\zeta^a\zeta^b)\{\eta_{ab}[x^a-x_0^a][x^b-x_0^b]-\eta_{ab}\zeta^a\zeta^b\}}{(\eta_{ab}[x^a-x_0^a][x^b-x_0^b]-\eta_{ab}\zeta^a\zeta^b)^2+4(\eta_{ab}[x^a-x_0^a]\zeta^b)^2}; \tag{22}$$

$$\phi_2 = \frac{2\phi_0(\eta_{ab}\zeta^a\zeta^b)\{\eta_{ab}[x^a-x_0^a]\zeta^b\}}{(\eta_{ab}[x^a-x_0^a][x^b-x_0^b]-\eta_{ab}\zeta^a\zeta^b)^2+4(\eta_{ab}[x^a-x_0^a]\zeta^b)^2}. \tag{23}$$

We can without loss of generality translate $x_0 \to 0$ and go to the zero-momentum frame $\zeta^a = (a, 0, 0, 0)$, then

$$\phi_1 = \frac{\phi_0 a^2\{t^2-r^2-a^2\}}{(t^2-r^2-a^2)^2+4a^2t^2}, \qquad \phi_2 = \frac{\phi_0 a^2 2at}{(t^2-r^2-a^2)^2+4a^2t^2}. \tag{24}$$

The stress–energy tensor and divergence for a real scalar field simplify

$$T_{ab} = \phi_a\phi_b - \frac{1}{2}\eta_{ab}|\nabla\phi|^2, \qquad \nabla_a T^{ab} \equiv \Delta\phi\nabla_b\phi, \tag{25}$$

with the divergence vanishing by the equations of motion. The calculation for the energy–momentum 4-vector now yields:

$$P_1^a = \frac{1}{4}\pi^2|\phi_0|^2\zeta^a = P_2^a. \tag{26}$$

S0375-9601(03)01051-X/SCO AID:12659 Vol.●●●(●●●)
ELSGMLTM(PLA):m3 v 1.164 Prn:18/07/2003; 12:09
pla12659
P.5 (1-6)
by:Vita p. 5

The action integral for both of these field configurations is still zero.

4. Null ζ

Let us now return to the original scalar complex wavelet. Suppose the 4-vector ζ is null. Then because the numerator vanishes identically the original definition above gives $\phi \equiv 0$. We should at a minimum change our field definition to read

$$\phi(x) = -\frac{\psi_0}{\eta_{ab}[x^a - x_0^a - i\zeta^a][x^b - x_0^b - i\zeta^b]}. \tag{27}$$

Then without loss of generality we go into the frame

$$\zeta^a = (a, 0, 0, a), \tag{28}$$

and then

$$\phi(x) = -\frac{\psi_0}{[t - ia]^2 - x^2 - y^2 - [z - ia]^2} = \frac{\psi_0}{r^2 - t^2 + 2ia(t - z)}. \tag{29}$$

Note

$$\left|\phi(x)\right| = \frac{\psi_0}{\sqrt{(r^2 - t^2)^2 + 4a^2(t - z)^2}}. \tag{30}$$

The denominator now vanishes when $z - t$ and $x = y = 0$, so that the field is divergent on the beam axis. Suppose we write $R = \sqrt{x^2 + y^2}$ then

$$\phi(x) = \frac{\psi_0}{R^2 + z^2 - t^2 + 2ia(t - z)}. \tag{31}$$

So we see that the field drops off as $1/R^2$ as we move away from the beam axis. (And more critically, the field blows up as $1/R^2$ as we approach the beam axis.) Attempts at calculating the energy and action now lead to divergent integrals. In other words, despite the fact that it still solves the wave equation, for null ζ this is not a particularly useful field configuration.

5. Spacelike ζ

For the complex scalar wavelet, suppose the 4-vector ζ is spacelike. Then without loss of generality we go into the infinite velocity frame where

$$\zeta^a = (0, 0, 0, a), \tag{32}$$

and then

$$\phi(x) = \frac{\phi_0 a^2}{t^2 - x^2 - y^2 - [z - ia]^2} = -\frac{\phi_0 a^2}{r^2 - t^2 - a^2 - 2iaz}. \tag{33}$$

Note

$$\left|\phi(x)\right| = \frac{\phi_0 a^2}{\sqrt{(r^2 - t^2 - a^2)^2 + 4a^2 z^2}}. \tag{34}$$

The denominator now vanishes when $z = 0$ and $x^2 + y^2 = a^2 + t^2$. That is, the field is divergent on a time-dependent circle orthogonal to the beam axis. There is a short distance singularity as one approaches this circle, and the energy and action integrals diverge. Despite the fact that it still solves the wave equation, for spacelike ζ this is not a useful field configuration.

6. Discussion

A similar construction can be performed for the Maxwell field. However, because the optics and engineering literature generally does not use manifestly Lorentz covariant notation, it can be very time consuming to calculate the 4-momentum of a specific pulse. Indeed only very recently [6] has Lekner provided specific and explicit computations of both energy and momentum (as well as the angular momentum) for a pulse similar to that considered above. As expected (once one has the covariant perspective advocated in this article) for timelike ζ the momentum is less than the energy, indicating the existence of a zero-momentum frame for the pulses of this type [6]. Similarly, constructing a Yang–Mills wavelet is straightforward: let $\mathbf{\Lambda}$ be any constant matrix in the center of the gauge group and set $\mathbf{A}^a = \mathbf{\Lambda} A^a$, where $A^a(x)$ is a pulse-like solution of the Maxwell field.

The physical wavelet discussed in this Letter is important because it represents a very simple extended field configuration of a type not commonly encountered. The wavelet is neither a soliton, nor an instanton, nor a sphaleron; though it shares properties with all three of these extended objects:

- Like the soliton it lives in physical time (Minkowski space, not Euclidean space), and possesses a well-defined 4-velocity.
- Like the instanton it "dies away" in the infinite past and future.
- Like the instanton it possesses a continuously adjustable scale parameter.
- Like the sphaleron it is unstable to dispersal.

Because the wavelet fields are bounded and finite energy, wavelet configurations *will* be classically excited at any finite temperature. Because the wavelet configuration has zero action, arbitrarily complicated combinations of these physical wavelets can be added to the field configurations appearing in Feynman's path integral without modifying the phase—quantum mechanically there is no "cost" in adding these configurations to the Lorentzian path integral and they *will* contribute.

Other "localized waves" might be interesting in specific applications but the particular example discussed in this Letter is important because of its extreme simplicity and pleasant behaviour.

Acknowledgements

I wish to thank John Lekner for stimulating my interest in these issues. Research supported by the Marsden Fund administered by the Royal Society of New Zealand.

References

[1] R.W. Ziolkowski, J. Math. Phys. 26 (1985) 861.
[2] M.K. Tippet, R.W. Ziolkowski, J. Math. Phys. 32 (1991) 488.
[3] J. Lu, J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control 39 (1992) 19.
[4] J. Lu, H. Zou, J.F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control 42 (1995) 850.
[5] G. Kaiser, math-ph/0303027.
[6] J. Lekner, J. Opt. A 5 (2003) L15, physics/0304022.
[7] R. Streater, A.S. Wightman, PCT, Spin & Statistics, and All That, Benjamin, New York, 1964.