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Topological Proofs of Some Rasiowa-Sikorski Lemmas

In memory of Leo Esakia

Abstract. We give topological proofs of Görnemann's adaptation to Heyting algebras of the Rasiowa-Sikorski Lemma for Boolean algebras; and of the Rauszer-Sabalski generalisation of it to distributive lattices. The arguments use the Priestley topology on the set of prime filters, and the Baire category theorem.

This is preceded by a discussion of criteria for compactness of various spaces of subsets of a lattice, including spaces of filters, prime filters etc.

Keywords: Rasiowa-Sikorski Lemma, Baire category theorem, Priestley topology, compact, dense, lattice, distributive, Heyting algebra, prime filter, join, meet.

1. Introduction

The famous eponymous Lemma presented by Rasiowa and Sikorski in [11] stated that in a Boolean algebra, any non-unit element belongs to some prime ideal that preserves countably many prescribed joins (i.e. least upper bounds). Nowadays it is usually stated in the equivalent form that any non-zero element belongs to some ultrafilter that preserves countably many prescribed meets (greatest lower bounds). The result was derived as part of an algebraic proof of the completeness theorem for first-order logic, applying the Lemma to the Lindenbaum-Tarski algebra of a countable first-order language and using the resulting ultrafilter to construct a Tarskian model falsifying some given non-theorem.

Rasiowa and Sikorski's proof of their Lemma was topological, using a Baire category argument in the Stone space of a Boolean algebra, showing that the set of prime ideals preserving the prescribed joins is dense in this space. Tarski then provided a purely Boolean-algebraic approach. This was described in a review of [11] by Feferman [2], who suggested that the Rasiowa-Sikorski completeness method was "a distinct advance over all preceding proofs".[1]

[1]Texbook expositions of this completeness proof occur in [12] and [1], with the latter giving Tarski's version of the Rasiowa-Sikorski Lemma.

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Görnemann in [6] developed a result about the existence of prime filters that preserve countably many prescribed joins and meets in a Heyting algebra. She used this to axiomatise the extension of intuitionistic first-order logic characterised by the Kripke models with constant domains of individuals. The hypotheses of her result included closure conditions on these joins and meets mediated by the Heyting implication operation (see Theorem 4.1 below). Its proof was algebraic, and involved embedding the Heyting algebra into a minimal Boolean extension and applying the Rasiowa-Sikorski Lemma to the later. Rauszer and Sabalski [13] then gave another algebraic version of Görnemann's work that was internal to the Heyting algebra and its quotients. From this they extracted in [14] a result for arbitrary distributive lattices, stating that if element a is not below b, then there exists a prime filter containing a but not b and preserving countably many prescribed joins and meets, *provided* those joins and meets obey infinitary distributive laws.

The purpose of this paper is to put the topology back into the picture, by giving topological arguments for these results of Görnemann and Rauszer-Sabalski. Instead of using Stone's topology on the set of prime ideals of a distributive lattice [16], we use the topology of Priestley [9], which has the advantage of being Hausdorff as well as compact, and therefore obeys the Baire category theorem: the intersection of countably many open dense sets is dense. The set of prime filters preserving a given join or meet is open, and dense if the infinitary distributive laws hold, so the set of prime filters preserving countably many of them is dense. Moreover, the closure hypotheses in Görnemann's theorem are seen to underpin these density properties in the Heyting algebra case.

As preparation for this work, we begin with a discussion of spaces whose points are subsets of a given set, with the Priestley-style topology. The space of all subsets is compact and Hausdorff, so its compact subsets are precisely those that are closed. We study a variety of properties of subsets of a lattice (meet-closed, order closed, filter, prime filter etc.) that define a collection of subsets that is closed, hence forms a compact space.

Thus we show that the Rasiowa-Sikorski topological method extends naturally to the setting of distributive lattices and Heyting algebras, and does so in a way that is conceptually elegant and may point the way towards similar results for other kinds of structure used in algebraic logic.

2. Compact Spaces of Subsets

Let L be a non-empty set, with powerset $\mathcal{P}(L)$, and let S be a collection of subsets of L, i.e. $S \subseteq \mathcal{P}(L)$. For each $a \in L$, define:

$$|a|_S = \{p \in S : a \in p\}.$$
$$-|a|_S = S - |a|_S = \{p \in S : a \notin p\}.$$

In the case $S = \mathcal{P}(L)$, we will write $|a|_{\mathcal{P}(L)}$ more briefly as $|a|_L$.

Now let T_S be the topology on S generated by the collection

$$\{|a|_S, -|a|_S : a \in L\}.$$

The members of this collection are the *subbasic* T_S-open sets. They are both open and closed, and ensure that (S, T_S) is a Hausdorff space. A *basic* open set is one that is the intersection of finitely many subbasic sets. Any member of T_S is a union of basic open sets.

A basic set has the typical form

$$|a_0|_S \cap \cdots \cap |a_{n-1}|_S \cap -|b_0|_S \cap \cdots \cap -|b_{m-1}|_S \tag{2.1}$$

for some $n, m < \omega$ and some elements $a_i, b_j \in L$. Here we allow that $n = 0$, in which case there are no expressions $|a_i|_S$ in (2.1), or that $m = 0$, in which case there are no occurrences of $-|b_j|_S$.

Note that if $S \subseteq S' \subseteq \mathcal{P}(L)$, then T_S is the subspace topology induced on S by $T_{S'}$.

We will call T_S the *(generalized) Priestley topology* on S. It is the only kind of topology we discuss. It has the same definition as the well-known topology on the set of prime filters of a distributive lattice introduced by Priestley [9]. Under the correspondence between subsets and their characteristic functions, S corresponds to a set S^χ of functions from L to the discrete space $\{0, 1\}$, and T_S corresponds to the subspace topology induced on S^χ by the product topology on the set of all such functions (see the proof of Theorem 2.1 below). When L is a bounded lattice and S is its set of proper prime filters, then S^χ is the set of homomorphisms from L to $\{0, 1\}$ as a bounded lattice.

THEOREM 2.1. *S is compact if, and only if, it is closed in the Priestley topology on $\mathcal{P}(L)$.*

PROOF. It suffices to show that $\mathcal{P}(L)$ itself is compact in its Priestley topology, because in any compact Hausdorff space, the compact subsets are precisely the closed sets.

Now under the bijection between $\mathcal{P}(L)$ and the set 2^L of functions from L to $2 = \{0, 1\}$, the Priestley topology on $\mathcal{P}(L)$ corresponds to the product topology on 2^L when 2 is given the discrete topology. 2^L is compact by Tychonoff's Theorem, so $\mathcal{P}(L)$ is compact.

However, it is instructive to give a direct proof of the compactness of $\mathcal{P}(L)$. By the Alexander Subbase Lemma, it suffices to show that if $\mathcal{C}$ is any cover of $\mathcal{P}(L)$ by subbasic open sets, then $\mathcal{C}$ has a finite subcover. Given such a $\mathcal{C}$, let $p = \{a \in L : -|a|_L \in \mathcal{C}\}$. Then as $\mathcal{C}$ covers $\mathcal{P}(L)$, some member of $\mathcal{C}$ contains p. Now if we had $p \in -|a|_L \in \mathcal{C}$, then $a \notin p$, implying that $-|a|_L \notin \mathcal{C}$, which is contradictory. So we must conclude that there is some $a \in L$ with $p \in |a|_L \in \mathcal{C}$. Then $a \in p$, hence $-|a|_L \in \mathcal{C}$, so $\mathcal{C}$ has the finite subcover $\{|a|_L, -|a|_L\}$ of $\mathcal{P}(L)$, as desired. □

There are numerous conditions that define a closed, hence compact, subset of $\mathcal{P}(L)$. Here are some that are relevant to our concerns:

Theorem 2.2. *For the following conditions* (*i*) *on* p, *and for each* $I \subseteq \{1, \ldots, 5\}$, *the set*

$$S_I = \{p \in \mathcal{P}(L) : p \text{ satisfies } (i) \text{ for all } i \in I\}$$

is compact in the Priestley topology.

(1) *$d \in p$; where d is some designated member of L.*

(2) *$d' \notin p$; where d' is some designated member of L.*

(3) *If $a \sqsubseteq b$ and $a \in p$, then $b \in p$; where $\sqsubseteq$ is some specified binary relation on L.*

(4) *If $a \in p$ and $b \in p$, then $a \sqcap b \in p$; where $\sqcap$ is some specified binary operation on L.*

(5) *If $a \sqcup b \in p$, then $a \in p$ or $b \in p$; where $\sqcup$ is some specified binary operation on L.*

Proof. Since $S_I = \bigcap\{S_{\{i\}} : i \in I\}$ and any intersection of closed sets is closed, hence is compact in the Priestley topology, it suffices to show that each $S_{\{i\}}$ is a closed subset of $\mathcal{P}(L)$.

For (1) and (2), observe that $S_{\{1\}} = |d|_L$ and $S_{\{2\}} = -|d'|_L$, which are closed by definition. For the other cases, we show that $S_{\{i\}}$ is closed by showing that its complement is open, i.e. that if $p \notin S_{\{i\}}$, then p has an open neighbourhood disjoint from $S_{\{i\}}$.

(3): If $p \notin S_{\{3\}}$, then there are $a, b \in L$ with $a \sqsubseteq b$, $a \in p$, but $b \notin p$. Then $|a|_L - |b|_L$ is an open neighbourhood of p disjoint from $S_{\{3\}}$.

(4): If $a, b \in p$ but $a \sqcap b \notin p$, then $(|a|_L \cap |b|_L) - |a \sqcap b|_L$ is an open neighbourhood of p disjoint from $S_{\{4\}}$.

(5): If If $a \sqcup b \in p$ but $a \notin p$ and $b \notin p$, then $(|a \sqcup b|_L - |a|_L) - |b|_L$ is an open neighbourhood of p disjoint from $S_{\{5\}}$. □

Evidently there are many other such closure conditions on p, induced by various finitary relations and operations on L, that specify compact subsets of $\mathcal{P}(L)$. For example, [4, Section 4.1] lists more than a dozen conditions, related to "deductive" filters on residuated lattices, each of which defines a closed and hence compact set. On the other hand, here is a case of an important condition that defines an open subset that may not be compact:

THEOREM 2.3. *The set $S_{ne} = \{p : p \neq \emptyset\}$ of **non-empty** subsets of L is open in the Priestley topology, and is compact iff L is finite.*

PROOF. S_{ne} is the open set $\bigcup\{|a|_L : a \in L\}$.

If S_{ne} is compact, then it is closed, and so its complement $\{\emptyset\}$ is open. Thus the singleton $\{\emptyset\}$ must be a basic open set. But $\emptyset$ does not belong to any set of the form $|a|_L$, so we must have $\{\emptyset\} = -|a_1|_L \cap \cdots \cap -|a_n|_L$ for some $n < \omega$ and some a_i. Now if there were some $b \in L - \{a_1, \ldots, a_n\}$, then $\{b\}$ would belong to each set $-|a_i|_L$ but not belong to $\{\emptyset\}$, which is contradictory. Hence $L = \{a_1, \ldots, a_n\}$.

Conversely, if L is finite, then S_{ne} is finite, hence is compact. □

Analogously, the set $\{p : p \neq L\}$ of *proper* subsets of L is open, being equal to $\bigcup\{-|a|_L : a \in L\}$, and can be shown to be compact iff L is finite.

Our principal interest is in conditions that define p to be some kind of lattice filter. So now let $\mathbf{L} = (L, \sqsubseteq, \sqcap, \sqcup)$ be a lattice under a partial order $\sqsubseteq$, with binary meet operation $\sqcap$ and joint operation $\sqcup$. We write $a \sqsubset b$ when $a \sqsubseteq b$ and $a \neq b$. Recall that an *upset* is a set $p \subseteq L$ that is closed upwards under $\sqsubseteq$, in the sense that it satisfies (3) of Theorem 2.2. A *filter* of $\mathbf{L}$ is a non-empty upset that is closed under $\sqcap$; a *prime* filter is one satisfying (5) of Theorem 2.2; and a *proper* filter is one that is a proper subset of L.[2]

Putting $I = \{3, 4\}$ in Theorem 2.2 shows that the $\sqcap$-closed upsets form a compact subset of $\mathcal{P}(L)$. But the set of filters of $\mathbf{L}$ need not be compact. The point is that filters are standardly required to be *non-empty*, a condition that we have just shown may not determine a closed subset of $\mathcal{P}(L)$.

[2]Some authors include "proper" in the definition of "prime ideal", but our discussion encourages preservation of the distinction.

Example 2.4. Non-Compact Filter Spaces

Let $\mathbf{L} = (L, \sqsubseteq)$ be any *linearly* ordered poset, with no greatest element (e.g. $\mathbf{L} = (\mathbb{R}, \leq)$). Then $\mathbf{L}$ is a distributive lattice in which $a \sqcap b = \min\{a, b\}$ and $a \sqcup b = \max\{a, b\}$.

Let S be the set of all filters of $\mathbf{L}$. Then S is also the set of all prime filters: every filter is prime because $a \sqcup b$ is equal either to a or b. Each filter contains some $a \in L$, hence belongs to $|a|_L$. Thus $\mathcal{C} = \{|a|_L : a \in L\}$ is an open cover of S. $\mathcal{C}$ is linearly order by set inclusion, since $\mathbf{L}$ is linearly ordered by $\sqsubseteq$, and $a \sqsubseteq b$ implies $|a|_L \subseteq |b|_L$. Hence the union of any finite subset $\mathcal{C}_0$ of $\mathcal{C}$ is equal to $|a_0|_L$ for some a_0. Then as $\mathbf{L}$ has no greatest element, $\{b : a_0 \sqsubset b\}$ is a (proper prime) filter that does not belong to $|a_0|_L$, so $\mathcal{C}_0$ does not cover S.

This shows that S has an open cover with no finite subcover, so is not compact. The construction shows also that the set of proper filters, the set of prime filters, and the set of proper prime filters all fail to be compact.

Equivalently, we could have shown this by proving that S is not closed. Indeed the empty set $\emptyset$ does not belong to S, but is a closure point of S. For, if U is any basic neighbourhood of $\emptyset$, then U is an intersection of finitely many subbasic sets, none of which can be of the form $|a|_L$, so we must have $U = -|a_1|_L \cap \cdots \cap -|a_n|_L$ for some n and some a_i. Then if $a_0 = a_1 \sqcup \cdots \sqcup a_n$, the filter $\{b : a_0 \sqsubset b\}$ contains none of the a_i's, so belongs to U. This shows that every open neighbourhood of $\emptyset$ contains a member of S, as required. □

The absence of a greatest element from $\mathbf{L}$ was crucial in this Example. If, on the contrary, $\mathbf{L}$ does have a greatest element 1, then its filters are characterised as those $\sqcap$-closed upsets that contain 1, a description implying that the set of all filters is compact. Furthermore, if $\mathbf{L}$ has a least element 0, then a filter p is proper iff $0 \notin p$. Combining these facts with cases of Theorem 2.2 yields:

Theorem 2.5. *If a lattice* $\mathbf{L}$ *has a greatest element, then its set of filters, and its set of prime filters, are each compact in the Priestley topology on* $\mathcal{P}(L)$*. If* $\mathbf{L}$ *is* ***bounded****, i.e. has both greatest and least elements, then its set of proper filters, and its set of proper prime filters, are also compact.*

In [5] we gave a proof that the set of filters of an ortholattice is compact. The proof used only general lattice principles, but depended on every set, including the empty set, having a smallest extension to a filter. That is certainly true for an ortholattice, since it has a greatest element, hence a smallest filter $\{1\}$.

To round out this discussion, we observe that the presence of suitable distinguished elements is necessary for compactness results like those of Theorem 2.5:

THEOREM 2.6. *If S is any compact set of filters of a lattice $\mathbf{L}$, then:*

(1) *There exists an element d that belongs to every member of S, i.e. $S = |d|_S$.*

(2) *If every member of S is proper, then there exists an element d' that does not belong to any member of S, i.e. $S = -|d'|_S$.*

PROOF. (1): S has the open cover $\{|a|_L : a \in L\}$, so by compactness $S \subseteq \bigcup_{i \leq n} |a_i|_L$ for some $n \geq 1$ and some a_i. Let $d = a_1 \sqcup \cdots \sqcup a_n$. Then $a_i \sqsubseteq d$ and so $|a_i|_L \subseteq |d|_L$ for all $i \leq n$. Hence $S \subseteq |d|_L$, and so $S = |d|_S$.

(2): In this case S has the open cover $\{-|a|_L : a \in L\}$, and so $S \subseteq \bigcup_{i \leq n} -|a_i|_L$ for some $n \geq 1$ and some a_i. This time let $d' = a_1 \sqcap \cdots \sqcap a_n$. Then $d' \sqsubseteq a_i$ and so $-|a_i|_L \subseteq -|d'|_L$ for all $i \leq n$. Hence $S = -|d'|_S$. □

REMARK 2.7. In addition to T_S, we could consider the coarser topology C_S on S generated by the collection

$$\{|a|_S : a \in L\}.$$

This may be seen as a generalization of the Stone topology on the set of prime filters of a distributive lattice, or of the corresponding topology on the spectrum (set of prime ideals) of a commutative ring. The latter gives rise to the abstract notion of a *spectral space* [7], and the relationship between C_S and T_S parallels that between a spectral space and its *patch topology* [7, §2].[3]

However, whereas the patch topology of a spectral space is always compact, in our generalized setting compactness need not be preserved in passing from C_S to T_S. For examples of this, let $\mathbf{L} = (L, \sqsubseteq)$ be any lattice that has no least element, but does have a greatest element 1, and let S be the set of *principal* filters of $\mathbf{L}$. Thus the members of S are the sets $a{\uparrow} = \{b : a \sqsubseteq b\}$ for all $a \in L$, and every member of S is proper as $\mathbf{L}$ has no least element. C_S is compact in a strong sense, for if $\mathcal{C}$ is a cover of S by sets of the form $|a|_S$, then the principal filter $1{\uparrow} = \{1\}$ belongs to some $|a|_S \in \mathcal{C}$, hence $a = 1$ and so $S = |a|_S$ as 1 belongs to every filter.

On the other hand, T_S is not compact, or else by Theorem 2.6(2) there would be an element of L not in any member of S, which is evidently false.

[3]I thank the referee for drawing attention to this connection.

The proof of compactness of the patch topology of a spectral space [7, Theorem 1] uses the fact that a spectral space is *sober*. This means that every non-empty irreducible closed subset is the closure of a unique point, where an irreducible closed set is one that is not the union of two proper closed subsets. It is readily seen that the example (S, C_S) of the non-compact space of principal filters just discussed is not sober. Indeed S itself is irreducible, while the closure of any point $a{\uparrow}$ of S is the set $\{b{\uparrow} : a \sqsubseteq b\}$, which is not equal to S. To see why S is irreducible, observe that if B is any proper closed subset of S, then there is an open set of the form $|a_B|_S$ disjoint from B. Given a second proper closed subset B', with $|a_{B'}|_S$ disjoint from B', then $|a_B \sqcap a_{B'}|_S$ is disjoint from both B and B', and so the point $(a_B \sqcap a_{B'}){\uparrow}$ of S does not belong to $B \cup B'$.

3. Q-Filters of Distributive Lattices

If X is any subset of a lattice $\mathbf{L}$, we write $\bigsqcup X$ for the *join* of X, and $\bigsqcap X$ for its *meet*, when these entities exist in $\mathbf{L}$. A filter p is said to *preserve* an existing join $\bigsqcup X$ whenever

$$\bigsqcup X \in p \text{ implies } X \cap p \neq \emptyset. \tag{3.1}$$

For a given set S of filters, let $S_{\sqcup X} = \{p \in S : p \text{ preserves } \bigsqcup X\}$. Then

$$S_{\sqcup X} = -|\bigsqcup X|_S \cup \bigcup_{x \in X} |x|_S, \tag{3.2}$$

showing that $S_{\sqcup X}$ is an open set in the Priestley topology on S.

Dually, p preserves a meet $\bigsqcap X$ when

$$X \subseteq p \text{ implies } \bigsqcap X \in p. \tag{3.3}$$

The set $S_{\sqcap X} = \{p \in S : p \text{ preserves } \bigsqcap X\}$ is also open, as

$$S_{\sqcap X} = |\bigsqcap X|_S \cup \bigcup_{x \in X} -|x|_S. \tag{3.4}$$

For $a \in L$ and $X \subseteq L$, define $a \sqcap X = \{a \sqcap x : x \in X\}$ and $a \sqcup X = \{a \sqcup x : x \in X\}$. An existing join $\bigsqcup X$ is said to be *distributive* in $\mathbf{L}$ if, for all $a \in L$, $\bigsqcup(a \sqcap X)$ exists and

$$a \sqcap \bigsqcup X = \bigsqcup(a \sqcap X). \tag{3.5}$$

Thus $\mathbf{L}$ is a distributive lattice when each of its binary joins $b \sqcup c$ is distributive in this sense. Dually, a meet $\bigsqcap X$ is distributive in $\mathbf{L}$ if, for all $a \in L$, $\bigsqcap(a \sqcup X)$ exists and

$$a \sqcup \bigsqcap X = \bigsqcap(a \sqcup X). \tag{3.6}$$

Now let Q be a collection of subsets of $\mathbf{L}$, and put $Q_J = \{X \in Q : \bigsqcup X \text{ exists in } \mathbf{L}\}$ and $Q_M = \{X \in Q : \bigsqcap X \text{ exists in } \mathbf{L}\}$. Q will be called a *distributive family* if the join of each member of Q_J is distributive in the sense of (3.5) and the meet of each member of Q_M is distributive in the sense of (3.6). A *Q-filter* of $\mathbf{L}$ is a filter that preserves each join in Q_J as in (3.1) and preserves the meet of each member of Q_M as in (3.3).

It was shown algebraically by Rauszer and Sabalski [14] that if $\mathbf{L}$ is a distributive lattice, and Q is a *countable* distributive family, then if $a \not\sqsubseteq b$, there exists a prime Q-filter that contains a but not b. Here we give a topological proof of that result.

First, it is easy to reduce the problem to the case that $\mathbf{L}$ is bounded. For if $\mathbf{L}$ is not bounded, we can add a greatest element and/or a least element to $\mathbf{L}$ as required, forming a new lattice $\mathbf{L}^+$ that is still distributive. This expansion of $\mathbf{L}$ does not alter the join of any member of Q_J or the meet of any member of Q_M. Now if $a \not\sqsubseteq b$ in $\mathbf{L}$, the result for bounded distributive lattices gives a Q-filter p of $\mathbf{L}^+$ that contains a but not b. Then the restriction of p to $\mathbf{L}$ is the desired Q-filter of $\mathbf{L}$ that contains a but not b.

So from now on we assume that $\mathbf{L}$ is a distributive lattice with a greatest element 1 and a least element 0. Let S be the set of all proper prime filters of $\mathbf{L}$. Then $|1|_S = S$ and $|0|_S = \emptyset$. Standard properties of prime filters imply:

$$|a \sqcap b|_S = |a|_S \cap |b|_S. \tag{3.7}$$

$$|a \sqcup b|_S = |a|_S \cup |b|_S. \tag{3.8}$$

The basic existence result on prime filters gives that

$$|a|_S \subseteq |b|_S \text{ implies } a \sqsubseteq b \tag{3.9}$$

(equivalently, if $a \not\sqsubseteq b$, then there is a prime filter containing a but not b).

The presence of 1 and 0 allows the description of basic open sets to be reduced to a simple standard form:

LEMMA 3.1. *Every basic open set in the Priestley topology of S is of the form $|a|_S - |b|_S$ for some $a, b \in L$.*

Proof. If U is a basic open set, then as in (2.1), U has the form

$$\bigcap_{i<n} |a_i|_S \cap \bigcap_{j<m} -|b_j|_S = \bigcap_{i<n} |a_i|_S - \bigcup_{j<m} |b_j|_S \tag{3.10}$$

for some n, m. Define $a = \bigsqcap_{i<n} a_i$ and $b = \bigsqcup_{j<m} b_j$.

Now using property (3.7), when $n > 0$ we get $|a|_S = \bigcap_{i<n} |a_i|_S$. But this equation also holds when $n = 0$, since then $a = \bigsqcap \emptyset = 1$, and $\bigcap_{i<0} |a_i|_S = \bigcap \emptyset = S = |1|_S$.

When $m > 0$, (3.8) implies that $|b|_S = \bigcup_{j<m} |b_j|_S$. But this also holds when $m = 0$, since then $b = \bigsqcup \emptyset = 0$, and $\bigcup_{j<0} |b_j|_S = \bigcup \emptyset = \emptyset = |0|_S$.

Hence $U = |a|_S - |b|_S$. □

Lemma 3.2. *If X is a subset of $\mathbf{L}$, then:*

(1) *If $\bigsqcup X$ exists and is distributive in $\mathbf{L}$, then $S_{\sqcup X}$ is open and dense in the space (S, T_S).*

(2) *If $\bigsqcap X$ exists and is distributive in $\mathbf{L}$, then $S_{\sqcap X}$ is open and dense in (S, T_S).*

Proof. (1) $S_{\sqcup X}$ was shown to be open in (3.2). Density means that the closure of $S_{\sqcup X}$ is equal to S, or equivalently that the complement $-S_{\sqcup X}$ has empty interior, i.e. does not include any non-empty basic open set. To show this let U be a basic open included in $-S_{\sqcup X}$. By the previous Lemma, $U = |a|_S - |b|_S$ for some $a, b \in L$. Thus we have

$$|a|_S - |b|_S \subseteq -S_{\sqcup X} = |\bigsqcup X|_S \cap \bigcap_{x \in X} -|x|_S \tag{3.11}$$

(cf. (3.2)). Then for each $x \in X$, $|a|_S - |b|_S \subseteq -|x|_S$, and so

$$|a \sqcap x|_S = |a|_S \cap |x|_S \subseteq |b|_S,$$

hence $a \sqcap x \sqsubseteq b$ by (3.9). Thus b is an upper bound of $a \sqcap X$. Since $\bigsqcup X$ is distributive, this yields

$$a \sqcap \bigsqcup X = \bigsqcup (a \sqcap X) \sqsubseteq b.$$

Therefore $|a|_S \cap |\bigsqcup X|_S \subseteq |b|_S$, giving $U = |a|_S - |b|_S \subseteq -|\bigsqcup X|_S$. But $U \subseteq |\bigsqcup X|_S$ (cf. (3.11)), so we conclude that $U = \emptyset$.
This confirms that $-S_{\sqcup X}$ has empty interior.

(2) $S_{\sqcap X}$ is open by (3.4). To show it is dense, let $U = |a|_S - |b|_S$ be a basic open included in the complement of $S_{\sqcap X}$. Then

$$U \subseteq -S_{\sqcap X} = -|\bigsqcap X|_S \cap \bigcap_{x \in X} |x|_S.$$

Then for each $x \in X$, $|a|_S - |b|_S \subseteq |x|_S$, and so

$$|a|_S \subseteq |b|_S \cup |x|_S = |b \sqcup x|_S,$$

hence $a \sqsubseteq b \sqcup x$ by (3.9). Thus a is an lower bound of $b \sqcup X$. Since $\bigsqcap X$ is distributive, this yields

$$a \sqsubseteq \bigsqcap(b \sqcup X) = b \sqcup \bigsqcap X.$$

Therefore $|a|_S \sqsubseteq |b|_S \cup |\bigsqcap X|_S$, giving $U = |a|_S - |b|_S \subseteq |\bigsqcap X|_S$. But $U \subseteq -|\bigsqcap X|_S$, so we conclude that $U = \emptyset$.

□

Corollary 3.3. *If Q is a countable distributive family in $\mathbf{L}$, then the set S_Q of all proper prime Q-filters of $\mathbf{L}$ is dense in S.*

Proof. S_Q is the intersection

$$\bigcap\{S_{\sqcup X} : X \in Q_J\} \cap \bigcap\{S_{\sqcap X} : X \in Q_M\}$$

of countably many sets of the form $S_{\sqcup X}$ and $S_{\sqcap X}$, each of which is open and dense by Lemma 3.2.

Now the Hausdorff space S of all proper prime filters of $\mathbf{L}$ is compact [9] (see also Theorem 2.5). But it is a classical fact that compact Hausdorff spaces satisfy the *Baire category theorem*: the intersection of any countable collection of open dense sets is dense. Hence S_Q is dense. □

A proof of the Rauszer-Sabalski theorem now follows quickly: If Q is a countable distributive family, and $a \not\sqsubseteq b$ in $\mathbf{L}$, then $|a|_S \not\subseteq |b|_S$ by (3.9). Hence the open set $|a|_S - |b|_S$ is non-empty, and so must intersect the dense set S_Q. Any member of $(|a|_S - |b|_S) \cap S_Q$ is a prime Q-filter containing a but not b, as required. □

Remark 3.4. We could have given this proof by taking S to be the set of all prime filters, including the improper filter L. Then $\mathbf{L}$ would not require a least element, and the basic T_S-open set would be those of the form $|a|_S - |b|_S$ and $|a|_S$. Lemma 3.2 would still hold, with some adjustment to the proof to allow for the presence of L in S.

Also, we could consider the case of S being the set of all prime filters that contain some distinguished element d, which need not be a greatest element. The argument would still work, provided a suitable version of the existence property (3.9) held, namely that if $a \not\sqsubseteq b$, then there is a prime filter containing d and a, but not b.

4. Q-Filters of Heyting Algebras

Let $\mathbf{L} = (L, \sqsubseteq, \sqcap, \sqcup, 0, \Rightarrow)$ be a Heyting algebra, a lattice with least element 0 and a binary *implication* operation $\Rightarrow$ that is residual to $\sqcap$, i.e.

$$a \sqcap b \sqsubseteq c \quad \text{iff} \quad a \sqsubseteq b \Rightarrow c.$$

Then $\mathbf{L}$ is distributive and has a greatest element 1, equal to $a \Rightarrow a$ for any $a \in L$. As in the previous section, we take S to be the space of all proper prime filters of $\mathbf{L}$, and S_Q to be the set of all proper prime Q-filters, where $Q \subseteq \mathcal{P}(L)$.

Define $a \Rightarrow X = \{a \Rightarrow x : x \in X\}$ and $X \Rightarrow a = \{x \Rightarrow a : x \in X\}$, for any $a \in L$ and $X \subseteq L$. The following are well known facts about Heyting algebras [12, IV.7]:

- If $\bigsqcup X$ exists, then it is distributive, i.e. for all $a \in L$, (3.5) holds.
- If $\bigsqcup X$ exists, then for all $a \in L$, $\bigsqcap(X \Rightarrow a)$ exists, and

$$\bigsqcap(X \Rightarrow a) = (\bigsqcup X) \Rightarrow a. \tag{4.1}$$

- If $\bigsqcap X$ exists, then for all $a \in L$, $\bigsqcap(a \Rightarrow X)$ exists, and

$$\bigsqcap(a \Rightarrow X) = a \Rightarrow (\bigsqcap X). \tag{4.2}$$

However, an existing meet $\bigsqcap X$ need not be distributive in the sense of (3.6). So a family Q is distributive iff the meet of every member of Q_M is distributive. We employ combinations of the above operations; in particular using

$$a \sqcap X \Rightarrow b = \{a \sqcap x \Rightarrow b : x \in X\}, \text{ and}$$
$$a \Rightarrow b \sqcup X = \{a \Rightarrow b \sqcup x : x \in X\}.$$

We now review the representation of $\mathbf{L}$ as an algebra of subsets of its prime filter space, due originally to Stone [16], describing the version that underlies the equivalence of algebraic and Kripkean semantics for intuitionistic propositional logic [3, §1.6]. Let $Up(S)$ be the set of upsets of the partially ordered set $(S, \subseteq)$. To spell this out, for each $p \in S$, define $[p) = \{q \in S : p \subseteq q\}$. Then a member of $Up(S)$ is a set $U \subseteq S$ such that in general $p \in U$ implies $[p) \subseteq U$. $Up(S)$ is a complete Heyting algebra with greatest and least elements S and $\emptyset$, in which all joins and meets are given by set union and intersection, respectively. Its Heyting implication operation $\Rightarrow_S$ is given by

$$U \Rightarrow_S V = \{p \in S : [p) \cap U \subseteq V\}.$$

The function $f_S : L \to Up(S)$ defined by $f_S(a) = |a|_S$ is an injective Heyting algebra homomorphism, giving a concrete representation of $\mathbf{L}$. Results (3.7)–(3.9) imply that f_S preserves binary meets and joins and is injective; and since $|0|_S = \emptyset$ it preserves least elements. Preservation of implications, meaning that $|a \Rightarrow b|_S = |a|_S \Rightarrow_S |b|_S$, amounts to the fact that for any $p \in S$,

$$a \Rightarrow b \in p \quad \text{iff} \quad [p) \cap |a|_S \subseteq |b|_S, \tag{4.3}$$

a fact that we will use repeatedly.

Now suppose we replace S here by some S_Q, $Up(S)$ by the complete Heyting algebra $Up(S_Q)$ of upsets of $(S_Q, \subseteq)$, and f_S by the function $f_Q : L \to Up(S_Q)$ having $f_Q(a) = |a|_{S_Q}$. Then f_Q preserves binary joins and meets, and least elements. Since each member of S_Q is a Q-filter, f_Q also preserves the joins of all members of Q_J and the meets of all members of Q_M. If Q is a countable distributive family, then the Rauszer-Sabalski theorem for distributive lattices holds for $\mathbf{L}$ and Q, and can be stated in the form

$$|a|_{S_Q} \subseteq |b|_{S_Q} \text{ implies } a \sqsubseteq b.$$

This readily implies that f_Q is injective.

Preservation of Heyting implications by f_Q requires that $|a \Rightarrow b|_{S_Q} = |a|_{S_Q} \Rightarrow_Q |b|_{S_Q}$, where the implication $\Rightarrow_Q$ of $Up(S_Q)$ has

$$U \Rightarrow_Q V = \{p \in S_Q : [p) \cap U \subseteq V\}$$

for all upsets U, V of $(S_Q, \subseteq)$. The inclusion $|a \Rightarrow b|_{S_Q} \subseteq |a|_{S_Q} \Rightarrow_Q |b|_{S_Q}$ holds for any Q, for if $p \in |a \Rightarrow b|_{S_Q}$ and $p \subseteq q \in |a|_{S_Q}$, we have $a, a \Rightarrow b \in q$, and so as $a \sqcap (a \Rightarrow b) \sqsubseteq b$ we get $q \in |b|_{S_Q}$.

The reverse inclusion however requires additional constraints on Q. The result we are concerned with here is:

THEOREM 4.1. *Let Q be a countable distributive family in a Heyting algebra $\mathbf{L}$ such that for all $a, b \in L$ and $X \subseteq L$:*

- *$X \in Q_J$ implies $a \sqcap X \Rightarrow b \in Q_M$; and*

- *$X \in Q_M$ implies $a \Rightarrow b \sqcup X \in Q_M$.*

Then $|a|_{S_Q} \Rightarrow_Q |b|_{S_Q} \subseteq |a \Rightarrow b|_{S_Q}$ for any $a, b \in L$. That is, if p is any prime Q-filter with $a \Rightarrow b \notin p$, then there exists a prime Q-filter q such that $p \subseteq q$, with $a \in q$ and $b \notin q$. □

A variant of this result was derived by Görnemann [6], working with prime Q-filters, not of $\mathbf{L}$ but of the minimal Boolean extension $\mathbf{B}(\mathbf{L})$ of $\mathbf{L}$, and directly applying the Rasiowa-Sikorski Lemma in $\mathbf{B}(\mathbf{L})$ to embed $\mathbf{L}$ into the Heyting algebra of upsets of a certain pre-ordering of the prime Q-filters of $\mathbf{B}(\mathbf{L})$. Restricting such filters to $\mathbf{L}$ produces Theorem 4.1 as a corollary. Later Rauszer-Sabalski [13] gave an alternative algebraic proof of this result by working within $\mathbf{L}$ and its quotient algebras. Here we will show that the Baire category approach gives an elegant proof, one showing that the two closure conditions on Q_J and Q_M in the Theorem ensure the density of certain sets of Q-filters preserving $\bigsqcup X$ and $\bigsqcap X$, respectively.

The key to this is that for each $p \in S$, the set $[p) = \{q \in S : p \subseteq q\}$ is closed in the Priestley topology on S, since it is equal to $\bigcap\{|a|_S : a \in p\}$, and is therefore compact. We view $[p)$ as a subspace of S, with its open sets being the sets $[p) \cap U$ with U open in S. $[p)$ is a compact Hausdorff space, so satisfies the Baire category theorem.

Lemma 4.2. *Let p be a Q-filter, where Q is such that*

$$X \in Q_J \text{ implies } a \sqcap X \Rightarrow b \in Q_M.$$

Then for any $X \in Q_J$, $[p) \cap S_{\sqcup X}$ is open and dense in $[p)$.

Proof. This refines Lemma 3.2(1). Let $X \in Q_J$. Since $S_{\sqcup X}$ is open in S (3.2), $[p) \cap S_{\sqcup X}$ is open in $[p)$. For density, we show that $[p) - S_{\sqcup X}$ has empty interior. To this end, let U be any basic open subset of

$$[p) - S_{\sqcup X} = [p) \cap |\bigsqcup X|_S \cap \bigcap_{x \in X} -|x|_S.$$

Now $U = [p) \cap |a|_S - |b|_S$ for some $a, b \in L$. So for all $x \in X$,

$$[p) \cap |a|_S - |b|_S \subseteq -|x|_S,$$

and so

$$[p) \cap |a \sqcap x|_S = [p) \cap |a|_S \cap |x|_S \subseteq |b|_S,$$

hence $a \sqcap x \Rightarrow b \in p$ by (4.3). This shows that $a \sqcap X \Rightarrow b \subseteq p$.

Now $\bigsqcup X$ exists, and so $\bigsqcup(a \sqcap X)$ exists and equals $a \sqcap \bigsqcup X$ (as all joins are distributive in a Heyting algebra). Hence by (4.1), $\bigsqcap(a \sqcap X \Rightarrow b)$ exists and equals $\bigsqcup(a \sqcap X) \Rightarrow b$, which is $a \sqcap \bigsqcup X \Rightarrow b$.

But $a \sqcap X \Rightarrow b \in Q_M$ by the hypothesis of the Lemma, $a \sqcap X \Rightarrow b \subseteq p$ as above, and p is a Q-filter, so

$$a \sqcap \bigsqcup X \Rightarrow b = \bigsqcap(a \sqcap X \Rightarrow b) \in p.$$

Therefore $[p) \cap |a|_S \cap |\bigsqcup X|_S \subseteq |b|_S$ by (4.3). Hence

$$U = [p) \cap |a|_S - |b|_S \subseteq -|\bigsqcup X|_S.$$

But $U \subseteq [p) - S_{\sqcup X} \subseteq |\bigsqcup X|_S$, so $U = \emptyset$. □

LEMMA 4.3. *Let p be a Q-filter, where Q is a distributive family such that*

$$X \in Q_M \textit{ implies } a \Rightarrow b \sqcup X \in Q_M.$$

Then for any $X \in Q_M$, $[p) \cap S_{\sqcap X}$ is open and dense in $[p)$.

PROOF. This refines Lemma 3.2(2). Let $X \in Q_M$. Since $S_{\sqcap X}$ is open in S (3.4), $[p) \cap S_{\sqcap X}$ is open in $[p)$. For density, let $U = [p) \cap |a|_S - |b|_S$ be a basic open subset of

$$[p) - S_{\sqcap X} = [p) \cap \bigcap_{x \in X} |x|_S - |\bigsqcap X|_S.$$

Then for all $x \in X$, $[p) \cap |a|_S - |b|_S \subseteq |x|_S$, and so

$$[p) \cap |a|_S \subseteq |b|_S \cup |x|_S = |b \sqcup x|_S,$$

hence $a \Rightarrow b \sqcup x \in p$ by (4.3). This shows that $a \Rightarrow b \sqcup X \subseteq p$.

Now as $\bigsqcap X$ exists and is distributive, $\bigsqcap(b \sqcup X)$ exists and equals $b \sqcup \bigsqcap X$. Hence by (4.2), $\bigsqcap(a \Rightarrow b \sqcup X)$ exists and equals $a \Rightarrow \bigsqcap(b \sqcup X)$, which is $a \Rightarrow b \sqcup \bigsqcap X$.

But $a \Rightarrow b \sqcup X \in Q_M$ by the hypothesis of the Lemma, $a \Rightarrow b \sqcup X \subseteq p$ as above, and p is a Q-filter, so

$$a \Rightarrow b \sqcup \bigsqcap X = \bigsqcap(a \Rightarrow b \sqcup X) \in p.$$

Therefore $[p) \cap |a|_S \subseteq |b|_S \cup |\bigsqcap X|_S$ by (4.3). Hence

$$U = [p) \cap |a|_S - |b|_S \subseteq |\bigsqcap X|_S.$$

But $U \subseteq [p) - S_{\sqcup X} \subseteq -|\bigsqcap X|_S$, so $U = \emptyset$. This shows that $[p) - S_{\sqcap X}$ has empty interior. □

The proof of Theorem 4.1 now follows from these last two Lemmas. Given the hypotheses of the Theorem, we see that the set of Q-filters in $[p)$, which is

$$[p) \cap S_Q = \bigcap\{[p) \cap S_{\sqcup X} : X \in Q_J\} \cap \bigcap\{[p) \cap S_{\sqcap X} : X \in Q_M\},$$

is the intersection of countably many sets that are open and dense in $[p)$. Therefore $[p) \cap S_Q$ is dense in $[p)$.

Now if $a \Rightarrow b \notin p$, then $[p) \cap |a|_S \not\subseteq |b|_S$ by (4.3), so $U = [p) \cap |a|_S - |b|_S$ is a non-empty open subset of $[p)$, hence must intersect the dense set $[p) \cap S_Q$. Any member of $U \cap S_Q$ is a prime Q-filter extending p and containing a but not b, as required. □

In conclusion, we comment on the fact that the original Rasiowa-Sikorski Lemma of [11] is simpler in its formulation than Theorem 4.1. It just refers to the existence of prime filters in a Boolean algebra that preserve a countable set of joins. It does not simultaneously refer to a set of meets, let alone impose the distributivity conditions of (3.5) and (3.6) – which are automatic in a Boolean algebra – or the two conditions involving the implication operation $\Rightarrow$ stated in Theorem 4.1.

The point is that in applying such results to completeness theorems for first-order logics, the joins are used to interpret the existential quantifier $\exists$ while the meets interpret the universal quantifier $\forall$. For classical logic only one of the quantifiers need be taken as primitive: $\exists$ is taken as primitive in [11], along with the negation and disjunction connectives. Thus only a result about preservations of joins was needed, and no conditions as in Theorem 4.1 were required for the definable implication connective. Of course it is a different story for intuitionistic first-order logic, where all of the connectives and the quantifiers are independent.

Rasiowa [10] gave a completeness theorem for intuitionistic first-order logic with respect to algebraic models (Heyting-algebra-valued models), using the MacNeille completion and results of McKinsey and Tarski [8] to show that the characterization could be confined to models on complete Heyting algebras (this theory is further elaborated in [12]). One could also use this approach to show the completeness of classical first-order logic with respect to complete-Boolean-algebra-valued models. But the essential additional contribution of the Rasiowa-Sikorski Lemma was to show that a Boolean-valued model can be factored though a join-preserving ultrafilter to obtain a 2-valued model, hence a model in the sense of Tarski, and that this procedure provides sufficiently many such 2-valued models to yield a proof of Gödel's completeness theorem for classical first-order logic.

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