CIRCUITS AND COCIRCUITS IN REGULAR MATROIDS

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Abstract. A classical result of Dirac's shows that, for any two edges and any $n-2$ vertices in a simple n-connected graph, there is a cycle that contains both edges and all $n-2$ of the vertices. Oxley has asked whether, for any two elements and any $n-2$ cocircuits in an n-connected matroid, there is a circuit that contains both elements and that has a non-empty intersection with all $n-2$ of the cocircuits. By using Seymour's decomposition theorem and results of Oxley and Denley and Wu, we prove that a slightly stronger property holds for regular matroids.

1. Introduction

The cocircuits of the graphic matroid, $M(G)$, are exactly the minimal edge cut-sets of G. In particular, the set of edges incident with a vertex always contains a cocircuit in the associated polygon matroid. For this reason it has become common, when seeking a matroid analogue of a graph-theoretical result, to replace the word "vertex" with the word "cocircuit" whenever it appears in the statement of the result, and to see whether the new statement holds for matroids in general.

For example, Oxley [5] asked whether the following natural matroid analogue of a result by Dirac is true. Dirac proved that for any two edges and any $n-2$ vertices in a simple n-connected graph, there is a cycle that contains both of the edges and that meets all $n-2$ of the vertices [2].

Problem 1.1. Suppose that M is an n-connected matroid, where $n \geq 2$. Let e and f be distinct elements of M and let $C_1^*, \ldots, C_{n-2}^*$ be cocircuits. Does there always exist a circuit, C, of M, such that e, $f \in C$, and $C \cap C_i^* \neq \varnothing$ for all $i \in \{1, \ldots, n-2\}$?

It is clear that the answer to this question is affirmative when $n = 2$. Oxley has proved that this is also the case when $n = 3$.

Theorem 1.2. [5, Theorem 1.1] *Suppose that M is a 3-connected matroid. For every pair, $\{e, f\}$, of distinct elements of $E(M)$ and every cocircuit, C^*, of M, there is a circuit that contains $\{e, f\}$ and meets C^*.*

Denley and Wu [1] prove that the matroid property described in Problem 1.1 holds for the class of graphic matroids. In fact they prove a stronger result. If S is a set of paths in a graph, and the members of S are pairwise vertex-disjoint, then we shall say that S is a set of *independent paths*.

Theorem 1.3. [1, Theorem 2.1] *Let G be an n-connected graph, where $n \geq 2$. Let S be a set of independent paths of total length s, and let T be a set of t cocircuits, where $s + t = n$ and $t \geq 1$. Then there is a cycle of G that contains each path of S and meets every cocircuit in T.*

We will prove that the property described in Problem 1.1 holds for the class of regular matroids. In fact we will show that it holds even using a slightly stronger version of connectivity.

Recall that a *k-separation* of a matroid, M, is a partition, (X, Y), of $E(M)$, such that $\min\{|X|, |Y|\} \geq k$, and $r(X) + r(Y) - r(M) \leq k - 1$. The matroid M is said to be *n-connected* if M has no k-separation where $k < n$. The *Tutte connectivity* (often just the *connectivity*) of M, denoted by $\lambda(M)$, is the greatest integer, n, such that M is n-connected. If there exists no such greatest integer, then $\lambda(M)$ is taken to be infinite.

Tutte connectivity fails to completely capture the notion of graph connectivity, since an n-connected matroid with at least $2(n-1)$ elements has no circuit of size less than n. The next notion of connectivity that we consider is a direct analogue of graph connectivity. For an integer $k \geq 1$, a *vertical k-separation* of a matroid, M, is a partition, (X, Y), of $E(M)$ such that $\min\{r(X), r(Y)\} \geq k$, and $r(X) + r(Y) - r(M) \leq k - 1$. If $r(X)+r(Y)-r(M) = k-1$, then we shall say that (X, Y) is an *exact vertical k-separation*. We say that M is *vertically n-connected* if M has no vertical k-separations where $k < n$, and the *vertical connectivity of M*, denoted by $\kappa(M)$, is the greatest integer, n, for which M is vertically n-connected. If no such integer exists then we define $\kappa(M)$ to be equal to $r(M)$.

Proposition 1.4. [4, Theorem 8.2.5] *If G is a connected graph, then $\kappa(M(G))$ is equal to the connectivity of G.*

For the sake of brevity, we shall assign names to two properties of classes of matroids.

Definition 1.5. Suppose that $\mathcal{M}$ is a class of matroids and that $n \geq 2$ is an integer. We shall say that $\mathcal{M}$ has the *$\mathfrak{A}(n)$-property* (respectively, the *$\mathfrak{B}(n)$-property*) if, whenever e and f are elements of M, an n-connected member (respectively, a simple, vertically n-connected member) of $\mathcal{M}$, and $C_1^*, \ldots, C_{n-2}^*$ are cocircuits of M, then there exists a circuit, C, of M, such that $e, f \in C$ and $C \cap C_i^* \neq \varnothing$ for all $i \in \{1, \ldots, n-2\}$.

If $\mathcal{M}$ has the $\mathfrak{A}(n)$-property (respectively, the $\mathfrak{B}(n)$-property) for all $n \geq 2$, we shall say that $\mathcal{M}$ has the *$\mathfrak{A}$-property* (respectively, the *$\mathfrak{B}$-property*). Thus it is known that the class of all matroids has the $\mathfrak{A}(2)$- and $\mathfrak{A}(3)$-properties, and, furthermore, the class of graphic matroids has the $\mathfrak{A}$-property. If $\mathcal{M}$ has the $\mathfrak{A}(n)$-property (respectively, the $\mathfrak{B}(n)$-property), and $M \in \mathcal{M}$, then we may abuse the terminology slightly and say that M has the respective property.

The next result shows the relation between the two properties.

Proposition 1.6. *Suppose that $\mathcal{M}$ is a class of matroids and that $n \geq 2$ is an integer. If $\mathcal{M}$ has the $\mathfrak{B}(n)$-property, then it also has the $\mathfrak{A}(n)$-property.*

Proof. We may assume that $n > 2$, for otherwise $\mathcal{M}$ certainly has the $\mathfrak{A}(n)$-property. Let M be an n-connected matroid in $\mathcal{M}$. Suppose that $e, f \in E(M)$, and $C_1^*, \ldots, C_{n-2}^*$ are cocircuits of M. If M is not uniform, then $\kappa(M) \geq \lambda(M)$ [4, Theorem 8.2.6]. Therefore $\kappa(M) \geq \lambda(M) \geq n$, so, in this case, M is vertically n-connected and simple (since $n > 2$, and the assumption that M is n-connected and not uniform leads easily to the conclusion that $|E(M)| \geq 4$). Since M has the $\mathfrak{B}(n)$-property, it follows that some circuit of M contains both e and f and meets every one of the cocircuits.

It is straightforward to show that, if M is uniform, then every cocircuit of M meets every circuit of M, so the result follows. □

Thus, if a class of matroids has the $\mathfrak{B}$-property, it also has the $\mathfrak{A}$-property.

It is easy to demonstrate that, for $n \in \{2, 3\}$, a simple, vertically n-connected matroid is also n-connected. Thus the class of all matroids has the $\mathfrak{B}(2)$- and $\mathfrak{B}(3)$-properties. Furthermore, it follows from Theorem 1.3 and Proposition 1.4 that the class of graphic matroids has the $\mathfrak{B}$-property.

Our main result shows that the class of regular matroids has the $\mathfrak{B}$-property.

Theorem 1.7. *Suppose that M is a simple, vertically n-connected regular matroid, where $n \geq 2$. If e and f are elements of M, and $C_1^*, \ldots, C_{n-2}^*$ are cocircuits of M, then there exists a circuit, C, of M such that $e, f \in C$ and $C \cap C_i^* \neq \varnothing$ for all $i \in \{1, \ldots, n-2\}$.*

We prove Theorem 1.7 by showing that the class of cographic matroids has the $\mathfrak{B}$-property. Theorem 1.7 then follows easily, as a consequence of Seymour's famous decomposition theorem for regular matroids [6], and the results by Oxley [5], and Denley and Wu [1].

2. Proof of the main result

Seymour's decomposition theorem states that any regular matroid can be constructed using direct sums, and 2- and 3-sums, starting with graphic and cographic matroids and isomorphic copies of the self-dual binary matroid R_{10}. A GF(2)-representation of R_{10} is given by the matrix $[I_5|A]$, where I_5 is the 5×5 identity matrix and A is the matrix shown in Figure 1.

The following proposition is a consequence of Seymour's decomposition theorem.

Proposition 2.1. *If M is a simple, vertically 4-connected regular matroid, then M is either graphic or cographic, or isomorphic to R_{10}.*

Proof. Assume that M is a simple, vertically 4-connected regular matroid. Then M is also vertically 3-connected. We have already noted that a simple, vertically 3-connected matroid must be 3-connected.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

FIGURE 1. The matrix $[I_5|A]$ represents R_{10} over GF(2).

We will be done if we can show that, for every 3-separation, (X, Y), of M, either $|X| = 3$ or $|Y| = 3$, for in this case M is said to be *internally* 4*-connected*, and regular matroids that are 3-connected and internally 4-connected are either graphic, cographic, or isomorphic to R_{10} [4, Corollary 13.2.6].

Therefore, suppose that (X, Y) is a 3-separation, and that $\min\{|X|, |Y|\} \geq 4$. It cannot be the case that (X, Y) is a vertical 3-separation of M, so either $r(X) < 3$, or $r(Y) < 3$. Without loss of generality we will assume the former. Since M is simple, $r(X) = 2$. But this implies that M contains a rank-2 flat containing at least four elements, and hence M has a $U_{2,4}$-minor. This contradicts our assumption that M is regular and completes the proof. □

Since we know that the class of all matroids has the $\mathfrak{B}(2)$- and $\mathfrak{B}(3)$-properties, and that graphic matroids have the $\mathfrak{B}$-property, to prove Theorem 1.7 it will suffice to prove that the class of cographic matroids has the $\mathfrak{B}$-property, and that R_{10} has the $\mathfrak{B}(n)$-property for all $2 \leq n \leq \kappa(R_{10})$. We will complete the latter task first.

It is easy to see that the first four columns of the matrix $[I_5|A]$, along with the sixth and ninth columns, form a spanning circuit of R_{10}. Since the automorphism group of R_{10} acts transitively upon ordered pairs of elements [6], it follows that every pair of elements in R_{10} is contained in a spanning circuit. It is straightforward to show that every spanning circuit meets every cocircuit, so R_{10} has the $\mathfrak{B}(n)$-property whenever $2 \leq n \leq \kappa(R_{10})$.

Before we prove that the class of cographic matroids has the $\mathfrak{B}$-property, we will discuss some preliminary material on vertical connectivity.

Unlike Tutte connectivity, vertical connectivity is not invariant under duality: the dual of a vertically n-connected matroid need not be vertically n-connected. It is not difficult to show that (X, Y) is an exact vertical k-separation of M if and only if $r(X) + r(Y) - r(M) = k - 1$, and X and Y both contain cocircuits of M. Thus, if (X, Y) is a partition of $E(M)$ such that

$$r^*(X) + r^*(Y) - r(M^*) = r(X) + r(Y) - r(M) \leq k - 1$$

and X and Y both contain circuits of M, then (X, Y) is a vertical k'-separation of M^* for some $k' \leq k$. We shall call such a separation a *cyclic k-separation of M*. We say that M is *cyclically n-connected* if M has no cyclic k-separations where $k < n$. Thus M is cyclically n-connected if and only if M^* is vertically n-connected.

We shall say that a graph, G, is *cyclically n-connected* if $M(G)$ is cyclically n-connected. Thus, if $G = (V, E)$ is a cyclically n-connected graph, and X is a subset of E such that

$$r_{M(G)}(X) + r_{M(G)}(E - X) - r(M(G)) < n - 1,$$

then it must be the case that no more than one of X and $E - X$ contains a cycle.

If X is a set of edges in G, let $V(X)$ denote the set of vertices incident with edges in X. Since $r_{M(G)}(X)$ is equal to $|V(X)| - \omega(X)$, where $\omega(X)$ is the number of connected components in the subgraph induced by X, it follows that, if G is connected, then

$$r_{M(G)}(X) + r_{M(G)}(E - X) - r(M(G)) \leq |V(X) \cap V(E - X)| - 1.$$

We shall prove the following theorem.

Theorem 2.2. *Suppose that G is a cyclically n-connected graph (where $n \geq 3$), and that $M(G)$ is cosimple. For every pair of edges, e and f, and every set of cycles, $\{C_1, \ldots, C_{n-2}\}$, there exists a cocircuit of $M(G)$ that contains both e and f and has edges in common with all the cycles in $\{C_1, \ldots, C_{n-2}\}$.*

Proving this theorem will clearly establish that cographic matroids have the $\mathfrak{B}$-property. Theorem 2.2 will follow almost immediately from the next lemma.

Lemma 2.3. *Suppose that $G = (V, E)$ is a cyclically n-connected graph (where $n \geq 3$), and that $M(G)$ is cosimple. Let e and f be edges of G, and let $C_1, \ldots, C_{n-2}$ be cycles of G. Suppose there is a cocircuit of $M(G)$ that contains e and f and has edges in common with all the cycles in $\{C_1, \ldots, C_{n-3}\}$. Then there exists a cocircuit of $M(G)$ that contains e and f and has edges in common with all the cycles in $\{C_1, \ldots, C_{n-2}\}$.*

Proof. We will assume that G has no isolated vertices. Then it is clear that G must be connected. Every cocircuit of $M(G)$ corresponds to a minimal edge cut-set of G, so there exists a partition, (A, B), of V, such that $G[A]$ and $G[B]$, the subgraphs induced by A and B, are connected, and $\Delta(A, B)$, the set of edges joining vertices in A to vertices in B, contains both e and f and shares edges with all the cycles in $\{C_1, \ldots, C_{n-3}\}$. Suppose that the lemma is false, so there exists no partition, (A', B'), of V, such that $G[A']$ and $G[B']$ are connected and $\Delta(A', B')$ contains both e and f and edges from every cycle in $\{C_1, \ldots, C_{n-2}\}$. Then, in particular, C_{n-2} has no edges in common with $\Delta(A, B)$. We may assume that the vertices of C_{n-2} are contained in A. Among such partitions, let (A, B) have the property that $|A|$ is as small as possible.

Let $\delta(A)$ be the set of vertices in A that are adjacent to vertices in B. A vertex, $v \in \delta(A)$, is *bad* if v is incident with either e or f, or if there is some cycle $C_i \in \{C_1, \ldots, C_{n-3}\}$ such that $V(C_i) \cap A = \{v\}$. (Note that every cycle in $\{C_1, \ldots, C_{n-3}\}$ has vertices in both A and B.) We observe that there can be at most $n-1$ bad vertices in $\delta(A)$. Any vertex in $\delta(A)$ that is not bad is *good*.

If v is a vertex of the graph, H, and deleting v and all the edges incident with it from H increases the number of connected components, then v is a *cut-vertex of* H.

2.3.1. Every good vertex in $\delta(A)$ is a cut-vertex of $G[A]$.

Proof. Suppose that $v \in \delta(A)$ is good, and is not a cut-vertex of $G[A]$. Consider the partition $(A - v,\, B \cup v)$. Both $G[A - v]$ and $G[B \cup v]$ are connected, both the edges e and f are contained in the cocircuit $\Delta(A - v,\, B \cup v)$, and every circuit in $\{C_1, \ldots, C_{n-3}\}$ has vertices in both $A - v$ and $B \cup v$. Therefore the minimality of A is contradicted. □

A *block* of a graph is a maximal connected subgraph that does not have a cut-vertex. Suppose that the graph, H, has blocks $B_1, \ldots, B_m$ and cut-vertices $c_1, \ldots, c_n$. The *block-cutpoint* graph of H, denoted by $\mathrm{bc}(H)$, has $\{B_1, \ldots, B_m\} \cup \{c_1, \ldots, c_n\}$ as its vertices. Two vertices are adjacent in $\mathrm{bc}(H)$ if and only if they correspond to a block and a cut-vertex contained in that block.

Let B_0 be the block of $G[A]$ that contains the cycle C_{n-2}, and let X be the set of edges of B_0.

2.3.2. The number of vertices in $V(X) \cap V(E - X)$ does not exceed the number of bad vertices in $\delta(A)$.

Proof. Suppose that $V(X) \cap V(E - X)$ contains exactly t vertices, $v_1, \ldots, v_t$. For $1 \leq i \leq t$, the vertex v_i is either a cut-vertex of $G[A]$, or a member of $\delta(A)$, perhaps both. By Claim 2.3.1, any vertex in $\{v_1, \ldots, v_t\}$ that is not a cut-vertex of $G[A]$ must be a bad vertex in $\delta(A)$. Therefore, if none of the vertices in $\{v_1, \ldots, v_t\}$ is a cut-vertex of $G[A]$, then Claim 2.3.2 is proved. Thus we will assume that $\{v_1, \ldots, v_t\}$ contains at least one cut-vertex of $G[A]$. By relabelling, we will suppose that v_i is a cut-vertex of $G[A]$ if and only if $i \leq k$, for some integer $k \in \{1, \ldots, t\}$.

For each $i \in \{1, \ldots, k\}$ consider a maximal path in $\mathrm{bc}(G[A])$ that has B_0 as one of its end-vertices, and that passes through the cut-vertex v_i. Let the other end-vertex of this path be B_i (the degree-one vertices of $\mathrm{bc}(G[A])$ correspond to blocks). It is clear that the blocks $B_1, \ldots, B_k$ are pairwise distinct, since $\mathrm{bc}(G[A])$ is a tree [3, Theorem 4.4].

For $1 \leq i \leq k$, let c_i be the unique cut-vertex of $G[A]$ that is incident with the block B_i. Suppose that B_i contains no vertex in $\delta(A)$, other than perhaps c_i. Now B_i must contain a cycle, for otherwise B_i contains only a single edge, and in this case $M(G)$ is not cosimple. Let X' be the edge set of B_i. Then both X' and $E - X'$ contain cycles (since $E - X'$ contains C_{n-2}) and, since $|V(X') \cap V(E - X')| = 1$, it follows that $(X',\, E - X')$ is a cyclic

1-separation. This contradiction shows that there must exist a vertex, u_i, in B_i that is in $\delta(A)$ and is not a cut-vertex of $G[A]$.

The set $\{u_1, \ldots, u_k\} \cup \{v_{k+1}, \ldots, v_t\}$ contains t members of $\delta(A)$, none of which are cut-vertices of $G[A]$. Hence it follows from Claim 2.3.1 that each of these vertices is a bad vertex and Claim 2.3.2 follows. □

Claim 2.3.2, and our observation that there are at most $n-1$ bad vertices in $\delta(A)$, implies that $t = |V(X) \cap V(E-X)| \leq n-1$. Since G is cyclically n-connected, $E-X$ can contain no cycle. Thus, in particular, every cycle in $\{C_1, \ldots, C_{n-3}\}$ contains at least one edge of X. But this implies that there can be at most two bad vertices in $\delta(A)$, namely, those incident with e and f. This in turn implies that $t \leq 2$.

Now $E-X$ contains no cycle, and so is a forest. But the subgraph generated by $E-X$ can contain no degree-one vertices, other than those incident with some edge in X, for otherwise $M(G)$ would fail to be cosimple. Since $|V(X) \cap V(E-X)| \leq 2$ it follows that the subgraph induced by $E-X$ contains at most two degree-one vertices, and is therefore a path. But G has minimum degree at least three, so $E-X$ must contain only one edge. However B contains at least one vertex, and all the edges incident with that vertex are in $E-X$. This contradiction completes the proof of the lemma. □

Proof of Theorem 2.2. We will prove the theorem by induction on n. We know that the class of all matroids has the $\mathfrak{B}(3)$-property, so, in particular, it follows that Theorem 2.2 holds when $n = 3$. Assume that $n > 3$, and that the theorem holds for $n-1$. Let G be a graph such that $M(G)$ is cosimple and cyclically n-connected. Suppose that e and f are edges of G, and that $\{C_1, \ldots, C_{n-2}\}$ is a collection of cycles. By the inductive hypothesis there is a cocircuit that contains both e and f and meets all the cycles in $\{C_1, \ldots, C_{n-3}\}$. The proof of the theorem now follows from an application of Lemma 2.3.

Theorem 1.7 follows from Theorem 2.2, Proposition 2.1, and Theorem 1.3.

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