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Quantifying energy condition violations in traversable wormholes

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Abstract. The 'theoretical' existence of traversable Lorentzian wormholes in the classical, macroscopic world is plagued by the violation of the well-known energy conditions of general relativity. In this brief article we show: (i) how the extent of violation can be quantified using certain volume integrals and (ii) whether this 'amount of violation' can be minimised for some specific cut-and-paste geometric constructions. Examples and possibilities are also outlined.

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1. The problem and our attitude

It is well-known by now that the 'theoretical' existence of traversable Lorentzian wormholes is plagued by the violation of the energy conditions of general relativity [1,2]. Researchers have come up with a variety of proposals, most of which gain support from the fact that quantum expectation values of the stress energy tensor can often become negative [3,4]. The experimentally verified case of the Casimir effect [5] is often cited as a 'proof' of the existence of 'matter' with 'negative energy density' though in experiments on the Casimir effect the quantity measured is the force (and hence the pressure) of the 'fluctuating vacuum' between parallel metallic plates.

Leaving aside the question about whether wormholes exist or whether negative energy is justifiable we prefer to adopt a somewhat braver attitude based on some recent results in other areas of physics. For instance, before it was actually seen in the laboratory one never believed that 'negative group velocity' [6] or 'negative refractive index' [7] could be real. Theoretically however, these esoteric concepts were outlined decades ago and largely forgotten. The same also holds good for the

Casimir effect. It is true that today, negative energy or wormholes are esoteric ideas. But, following the abovementioned realisation of negative v_g and n it may not be too outrageous to say that exotic things of today might be a reality (in some now-inconceivable form) tomorrow. Another example is 'dark energy' which seems to dominate 70% of the matter in the universe today [8]. Dark energy has 'negative' pressure which is indeed counterintuitive but largely in vogue amongst today's cosmolgists. Furthermore, particle theorists seem to be happy with a negative cosmological constant [9] which helps them solve the so-called hierarchy problem. So, why not wormholes with 'negative energy'?

Of course, negative energy or negative energy density is problematic. But then, one must ask the question 'how much negative energy'– or is there a way to quantify the amount of violation? There have been attempts at such quantification through the so-called 'quantum inequalities' which are essentially similar to the energy–time uncertainty relations [10]. Here we propose a quantifier in terms of a spatial volume integral [11]. Using this we can show that certain 'cut-and-paste constructions' allow us to reduce this 'amount of violation' to arbitrarily small values. We provide an example of such a construction and conclude with some open questions [11].

2. The energy conditions and the 'volume integral quantifier'

Let us begin by discussing some of the energy conditions in the literature. We can classify them as 'local' and 'global' conditions. Among local conditions we have the weak energy condition (WEC) and the null energy condition (NEC) which are stated as (for a diagonal energy momentum tensor with energy density ρ and pressures p_i $(i = 1, 2, 3)$):

$$\rho \geq 0, \quad \rho + p_i \geq 0 \quad \text{(WEC)}; \quad \rho + p_i \geq 0 \quad \text{(NEC)}. \tag{1}$$

Other local conditions include the strong energy condition (SEC) and the dominant energy condition (DEC) (for a discussion on these, see [2]). On the other hand, global conditions involve line integrals along complete null or time-like geodesics and therefore yield numbers. For example, the averaged null energy condition (ANEC) is given as

$$\int_{\lambda_1}^{\lambda_2} T_{ij} k^i k^j \mathrm{d}\lambda \geq 0, \tag{2}$$

where k^i is the tangent vector along a null geodesic and λ is the affine parameter labeling points on the geodesic. A useful discussion on the violation of the local and global energy conditions in the context of both classical (exotic) or quantum stress–energy can be found in [2,12].

Now consider a static spherically symmetric space-time with a line element given by

$$\mathrm{d}s^2 = -\exp[2\phi(r)]\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - b(r)/r} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{3}$$

Using the Einstein field equations, the components of the diagonal energy–momentum tensor in an orthonormal basis turn out to be (units – $G = c = 1$) [2]:

$$\rho = \frac{1}{8\pi}\frac{b'}{r^2}; \quad p_{\rm r} = \frac{1}{8\pi}\left[-\frac{b}{r^3} + 2\left\{1 - \frac{b}{r}\right\}\frac{\phi'}{r}\right], \tag{4}$$

$$p_{\rm t} = \frac{1}{8\pi}\left[\left\{1 - \frac{b}{r}\right\}\left[\phi'' + \phi'\left(\phi' + \frac{1}{r}\right)\right] - \frac{1}{2}\left(\frac{b}{r}\right)'\left(\phi' + \frac{1}{r}\right)\right], \tag{5}$$

where ρ, $p_{\rm r}$, and $p_{\rm t}$ are the energy density, the radial and tangential pressures respectively.

The ANEC integral along a radial null geodesic is

$$\begin{aligned} I &= \oint (\rho + p_{\rm r}) \exp(-2\phi)\, {\rm d}\lambda = \oint (\rho + p_{\rm r}) \exp(-\phi)\, {\rm d}\eta \\ &= -\frac{1}{4\pi}\oint \frac{2}{r}\frac{\rm d}{{\rm d}r}\left[{\rm e}^{-\phi}\sqrt{1 - \frac{b(r)}{r}}\right]{\rm d}r \\ &= -\frac{1}{4\pi}\oint \frac{1}{r^2}{\rm e}^{-\phi}\sqrt{1 - \frac{b}{r}}\,{\rm d}r < 0, \end{aligned} \tag{6}$$

where η is the proper radial distance and we have performed an integration by parts in the last step. Both the local and averaged energy conditions are violated by wormholes [1]. A quick way to see this is to note that for light rays the wormhole throat behaves like a diverging lens – light rays are not focused and therefore there must be a violation of the null convergence condition (or null energy condition, via the Einstein equations).

However, the ANEC is a line integral and therefore not very helpful for quantifying the 'amount of violation'. This prompts us to propose a 'volume integral quantifier' which amounts to calculating the following definite integrals (for the relevant coordinate domains):

$$\int \rho\, {\rm d}V; \quad \int (\rho + p_i){\rm d}V \tag{7}$$

with an appropriate choice of the integration measure ($4\pi r^2 {\rm d}r$ or $\sqrt{g}\, {\rm d}r\, {\rm d}\theta\, {\rm d}\phi$). We define the amount of violation as the extent to which these integrals can become negative. The important point which we shall demonstrate below is that even if the ANEC yields a constant negative number the volume integrals can be adjusted to become vanishingly small by appropriate choice of parameters.

Let us now focus on one such volume integral. Using the Einstein field equations it is easy to check that

$$\rho + p_{\rm r} = \frac{1}{8\pi r}\left\{1 - \frac{b}{r}\right\}\left[\ln\left(\frac{\exp(2\phi)}{1 - b/r}\right)\right]'. \tag{8}$$

Then integrating by parts

$$\oint (\rho + p_{\rm r})\, {\rm d}V = \left[(r-b) \ln \left(\frac{\exp(2\phi)}{1-b/r} \right) \right]_{r_0}^{\infty} - \int_{r_0}^{\infty} (1-b') \left[\ln \left(\frac{\exp(2\phi)}{1-b/r} \right) \right] {\rm d}r. \tag{9}$$

The boundary term at r_0 vanishes by our construction (recall that for a wormhole $b(r = r_0) = r_0$, where r_0 is the throat radius and hence the minimum value of r). The boundary term at infinity vanishes because of the assumed condition of asymptotic flatness. Then

$$\oint (\rho + p_{\rm r})\, {\rm d}V = - \int_{r_0}^{\infty} (1-b') \left[\ln \left(\frac{\exp(2\phi)}{1-b/r} \right) \right] {\rm d}r. \tag{10}$$

The value of this volume integral provides information about the 'total amount' of ANEC violating matter in the space-time. One should also calculate the other volume integrals though in most cases they do not provide any further information on the amount of violation.

3. An explicit 'cut-and-paste' example

Let us now look at a specific example. If we consider a line element for which the spatial metric is exactly Schwarzschild, that is $b(r) \to 2m = r_0$. Then $\rho = 0$ throughout the space-time and we simply get

$$\oint p_{\rm r}\, {\rm d}V = - \int_{r_0}^{\infty} \ln \left[\frac{\exp(2\phi)}{1-2m/r} \right] {\rm d}r. \tag{11}$$

Now assume that we have a wormhole whose field only deviates from Schwarzschild ($g_{00} \neq -(1-2m/r)$) in the region from the throat out to radius $a > 2m$. At $r = a$ we join this geometry to a Schwarzschild. We must take care of the matching conditions – details on these are available in [11]. It turns out that for this case, we can further simplify the above volume integral to

$$\oint p_{\rm r}\, {\rm d}V = - \int_{r_0}^{a} \ln \left[\frac{\exp(2\phi)}{1-2m/r} \right] {\rm d}r. \tag{12}$$

Under this same restriction the ANEC integral satisfies

$$I < -\frac{2}{4\pi} \int_a^{\infty} \frac{1}{r^2}\, {\rm d}r = -\frac{1}{2\pi\, a}, \tag{13}$$

which is strictly bounded away from zero. (Note that while evaluating the above from eq. (5) one has to be careful about the derivative discontinuity of ${\rm e}^{-\phi}\sqrt{1-b(r)/r}$ at $r = a$. As discussed below, this formula can safely be applied as a approaches $2m$ from the $a > 2m$ side.) Now

$$\int_{r_0}^{a} \ln \left[\frac{\exp(2\phi)}{1-2m/r} \right] {\rm d}r < \int_{r_0}^{a} \ln \left[\frac{\exp(2\phi_{\rm max})}{1-2m/r} \right] {\rm d}r. \tag{14}$$

Evaluating this last integral

$$\oint p_\mathrm{r}\, \mathrm{d}V > -(a-2m)\ln\left[\frac{\exp(2\phi_\mathrm{max})}{1-2m/a}\right] - 2m\ \ln\left(\frac{a}{2m}\right). \tag{15}$$

This is useful because it is an explicit lower bound on the total amount of radial stress in terms of ϕ_max and the size of the region of ANEC violating matter. Similarly

$$\oint p_\mathrm{r}\, \mathrm{d}V < -(a-2m)\ln\left[\frac{\exp(2\phi_\mathrm{min})}{1-2m/a}\right] - 2m\ \ln\left(\frac{a}{2m}\right). \tag{16}$$

This is now an upper bound in terms of ϕ_min and the size of the region of ANEC violating matter. If we now choose geometries such that ϕ_max and ϕ_min are not excessively divergent (no worse than $(a-2m)^{-\delta}$ with $\delta < 1$), we can take the limit $a \to 2m^+$ (the superscript + here means that a approaches $2m$ from the $a > 2m$ side) to obtain

$$\oint p_\mathrm{r}\, \mathrm{d}V \to 0. \tag{17}$$

We emphasize here that the ANEC integral does not go to zero as $a \to 2m^+$.

Furthermore, by considering a sequence of traversable wormholes with suitably chosen a and $\phi(r)$ (and $b(r) = 2m$) we can construct traversable wormholes with arbitrarily small quantities of ANEC-violating matter (with the ANEC line integral nevertheless remaining finite and negative). More examples are available in [11].

4. Remarks and conclusions

The above discussion shows that we are able to (i) quantify the amount of violation and (ii) construct examples for which the violation can be made very small. It is worthwhile to note that in both these constructions the local and averaged energy conditions still remain violated and that violation cannot be made to vanish! It might seem therefore that we have skirted the real issue by 'redefining' the notion of violation through these volume integrals. Therefore, to prove our point we must try to establish the fact that these volume integrals are the correct quantifiers (on physical grounds they do seem to be so) and all theorems which assume the validity of the averaged conditions can now be extended to include these volume averaged conditions. This is a task for the future.

Finally let us place our result in the context of the four great results of classical general relativity – the area increase theorem [13,14], the singularity theorem [13], the positive mass theorem [15] and the topological censorship theorem [16]. Each of these theorems do assume some form of an energy condition. The question is: if violations are small, can the conclusions of these theorems be evaded? It is worth noting at this point that the conclusions of the area increase and topological censorship thoerems can indeed be reversed by quantum-induced violations. In the case of the other two theorems the consequences of such microscopic violations may not lead to any drastic changes in their conclusions.

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