

STRUCTURE OF CUBIC LEHMAN MATRICES

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ABSTRACT. A pair (A, B) of square $(0, 1)$ -matrices is called a *Lehman pair* if $AB^T = J + kI$ for some integer $k \in \{-1, 1, 2, 3, \dots\}$, and the matrices A and B are called *Lehman matrices*. This terminology arises because Lehman showed that the rows of minimum weight in any non-degenerate minimally nonideal (mni) matrix M form a square Lehman submatrix of M . In this paper, we view a Lehman matrix as the bipartite adjacency matrix of a regular bipartite graph, focussing in particular on the case where the graph is cubic. From this perspective, we identify two constructions that generate cubic Lehman graphs from smaller Lehman graphs. The most prolific of these constructions involves repeatedly replacing suitable pairs of edges with a particular 6-vertex subgraph that we call a 3-rung ladder segment. Two decades ago, Lütolf & Margot initiated a computational study of mni matrices and constructed a catalogue containing (among other things) a listing of all cubic Lehman matrices with $k = 1$ of order up to 17×17 . We verify their catalogue (which has just one omission), and extend the computational results to 20×20 matrices. Of the 908 cubic Lehman matrices (with $k = 1$) of order up to 20×20 , only two do not arise from our 3-rung ladder construction. However these exceptions can be derived from our second construction, and so our two constructions cover all known cubic Lehman matrices with $k = 1$.

1. INTRODUCTION

This paper is concerned with certain square $(0, 1)$ -matrices that we call *Lehman matrices*¹, which are defined in the following way.

Definition 1.1. A pair (A, B) of square $(0, 1)$ -matrices of the same order is called a *Lehman pair* if $AB^T = J + kI$ for some integer $k \in \{-1, 1, 2, 3, \dots\}$, where J is the all-ones matrix. An individual matrix is called a *Lehman matrix* if it is in a Lehman pair.

These matrices (at least for $k > 0$) arise naturally in combinatorial optimization, and were also studied by Bridges & Ryser [1] as a generalization of the matrix equation defining the incidence matrix of a combinatorial design. We say that a $(0, 1)$ -matrix is r -regular if each of its rows and columns sum to r . Bridges & Ryser proved (again for $k > 0$) that if (A, B) is a Lehman pair, then there are integers r and s so that A is r -regular, B is s -regular, and $k = rs - n$. We show below that the same conclusion holds for $k = -1$,

¹As detailed below, this terminology differs slightly from that of some previous authors.

and say that A has *type* (n, r, s) or just that A is an (n, r, s) -Lehman matrix (and so B is an (n, s, r) -Lehman matrix). If $k = -1$, then we say that the Lehman pair is *negative* and that A and B are negative Lehman matrices, and analogously the Lehman pair and its matrices are *positive* if $k > 0$. A small Lehman pair is shown in Figure 1; in this case A is the point-line incidence matrix of the Fano plane and $B = A$ is the same matrix. In this example, $r = s = 3$ and $k = 2$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

FIGURE 1. The Fano plane gives a $(7, 3, 3)$ -Lehman matrix

The connection to combinatorial optimization arises from attempts to classify *minimally nonideal clutters*. Here, a *clutter* (also known as a *Sperner family*) is a pair $\mathcal{C} = (V, E)$ where V is a finite set and $E \subseteq 2^V$ is a set of subsets of V such that no element of E contains another. The elements of V are usually called the *vertices* of the clutter, and those of E the *hyperedges* (or just *edges*) of the clutter. A clutter can be represented by a $(0, 1)$ -matrix, with rows indexed by E , columns indexed by V , and where each row is the incidence vector of the corresponding hyperedge. Conversely, any $(0, 1)$ -matrix with the property that there is no row whose support contains the support of another row is a *clutter matrix* (i.e., the matrix of some clutter). We will often blur the distinction between a clutter and its matrix.

If \mathcal{C} is a clutter with an $m \times n$ clutter matrix A , then \mathcal{C} (and also A) is called *ideal* if the polyhedron

$$Q(A) = \{x \in \mathbb{R}^n : Ax \succeq \mathbf{1} \text{ and } x \succeq \mathbf{0}\}$$

has *integral* vertices. Here $\mathbf{0}$ and $\mathbf{1}$ represent the all-0 and all-1 vectors respectively and \succeq indicates that the inequality holds for each coordinate. If a clutter matrix A is ideal, then any *integer* program with coefficient matrix A has the same solutions as its *linear* program relaxation (where the integer requirement is dropped). As integer programs are computationally hard to solve and linear programs computationally feasible, this is a desirable situation, and hence one that we wish to better understand.

There are notions of *deletion* and *contraction*, and hence *minors*, for clutters that are reminiscent of the same notions for graphs or matroids. If \mathcal{C} is a clutter and v is a vertex of \mathcal{C} , then $\mathcal{C} \setminus v$ (\mathcal{C} delete v) is the clutter with vertex set $V \setminus \{v\}$ whose hyperedges are the hyperedges of \mathcal{C} that do not contain v . The clutter \mathcal{C}/v (\mathcal{C} contract v) is the clutter with vertex set $V \setminus \{v\}$ whose

hyperedges are the minimal sets (under inclusion) of the form $H \setminus \{v\}$ where H is a hyperedge of \mathcal{C} . In matrix terms, if A is a clutter matrix and c a column, then $A \setminus c$ is obtained by deleting any row that contains a 1 in column c , and then deleting the entire column. The contraction A/c is produced by first deleting column c , and then deleting any rows whose support is no longer minimal under set inclusion. Any clutter (or clutter matrix) obtained by a possibly-empty sequence of deletions and contractions is a *minor* of the original clutter (or clutter matrix).

Any minor of an ideal clutter is itself ideal, which raises the possibility of an *excluded-minor* characterisation of ideal clutters. Thus we define a clutter, or a clutter matrix, to be *minimally nonideal* (mni) if it is not ideal, but every proper minor is ideal. Lehman [9] proved the seminal result that if A is an mni clutter matrix, then either A belongs to a particular sporadic family (the degenerate projective planes) or the rows of A of minimum weight form a Lehman matrix as defined in Definition 1.1. Therefore we may assume that the first n rows of any mni clutter matrix of order $m \times n$ form a Lehman matrix. This raises the possibility of a two-stage approach to understanding mni matrices, namely first characterise Lehman matrices and then understand how additional rows can be added to a Lehman matrix to form a larger mni matrix. Unfortunately, this latter step appears to be extremely difficult because the property of being mni does not behave nicely under addition of rows. In particular, it is possible that adding a row to an mni matrix may result in one that is not mni, and conversely. Cornuéjols and Guenin [5] give a readable and comprehensive treatment of ideal clutters that provides useful additional background and a wider context to this work than we have given here.

More than 20 years ago, Lütolf & Margot [10] conducted a computational search based on these observations in order to provide a collection of small mni matrices. They observed that “*we lack a good understanding of the structure of mni matrices*”, and hoped to provide a significant number of examples of mni matrices in the hope that further study would shed light on their structure. For particular values of r , they implemented an *orderly algorithm* [12] to produce a complete list (up to permutations of rows and columns) of r -regular $(0, 1)$ -matrices and then extracted the Lehman matrices from this list. They identified the Lehman matrices that are already mni (without adding any rows) and used a heuristic search to produce non-square mni matrices by adding additional rows to each Lehman matrix. Their results mostly cover the cases where $r = 3$, the matrices have order at most 17×17 , and $k = 1$. The constraints on size and valency are consequences of the very rapid increase in the numbers of regular bipartite graphs as the size, and especially the valency, increases. Lehman matrices with $k > 1$ appear to be very rare, with the incidence matrices of projective planes being the only known infinite family and the adjacency matrices of the Moore graphs giving a handful of sporadic examples. We note that this takes the *usual* adjacency matrix, and then treats it as a clutter matrix.

In this paper, we consider in detail the structure of Lehman matrices by taking a *graphical perspective*. An r -regular $(0, 1)$ -matrix can be viewed as the *bipartite adjacency matrix* of an r -regular bipartite graph and vice versa, so we say that a graph is a *Lehman graph* if its bipartite adjacency matrix is a Lehman matrix. A matrix of order $n \times n$ corresponds to a bipartite graph of order $2n$ with n black and n white vertices. We primarily consider the case when $r = 3$, where both the theoretical and computational tools give us most traction, and we call these graphs *cubic Lehman graphs*. Figures 2 and 3 show all four cubic Lehman graphs on 22 vertices. It is immediately apparent that they are qualitatively rather similar and in particular all of them seem to be very “*ladder-like*”. The first graph of Figure 2 actually *is* the cubic Möbius ladder of order 22, while the others all appear to consist of *ladder segments* of varying lengths connected together. We shall see that this is no accident and that a single construction technique involving the replacing of suitable pairs of edges by 6-vertex ladder segments accounts for almost all of the known cubic Lehman graphs.

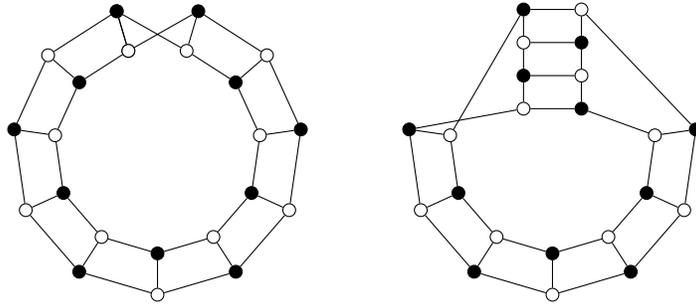


FIGURE 2. Two $(11, 3, 4)$ -Lehman graphs

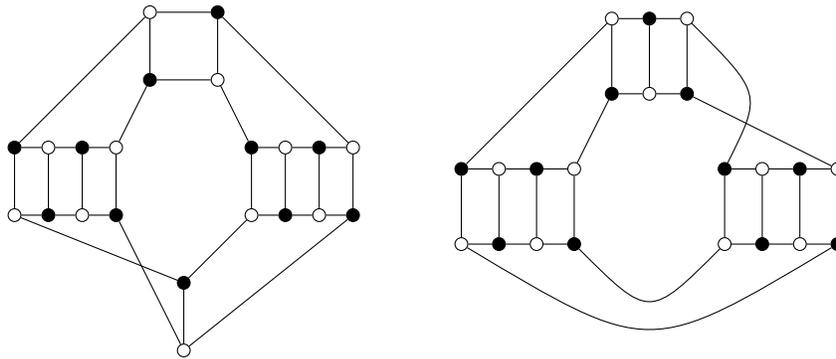


FIGURE 3. The other two $(11, 3, 4)$ -Lehman graphs

More precisely, we show that if a cubic Lehman graph with $k = \pm 1$ contains a ladder segment with 3 rungs, then it can be reduced to a smaller

cubic Lehman graph with the same k by removing the ladder segment and adding two edges to repair the regularity. We show that this process can be reversed, and used to construct huge numbers of cubic Lehman matrices with $k = \pm 1$ starting from the cubic planar ladder on 8 vertices (for $k = -1$) or cubic Möbius ladder on 10 vertices (for $k = 1$), and then repeatedly *inserting* 3-rung ladder segments.

We repeat, verify, and extend Lütolf & Margot’s computations, in the process discovering that their catalogue of 17×17 Lehman matrices omitted just one matrix — the graph corresponding to this matrix is shown in Figure 4. The sole omission is a Lehman graph of type $(17, 3, 6)$ that has no 4-rung ladder segment, but that does have 3-rung ladder segments. It is unclear as to how this graph/matrix was missed as the search described by Lütolf & Margot should certainly have constructed it at some stage.

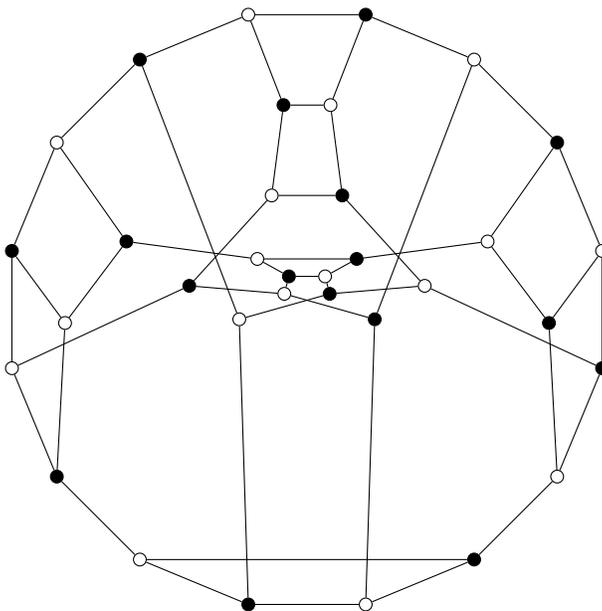


FIGURE 4. The “missing” Lehman graph on 34 vertices

The computations also give us some sense of how many of the small cubic Lehman graphs arise from ladder insertions, simply by testing which of them have a 3-rung ladder. Rather surprisingly, there are only *two* cubic Lehman graphs with $k = 1$ on up to 40 vertices (corresponding to 20×20 matrices) that do *not* have a 3-rung ladder segment (Figure 5 shows the smaller example). The smallest cubic Lehman graph with $k = 1$ is the Möbius ladder on 10 vertices, which is a $(5, 3, 2)$ -Lehman graph. Therefore all except two cubic Lehman graphs (with $k = 1$) on up to 40 vertices arise from the Möbius ladder on 10 vertices by iterated ladder insertion.

The two exceptional cubic Lehman graphs having no 3-rung ladder are also highly structured, in that their vertices can be partitioned into 4-cycles. Motivated by this example, we describe a second reduction operation, which involves replacing the 4-cycles with edges, thereby “compressing” a cubic Lehman graph with $k = 1$ into a smaller cubic Lehman graph, but this time with $k = -1$. Unlike ladder insertion and its reverse, this construction is applicable to Lehman graphs of higher valency. If an r -regular Lehman graph with $k = \pm 1$ can be partitioned into copies of the complete bipartite graph $K_{r-1,r-1}$ (which we denote *bicliques*) then each biclique can be compressed to a single edge leaving a smaller r -regular graph with $k = \mp 1$ (respectively). We call this operation *biclique compression* and we determine the circumstances under which it can be reversed (*biclique expansion*) thereby producing a second construction technique for Lehman graphs. The square mni matrices discovered by Wang [13] have the property that their vertices can be partitioned into copies of $K_{r-1,r-1}$ and so are instances of this construction.

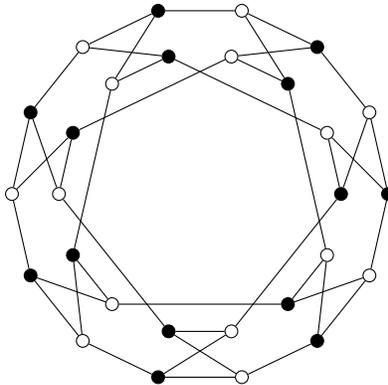


FIGURE 5. A $(14, 3, 5)$ -Lehman graph with no 3-rung ladder segment

We note that it is always possible to insert enough 3-rung ladder segments into a cubic Lehman graph with $k = 1$ to ensure that the vertices of the resulting graph can be partitioned into 4-cycles. Therefore every cubic Lehman graph with $k = 1$ can be obtained from a negative cubic Lehman graph by a combination of biclique expansion followed by 3-rung ladder reduction. In principle then, it suffices to characterise cubic negative Lehman graphs.

The paper is structured as follows: Section 2 contains all necessary background, definitions and notation for what follows. Section 3 gives a detailed analysis of the ladder reduction and insertion operations, while Section 4 does the same for biclique compression and expansion. Section 5 gives the results of a computer search for cubic Lehman graphs (with $k = \pm 1$) of order up to 20×20 . Subsequent analysis of the data reveals that all of these

Lehman graphs arise from the repeated application of our constructions (mostly ladder insertion) from a tiny number of base graphs.

Section 6 addresses the question of when the square submatrix formed by the minimum weight rows of a minimally nonideal matrix is the point-line incidence matrix of a projective plane. It is known that the point-line incidence matrix of the Fano plane (with no added rows) is mni. We conjecture that no other mni matrices, square or otherwise, can be obtained by adding (zero or more) rows to the point-line incidence matrix of a projective plane. We prove that the conjecture holds if the projective plane is the Fano plane $\text{PG}(2, 2)$ or the ternary plane $\text{PG}(2, 3)$.

2. PRELIMINARIES

The following theorem is mostly due to Bridges & Ryser [1]. We present a proof, even though it is directly modelled on theirs, because they do not consider the case $k = -1$ in either their theorem or proof.

Theorem 2.1. *Let A and B be $n \times n$ non-negative integral matrices with $n > 1$ such that $AB^T = J + kI$, where k is in $\{-1, 1, 2, 3, \dots\}$. Then $B^T A = AB^T$ and there are integers r, s such that A is r -regular, B is s -regular and $rs = n + k$.*

Proof. Let X be $J + kI$. It is straightforward to verify that

$$(1) \quad X^{-1} = \frac{1}{k}I - \frac{1}{k(n+k)}J.$$

From the hypotheses, we see that $I = (AB^T)X^{-1} = A(B^T X^{-1})$, and therefore $(B^T X^{-1})A = I$. We substitute (1) into this last equation, multiply out, and deduce that

$$(2) \quad B^T A = kI + \frac{1}{n+k}B^T J A = kI + \frac{1}{n+k}[f_i e_j]_{1 \leq i, j \leq n},$$

where f_i is the sum of row i in B^T , and e_j is the sum of column j in A . Since the trace of AB^T is equal to the trace of $B^T A$, it follows that

$$n(k+1) = \text{tr}(AB^T) = \text{tr}(B^T A) = nk + \frac{1}{n+k} \sum_{i=1}^n f_i e_i,$$

where the first equality is read off from the hypotheses, and the last comes from (2). We can now deduce that

$$(3) \quad \sum_{i=1}^n f_i e_i = n(n+k).$$

As $AB^T = J + kI$ is non-singular, we see that $f_i, e_j > 0$ for each i and j . Because $B^T A$ is integral, we can deduce from (2) that $n+k$ divides each $f_i e_j$. Therefore $f_i e_j \geq n+k$. Now (3) implies that $f_i e_i = n+k$ for each i . For distinct integers i and j , we therefore have $f_i e_i = n+k \leq f_i e_j$, which implies $e_i \leq e_j$. As i and j were arbitrary, this in turn implies that

$e_1 = \cdots = e_n$, and hence $f_1 = \cdots = f_n$. Let r be the common column sum of A , and let s be the common row sum of B^T . Thus $rs = n + k$, as asserted in the statement of the theorem, and by substituting $n + k$ for $f_i e_i$ in (2) it follows immediately that $B^T A = kI + J = AB^T$.

By repeating the arguments with A and B^T exchanged, we see that A has constant row sum, and B^T has constant column sum. Since the total sum of the entries in A is nr , each row sums to r . The same argument shows that each row or column of B^T sums to s and we are done. \square

Thus, if A is a Lehman matrix, then A has constant row- and column-sum, and moreover, the matrix B that satisfies $AB^T = J + kI$ is also a Lehman matrix. Given a non-singular $(0, 1)$ -matrix A and an integer k , the only possible matrix that might form a Lehman pair with A is

$$B = (A^{-1}(J + kI))^T$$

so (A, B) is a Lehman pair if and only if B is a $(0, 1)$ -matrix. However a matrix can belong to two different Lehman pairs — the bipartite adjacency matrix of the 6-cycle is a $(3, 2, 1)$ Lehman matrix and also a $(3, 2, 2)$ Lehman matrix.

Corollary 2.2. *Let A be a Lehman matrix satisfying $AB^T = J + kI$. Then $A^T B = J + kI$.*

Proof. Assume that $AB^T = J + kI$. Theorem 2.1 says that $B^T A = J + kI$. Therefore $J + kI = (J + kI)^T = (B^T A)^T = A^T B$. \square

Let G be a connected bipartite graph whose vertices are partitioned into two independent sets $G_B = \{b_1, b_2, \dots\}$ and $G_W = \{w_1, w_2, \dots\}$, which we refer to as the *black vertices* and the *white vertices* of G , respectively. Then the *bipartite adjacency matrix* of G is the matrix M with rows indexed by G_B and columns indexed by G_W where $M_{bw} = 1$ if and only if b is adjacent to w . Conversely, any $(0, 1)$ -matrix corresponds to a bipartite graph in the obvious fashion. If the matrix is a Lehman matrix, then its associated bipartite graph is regular. A bipartite graph is called a *Lehman graph* if its bipartite adjacency matrix is a Lehman matrix.

If v is a vertex in a loopless graph, let $N(v)$ stand for its *open neighbourhood*: that is, the set of vertices adjacent to v . The following proposition just reinterprets the definition of a Lehman matrix in graph-theoretical terms.

Proposition 2.3. *Let G be a regular bipartite graph with bipartition $\{G_B, G_W\}$. Then G is a Lehman graph if and only if there is some integer $k \in \{-1, 1, 2, \dots\}$ such that for every black vertex b , there is a set $\Gamma_G(b) \subseteq G_W$ of white vertices such that, for all $b' \in G_B$*

$$|\Gamma_G(b) \cap N(b')| = \begin{cases} k + 1, & b' = b; \\ 1, & b' \neq b. \end{cases}$$

Proof. If G is a Lehman graph with bipartite adjacency matrix A , then it belongs to a Lehman pair (A, B) and we can take the rows of B to be the (incidence vectors of the) sets $\Gamma_G(b)$. The Lehman condition for the matrix is then identical to the intersection conditions for the sets. The converse is very similar – if a collection of suitable sets $\{\Gamma_G(b)\}_{b \in G_B}$ exists, then the matrix B with the incidence vectors of these sets as its rows will form a Lehman pair with A . \square

We will call the set $\Gamma_G(b)$ the *mate* of b , and will drop the subscript G if the graph is uniquely determined by context. For both positive and negative Lehman graphs, each set $\Gamma(b)$ is a set of white vertices that *dominates* every black vertex other than b exactly once, while dominating b exactly $k + 1$ times. Figure 6 shows the mate of a vertex in the *Desargues graph*, which is a $(10, 3, 4)$ -Lehman graph. As $k = 3 \times 4 - 10 = 2$, the Desargues graph is one of the rare Lehman graphs with $k \neq \pm 1$. The set of four circled white vertices dominates the marked black vertex b three times, and all other black vertices exactly once each.

If G is a Lehman graph with bipartite adjacency matrix A , then the incidence vector x of the mate of the vertex b_i is the unique solution to $Ax = \mathbf{1} + ke_i$ (where e_i is the standard basis vector with a single 1 in the i th position). Therefore if any black vertex of a regular bipartite graph G has no mates or more than one mate, then G is not a Lehman graph. If every black vertex of G has *at least* one mate, then every black vertex must have *exactly* one mate, and the graph is a Lehman graph. By Corollary 2.2, we can swap the words “black” and “white” wherever they occur, and so every white vertex of a Lehman graph also has a unique mate. Our arguments in subsequent sections will largely be based around showing that a vertex in a candidate Lehman graph has too few, too many, or exactly the right number of mates.

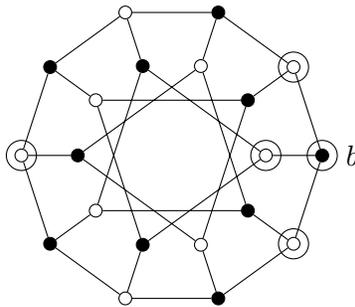


FIGURE 6. A mate in the unique $(10, 3, 4)$ -Lehman graph

Proposition 2.4. *Let G be a Lehman graph and suppose that b is a black vertex, and w is a white vertex. Then w is in the mate of b if and only if b is in the mate of w .*

Proof. Let A be the Lehman matrix associated with G , and let B be the matrix satisfying $AB^T = J + kI$. The rows of B are the incidence vectors of the mates of the black vertices, and so w is in the mate of b if and only if $B_{bw} = 1$. Now $A^T B = J + kI$, by Corollary 2.2, and so the rows of B^T are the incidence vectors of the mates of the white vertices. Thus b is in the mate of w if and only if $(B^T)_{wb} = 1$, which happens if and only if $B_{bw} = 1$. \square

The *Hadamard product* of two square matrices X and Y , written $X \circ Y$, is component-wise product of X and Y ; that is, $[X \circ Y]_{i,j} = [X]_{i,j}[Y]_{i,j}$ for all i and j . Let (A, B) be a Lehman pair satisfying $AB^T = J + kI$. Then $A \circ B$ is a $(k + 1)$ -regular matrix. Thus the bipartite graph corresponding to $A \circ B$ is $(k + 1)$ -regular. We call this the *auxiliary graph* of A (or B) and denote it $\text{aux}(A)$. If G is a cubic Lehman matrix with $k = 1$, then its auxiliary graph is 2-regular, and so the edges *not* in the auxiliary graph form a perfect matching of G . We will call the edges of this distinguished perfect matching the *rungs* of G . This terminology arises from the observation that if the graph actually is a ladder, either a cubic planar ladder or a cubic Möbius ladder, then the rungs of the Lehman graph are actually the rungs of the ladder in the normal graph-theoretical sense.

Figure 7 shows a $(14, 3, 5)$ -Lehman graph, with the diagram on the left highlighting the auxiliary graph and the diagram on the right highlighting the rungs.

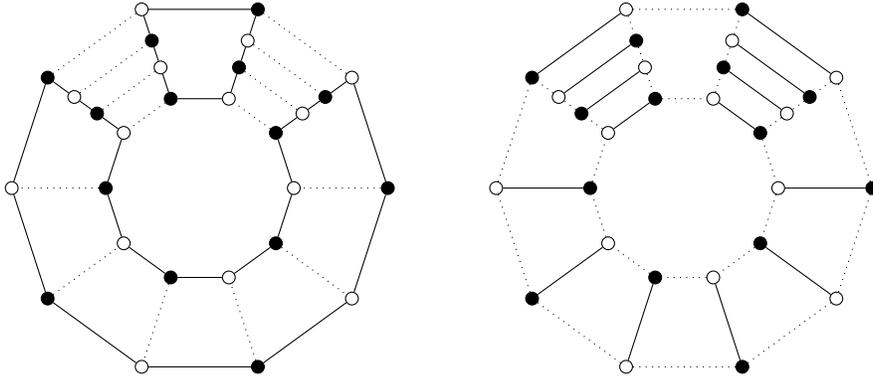


FIGURE 7. Auxiliary graph and rungs of a $(14, 3, 5)$ -Lehman graph

3. LADDER REDUCTION AND INSERTION

In this section, we describe the first of the two ways in which certain Lehman graphs can be reduced to smaller Lehman graphs, and when this operation can be reversed. This operation applies only to cubic Lehman graphs with $k \in \{-1, 1\}$.

A *3-rung ladder segment* is a 6-vertex induced subgraph isomorphic to the graph obtained from the cube Q_3 by deleting two adjacent vertices, along with their incident edges. Except for two small base cases, we show that it is always possible to delete a 3-rung ladder segment from a cubic Lehman graph and then add two edges to repair the regularity in such a way that the resulting graph is also a Lehman graph. This reduction produces Lehman graphs with six fewer vertices than the original graph, but with the same “sign” (i.e. positive or negative).

3.1. Ladder Reduction For Cubic Lehman Graphs. We define a reduction operation, which we denote *3-rung ladder reduction*, that preserves the property of being a cubic Lehman graph. More precisely, suppose that L is a 3-rung ladder segment in a cubic graph G , that there are four distinct vertices $\{w_L, b_L, w_R, b_R\}$ that are adjacent to, but outside, L (as shown in Figure 8), and that the pairs (b_L, w_R) and (w_L, b_R) are not edges of G . The dotted lines in the figure represent edges that may or may not be present. Then the *3-rung ladder reduction* of G with respect to L is the graph $G \downarrow L$ obtained from G by deleting the six vertices of L and then restoring 3-regularity by adding the edges (b_L, w_R) and (w_L, b_R) .

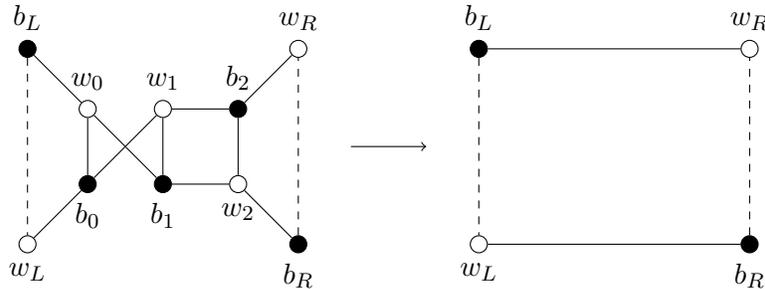
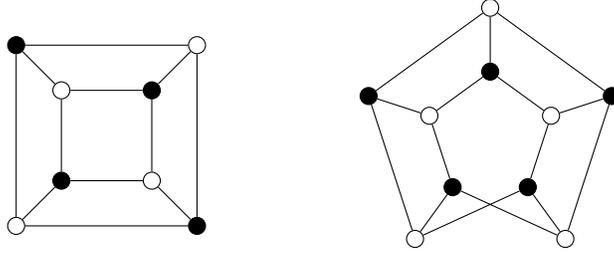


FIGURE 8. A 3-rung ladder reduction

The constraints on L and its vertices of attachment are simply the conditions required to ensure that $G \downarrow L$ is actually cubic. These conditions are necessary because there are two small Lehman graphs that each contain a 3-rung ladder, but which cannot be reduced with this operation. These graphs, illustrated in Figure 9 are the cubic planar ladder on 8 vertices (i.e. the cube) and the cubic Möbius ladder on 10 vertices. The cube cannot be reduced because the vertices $\{b_L, w_L, b_R, w_R\}$ are not distinct while the 10-vertex ladder cannot be reduced because b_L is already adjacent to w_R (similarly for w_L and b_R).

Our first lemma shows that for all larger cubic Lehman graphs, any 3-rung ladder segment automatically meets these additional constraints, and so is suitable for ladder reduction.

FIGURE 9. Lehman graphs of type $(4, 3, 1)$ and $(5, 3, 2)$

Lemma 3.1. *Suppose that G is a Lehman graph of type $(n, 3, s)$ where $k = 3s - n \in \{1, -1\}$ containing a 3-rung ladder L . If $k = -1$ and $s > 1$, or if $k = 1$ and $s > 2$, then the vertices $\{b_L, w_L, b_R, w_R\}$ are distinct, w_L is not adjacent to b_R , and w_R is not adjacent to b_L .*

Proof. First consider the case where $k = 1$ and suppose, for a contradiction, that G contains a 3-rung ladder segment L where $b_L = b_R$, as shown in Figure 10. (Here w_L and w_R may be the same or distinct.) The mate $\Gamma(b_1)$ contains two vertices from $\{w_0, w_1, w_2\}$. However any pair of vertices from that set dominates two vertices twice, and so cannot be contained in the mate of any black vertex. Therefore the four vertices $\{b_L, w_L, b_R, w_R\}$ are indeed distinct.

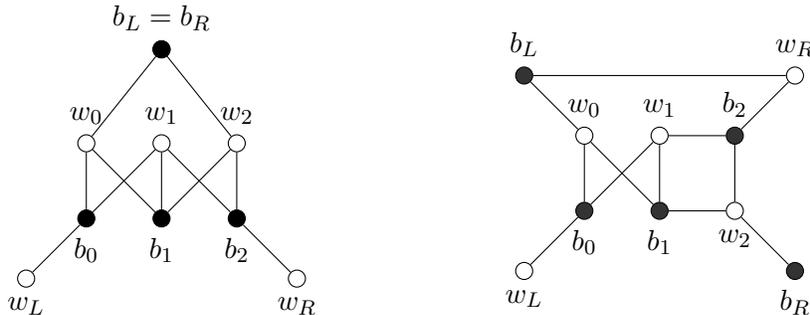


FIGURE 10. Configurations in a cubic Lehman graph

Next we will show that b_L is not adjacent to w_R . Again we proceed by contradiction starting from the second diagram of Figure 10. Suppose that $b \neq b_1$ and that $\Gamma(b)$ contains w_0 . Then $w_1, w_2 \notin \Gamma(b)$ and so to dominate b_2 it is forced that $w_R \in \Gamma(b)$ and as b_L is now twice-dominated, it follows that $b = b_L$. Thus w_0 is in at most two mates, hence it follows that $s \leq 2$, contradicting our assumptions. (We note that this configuration can occur when $s = 2$, as it is a subgraph of the cubic Möbius ladder on 10 vertices, which is a Lehman graph of type $(5, 3, 2)$.)

Next consider the case where $k = -1$ and suppose, again for a contradiction that G contains the configuration of Figure 10. Now we will consider the mates of the *white* vertices, which are sets of black vertices dominating each white vertex bar one exactly once each. The exceptional white vertex is not dominated at all. If a mate contains one vertex from $\{b_0, b_1, b_2\}$ then it cannot contain b_L for then we would have a vertex dominated twice. As w_1 is the only vertex whose mate contains no vertices from $\{b_0, b_1, b_2\}$ this means b_L is only in one mate. Therefore $s = 1$, $w_L = w_R$ and the entire graph is the cube.

Next we will show that b_L is not adjacent to w_R and again proceed by contradiction starting from the second diagram of Figure 10. Let b be an arbitrary black vertex and suppose that $w_0 \in \Gamma(b)$. Then $w_1, w_2, w_R \notin \Gamma(b)$ because that would cause either b_0, b_1 or b_L to be twice dominated. As $\Gamma(b)$ misses the entire neighbourhood of b_2 , it follows that $b = b_2$ and $s = 1$, contradicting our assumptions. \square

The main purpose of Lemma 3.1 is to ensure that for any Lehman graph other than the cube and the cubic Möbius ladder on 10 vertices, a 3-rung ladder reduction will at least give a cubic graph. Next we show that reducing 3-rung ladders also preserves the Lehman property.

Proposition 3.2. *Suppose that G is a Lehman graph of type $(n, 3, s)$ where $k = 3s - n \in \{1, -1\}$. Furthermore, assume that $s > 2$ if $k = 1$ and $s > 1$ if $k = -1$. If G contains a 3-rung ladder L , then $G \downarrow L$ is a Lehman graph of type $(n - 3, 3, s - 1)$.*

Proof. From Lemma 3.1, any 3-rung ladder L in G has the form depicted in Figure 8 where $\{w_L, w_R, b_L, b_R\}$ are distinct vertices and (w_L, b_L) and (w_R, b_R) are the only possible edges between vertices in this set. Therefore $G \downarrow L$ is at least a cubic graph.

To show that it is a Lehman graph, we show that each black vertex $b \in V(G \downarrow L)$ has a mate, and in fact we claim that

$$\Gamma_{G \downarrow L}(b) = \Gamma_G(b) \setminus \{w_0, w_1, w_2\}.$$

In other words, take the mates of all the vertices in G , completely throw away the mates of b_0, b_1 and b_2 and then just delete w_0, w_1 and w_2 from the remainder.

First we show that in both the positive and negative cases, the mate in G of any black vertex $b \notin \{b_0, b_1, b_2\}$ contains the vertex w_0 if and only if it contains w_R . This follows because if $w_0 \in \Gamma_G(b)$ then $w_1, w_2 \notin \Gamma_G(b)$ (as this would result in either b_0 or b_1 being twice-dominated). As $b \neq b_2$, the mate of b must dominate b_2 exactly once and so $w_R \in \Gamma_G(b)$. For the converse, note that if $w_R \in \Gamma_G(b)$, then $w_1, w_2 \notin \Gamma_G(b)$ because that would cause b_2 to be twice-dominated. In order to dominate b_1 , it must be the case that $w_0 \in \Gamma_G(b)$. Symmetrically, $w_2 \in \Gamma_G(b)$ if and only if $w_L \in \Gamma_G(b)$.

Next we will use this fact to show that for any $b \notin \{b_0, b_1, b_2\}$, the set $\Gamma_G(b) \setminus \{w_0, w_1, w_2\}$ is a mate for the vertex b in $G \downarrow L$. First observe that

$\Gamma_G(b)$ contains exactly one vertex in $\{w_0, w_1, w_2\}$. This follows because b_1 must be dominated at least once, and only $\Gamma_G(b)$ can contain either zero or two of these vertices (in the negative, positive case respectively). If $\Gamma_G(b)$ contains w_0 (the case w_2 is equivalent), then it also contains w_R , and although b_L is no longer dominated in $G\downarrow L$ by the deleted vertex w_0 , it is now dominated by w_R via the added edge. Hence $\Gamma_G(b)\setminus\{w_0, w_1, w_2\}$ dominates the black vertices in $G\downarrow L$ exactly the same number of times as $\Gamma_G(b)$ dominates the same vertices in G . If $\Gamma_G(b)$ contains w_1 , then it does not contain any of w_0, w_2, w_L , or w_R , for otherwise b_0 or b_2 is dominated twice, and $b \neq b_0, b_2$. Therefore every black vertex of $G\downarrow L$ is dominated by $\Gamma_G(b)\setminus\{w_0, w_1, w_2\}$ exactly the same number of times as the same vertex was dominated by $\Gamma_G(b)$ in G .

So we have shown that each black vertex has a mate, and therefore there is a solution to the defining (matrix) equation of a Lehman matrix and so $G\downarrow L$ is a Lehman graph. As $G\downarrow L$ has six fewer vertices than G , and each mate in $G\downarrow L$ has cardinality $s-1$, we see that its parameters are $(n-3, 3, s-1)$. \square

The consequence of this result is that any cubic Lehman graph on more than 10 vertices with $k=1$ and with a 3-rung ladder segment can be reduced to a smaller cubic Lehman graph with $k=1$. If the reduced graph itself has a 3-rung ladder segment, then the process can be iterated, ending with either the cubic Möbius ladder on 10 vertices or an “irreducible” Lehman graph with no 3-rung ladder segment. Our exhaustive computer search for cubic Lehman graphs with $k=1$ on up to 40 vertices show that there are just two irreducible graphs in this range, the smaller of which is shown in Figure 5.

3.2. Ladder Insertion For Cubic Lehman Graphs. Now we consider when the reverse operation of a 3-rung ladder reduction can be performed. The reverse operation consists of removing a non-incident pair of edges $e = (w_L, b_R)$ and $f = (w_R, b_L)$, adding a new 3-rung ladder segment L (again labelled as in Figure 8), and finally adding the edges (w_0, b_L) , (w_2, b_R) (w_L, b_0) and (w_2, b_R) . We denote the resulting graph by $G\uparrow\{e, f\}$ and call the pair of edges *expandable* if $G\uparrow\{e, f\}$ is a Lehman graph.

For a cubic Lehman graph with $k=1$, the partition of the edge set into the 2-regular auxiliary graph and its complementary perfect matching play a major role in determining when a 3-ladder expansion yields a larger Lehman graph. The next proposition gives some simple conditions sufficient to guarantee that $\{e, f\}$ is an expandable pair of edges. In this proof, we will frequently need to refer to the rows, columns and individual entries of several different matrices, so to avoid cramped subscripts we temporarily use more prominent notation. More precisely, if a matrix X has rows indexed by black vertices and columns by white vertices then $X(b, w)$ will refer to the (b, w) -entry of the matrix, $X(b)$ will refer to the *row* indexed by b and $X(w)$ will refer to the *column* indexed by w .

Proposition 3.3. *Let G be a $(3s-1, 3, s)$ -Lehman graph and let $e = (w_L, b_R)$ and $f = (w_R, b_L)$ be non-incident edges of G . If e and f are in the auxiliary graph of G , the mates of b_L and b_R are disjoint, and the mates of w_L and w_R are disjoint, then $G \uparrow \{e, f\}$ is a $(3(s+1) - 1, 3, s+1)$ -Lehman graph.*

Proof. Suppose that A and A' are the bipartite adjacency matrices of G and $G \uparrow \{e, f\}$ respectively. Simply translating the insertion operation into matrix terms, we see that A' is obtained from A by adding three additional rows and columns, and has the block form

$$A' = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix},$$

where A'_{11} is equal to A except that $A'_{11}(b_L, w_R)$ and $A'_{11}(b_R, w_L)$ have been changed from 1 to 0 (the dashed arrows in Figures 11 and 12 indicate that the original 1 has been “moved” leaving behind a 0 entry). Then A'_{12} is a $(3s-1) \times 3$ matrix with just two non-zero entries in the (b_L, w_0) and (b_R, w_2) positions, and A'_{21} is a $3 \times (3s-1)$ matrix with just two non-zero entries in the (b_0, w_L) and (b_2, w_R) positions. Finally A'_{22} is the 3×3 matrix shown in Figure 11.

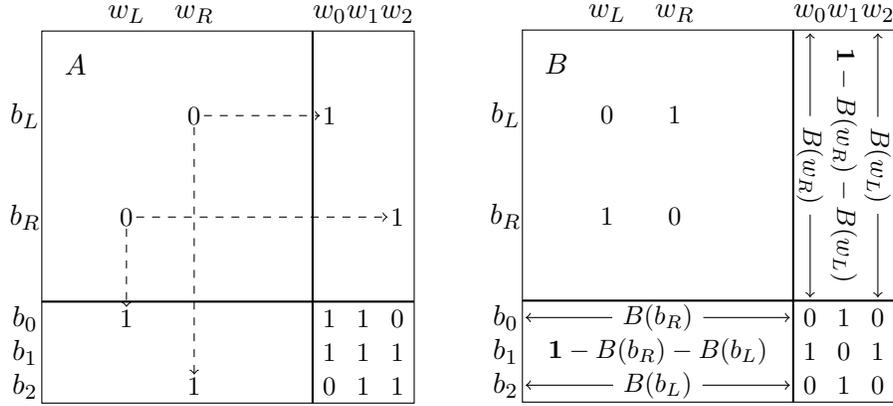


FIGURE 11. A' and B' after a 3-ladder insertion in a positive Lehman graph

Now we will define a matrix B' , and then prove that $A'B' = J + I$ as required. This is constructed from B (the partner of A in the Lehman pair (A, B)) by adding three additional rows and columns, and is illustrated in the second diagram of Figure 11. The upper left submatrix of B' is simply equal to B , while the new rows and columns of B' are defined in the following way: $B'(b_0)$ is obtained by duplicating $B(b_R)$ and extending it by adding three more coordinates equal to $(0, 1, 0)$. Thus, using block vector notation,

and with $\mathbf{1}$ being the all-ones vector, we have

$$\begin{aligned} B'(b_0) &= (B(b_R) \mid 0, 1, 0), \\ B'(b_1) &= (\mathbf{1} - B(b_L) - B(b_R) \mid 1, 0, 1), \\ B'(b_2) &= (B(b_L) \mid 0, 1, 0). \end{aligned}$$

The new columns are defined analogously using vectors $B(w_R)$, $\mathbf{1} - B(w_R) - B(w_L)$, and $B(w_L)$, and extending them as shown in Figure 11. The condition that the mates of b_L and b_R are disjoint ensures that $\mathbf{1} - B(b_R) - B(b_L)$ is a $(0, 1)$ -vector and that every column of B'_{21} sums to one, and analogously for the rows of B'_{12} . Note that because e and f are in the auxiliary graph, we know that w_R is in the mate of b_L , and w_L is in the mate of b_R . Thus $B(b_L, w_R) = B(b_R, w_L) = 1$. Since these mates are disjoint, we also see that $B(b_L, w_L) = B(b_R, w_R) = 0$.

Now we must show that for every pair of black vertices $b, b^* \in V(G) \cup \{b_0, b_1, b_2\}$ we have

$$A'(b) \cdot B'(b^*) = \begin{cases} 1, & b \neq b^*; \\ 2, & b = b^*. \end{cases}$$

We break into cases according to whether $b \in \{b_0, b_1, b_2\}$, $b \in \{b_L, b_R\}$, or $b \in V(G) \setminus \{b_L, b_R\}$.

Case 1: $b \in \{b_0, b_1, b_2\}$.

It is easy to check directly that when both $b, b^* \in \{b_0, b_1, b_2\}$ the dot products have the correct values, so we may take $b^* \in V(G)$. First assume that $b = b_1$, in which case the dot product $A'(b) \cdot B'(b^*) = B'(b^*, w_0) + B'(b^*, w_1) + B'(b^*, w_2)$ which is equal to one, as it is simply the row-sum of $B'_{12}(b^*)$. Next assume $b = b_0$, in which case the dot product $A'(b) \cdot B'(b^*) = B'(b^*, w_L) + B'(b^*, w_0) + B'(b^*, w_1)$. If $B'(b^*, w_L) = 1$, then $B'(b^*, w_0) = 0$, because the mates of b_L and b_R are disjoint. Similarly $B'(b^*, w_1) = 0$, so $A'(b) \cdot B'(b^*) = 1$. If $B'(b^*, w_L) = 0$, then exactly one of $B'(b^*, w_0)$ and $B'(b^*, w_1)$ is equal to one, so again the dot product takes the value one. By symmetry the same holds when $b = b_2$.

Case 2: $b \in \{b_L, b_R\}$.

Without loss of generality we assume that $b = b_L$. If $b^* \in V(G)$, then the dot product $A'(b) \cdot B'(b^*)$ will be equal to the dot product $A(b) \cdot B(b^*)$, except in the cases where the (b^*, w_R) -entry of B' is one (when the dot product will be reduced by one), and where the (b^*, w_0) -entry of B' is one (where it will be increased by one). By construction, either both or neither of these occur and so the net result is that $A'(b) \cdot B'(b^*) = A(b) \cdot B(b^*)$. If $b^* = b_0$, then $A'(b)$ and $B'(b^*)$ do not share a non-zero entry in the last three columns, or the w_L or w_R columns. It now follows that $A'(b) \cdot B'(b^*) = A(b_L) \cdot B(b_R) = 1$. Assume that $b^* = b_1$. Note that there is a single column from the last three columns in which $A'(b)$ and $B'(b^*)$ are both non-zero. In all columns other than the last three, $A'(b)$ contains two non-zeroes. One of these non-zeroes is in the same column as a non-zero of $B(b_R)$, and the other is in

the same column as a non-zero of $B(b_L)$. From these facts we see that $A'(b) \cdot B'(b^*) = 1$. Finally, if $b^* = b_2$, then the zero in $A'(b_L, w_R)$ means that $A'(b_L) \cdot B'(b^*) = A(b_L) \cdot B(b_L) - 1 = 1$. So now we have completed the analysis when $b = b_L$.

Case 3: $b \in V(G) \setminus \{b_L, b_R\}$.

The last three coordinates of $A'(b)$ are all zero and so any dot products involving $A'(b)$ can be rewritten using only $A(b)$. If $b^* \in V(G)$, then $A'(b) \cdot B'(b^*) = A(b) \cdot B(b^*)$, and because (A, B) is a Lehman pair, these dot products have the required values. If $b^* = b_0$, then $A'(b) \cdot B'(b^*) = A(b) \cdot B(b_R)$ which again has the required value because (A, B) is a Lehman pair. A symmetrical argument holds when $b^* = b_2$. Finally if $b^* = b_1$ then $A'(b) \cdot B'(b^*) = A(b) \cdot (\mathbf{1} - B(b_R) - B(b_L))$ which is equal to $3 - 1 - 1$ using the fact that A is cubic, and that the dot-products with the two rows of B are each equal to 1. \square

There is one simple situation where the conditions for inserting a 3-ladder are automatically satisfied, namely when there is a rung connecting a vertex of e to a vertex of f . To show this, we start with a lemma outlining how the mates of two vertices intersect.

Lemma 3.4. *Suppose that A is the matrix associated with a cubic Lehman graph, and that B is the $(0, 1)$ -matrix satisfying $AB^T = J + I$. Let $b_1 \neq b_2$ be distinct black vertices, let w be a white vertex and suppose that $A(b_1, w) = A(b_2, w) = 1$ (in other words, w is adjacent to b_1 and b_2). Then for all $w' \neq w$, either $B(b_1, w') = 0$ or $B(b_2, w') = 0$.*

Proof. If there is some w' such that $B(b_1, w') = B(b_2, w') = 1$, then the (w', w) -entry of $B^T A$ is at least 2 contradicting Theorem 2.1. \square

Corollary 3.5. *Let G be a $(3s - 1, 3, s)$ Lehman graph and let $e = (w_L, b_R)$ and $f = (w_R, b_L)$ be edges of G . If (w_L, b_L) is a rung of G , then $G \uparrow \{e, f\}$ is a $(3(s + 1) - 1, 3, s + 1)$ Lehman graph.*

Proof. By definition, the edges of G incident with (but not equal to) a rung are in the auxiliary graph, and so satisfy the first condition of Proposition 3.3. Now we show that the mates of b_L and b_R are disjoint. As $A(b_L, w_L) = A(b_R, w_L) = 1$ it follows from Lemma 3.4 that for any white vertex $w \neq w_L$, at least one of $\{B(b_L, w), B(b_R, w)\}$ is zero. As (b_L, w_L) is a rung, $B(b_L, w_L)$ is zero. This covers all white vertices and so no white vertex is in the mate of both b_L and b_R . A symmetric argument shows that the mates of w_L and w_R are disjoint. \square

The situation for negative cubic Lehman graphs is slightly different in that certain matrix entries must be 0 rather than 1, but it is very similar in style.

Proposition 3.6. *Let G be a $(3s + 1, 3, s)$ -Lehman graph and let $e = (w_L, b_R)$ and $f = (w_R, b_L)$ be edges of G . If w_L is not in the mate of b_R and w_R is*

not in the mate of b_L and the mates of b_L and b_R are disjoint, then $G \uparrow \{e, f\}$ is a $(3(s+1) + 1, 3, s+1)$ -Lehman graph.

Proof. As previously, let A, B and A', B' denote the matrices associated with G and $G \uparrow \{e, f\}$. Figure 12 shows how A' is related to A (which is the same as in positive case), and how B' is related to B (which is slightly different to the positive case). The arguments showing that the rows of A' and B' have the “right” dot product are entirely analogous to those given in Proposition 3.3 for the positive case. \square

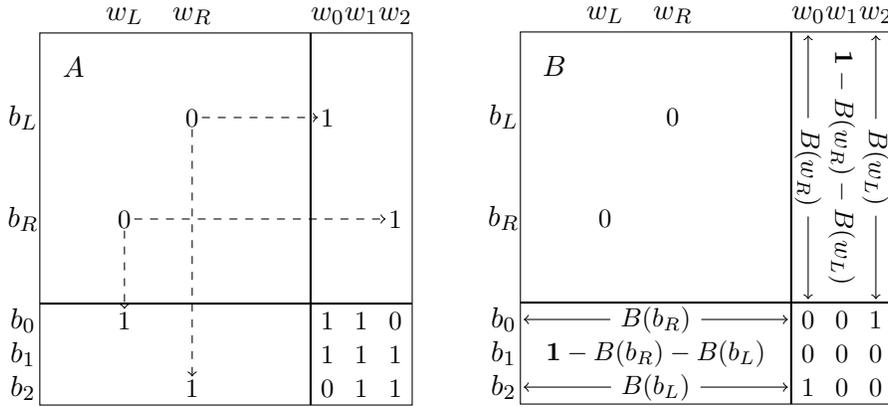


FIGURE 12. A' and B' after a 3-ladder insertion in a negative Lehman graph

4. BICLIQUE COMPRESSION AND EXPANSION

In this section we consider r -regular Lehman graphs whose vertex set can be partitioned into copies of $K_{r-1, r-1}$. By a slight abuse of notation, we will refer to these copies of $K_{r-1, r-1}$ as the *bicliques* of the graph. (Normally, ‘biclique’ refers to *any* maximal induced complete bipartite subgraph not just those of a particular valency.) We start by showing that any r -regular Lehman graph with a partition into bicliques must have $k \in \{-1, 1\}$, and moreover must have $r = 3$ if $k = -1$. Then each biclique can be compressed to an edge, yielding a smaller Lehman graph that is still r -regular, but of the opposite sign. The edges of the smaller graph corresponding to the compressed bicliques form a perfect matching. We then consider the reverse “expansion” operation, whereby the edges of a perfect matching are *expanded* into copies of $K_{r-1, r-1}$. We show that *any* perfect matching in a negative Lehman graph can be expanded to yield a larger Lehman graph with $k = 1$. In contrast, if G is a cubic Lehman graph with $k = 1$, then the only perfect matching that can be expanded to yield a Lehman graph is the perfect matching of rungs (that is, the set of edges not in the auxiliary graph of G).

4.1. Biclique compression. First we describe the process of *biclique compression*: Let G be an r -regular bipartite graph, and let \mathcal{P} be a partition of $V(G)$ into blocks each of which induces a copy of $K_{r-1, r-1}$. Each black vertex, b , of G is adjacent to the $r - 1$ white vertices in its own block, and exactly one additional vertex in a different block; we call this vertex the *out-neighbour* of b . Only Lehman graphs with certain parameters may possibly admit a partition into bicliques.

Lemma 4.1. *Let G be an r -regular Lehman graph with $r \geq 3$. If G has a partition into copies of $K_{r-1, r-1}$, then either $k = -1$ and $r = 3$, or $k = 1$.*

Proof. Suppose first that $k > 0$, and consider the mate of a black vertex b . As b has a unique out-neighbour, its mate $\Gamma(b)$ contains at least k white vertices from the block containing b . But now all the black vertices in this block (of which there are at least two) are dominated at least k times by $\Gamma(b)$ and so $k = 1$.

Now suppose that $k = -1$ and that $r > 3$. Let X be a block of \mathcal{P} , and let b_1, b_2 , and b_3 be distinct black vertices in X . Let w_1 be the out-neighbour of b_1 . Consider the mate, $\Gamma(b_2)$, of b_2 . It cannot contain any white vertex of X , since b_2 is not adjacent with a vertex in its mate. But b_1 is adjacent with a vertex in $\Gamma(b_2)$, so $w_1 \in \Gamma(b_2)$. Exactly the same argument shows that w_1 is in $\Gamma(b_3)$.

Now Proposition 2.4 says that b_2 and b_3 are both in the mate of w_1 . Therefore any white vertex in X is adjacent to two vertices in this mate, and this is a contradiction. \square

We now define a bipartite graph, $c(G, \mathcal{P})$, that will be the graph obtained by compressing each biclique to an edge. More formally, for each block $X \in \mathcal{P}$, the graph $c(G, \mathcal{P})$ contains a black vertex, b_X , and a white vertex, w_X . A black vertex b_X is adjacent to a white vertex w_Y if and only if $X = Y$, or there is a black vertex in X adjacent to a white vertex in Y . (Alternatively, we can see this as the image of G under the graph homomorphism that, for each block X , maps the white vertices of X to w_X and the black vertices of X to b_X .) The edges $\{\{b_X, w_X\} : X \in \mathcal{P}\}$ form a *perfect matching* in $c(G, \mathcal{P})$.

Proposition 4.2. *Let G be an r -regular Lehman graph with $k = 1$ and $r \geq 3$, and let \mathcal{P} be a partition of G into copies of $K_{r-1, r-1}$. Then $c(G, \mathcal{P})$ is r -regular.*

Proof. Certainly $c(G, \mathcal{P})$ is a bipartite graph with maximum degree at most r , and the number of white vertices is equal to the number of black vertices. Now it is easy to see that if $c(G, \mathcal{P})$ is not r -regular, then there is a block $X \in \mathcal{P}$, and distinct black vertices, $b_1, b_2 \in X$, such that b_1 and b_2 are adjacent with white vertices in the same block, Y . Assume that b_i is adjacent with $w_i \in Y$ for $i = 1, 2$, and note that $w_1 \neq w_2$.

Let n be the number of black vertices of G . Then Theorem 2.1 says that $rs = n + 1$, where s is the number of white vertices in each mate

$\Gamma(b)$. Moreover, each white vertex is in exactly s mates of black vertices. If $s = 1$, then each black vertex is adjacent with $n + 1$ white vertices, which is clearly impossible. Therefore $s > 1$. Hence we can choose a set $\Gamma(b)$ such that $w_1 \in \Gamma(b)$ and $b \neq b_1$. Now $\Gamma(b)$ does not contain a vertex in X , for otherwise b_1 would be adjacent with both that vertex, and with w_1 , which is impossible as b_1 is not b . Now b_2 must be adjacent with at least one vertex in $\Gamma(b)$, so it follows that w_2 is in $\Gamma(b)$. We choose a black vertex $b' \in Y$ that is not b . Now b' is adjacent with both w_1 and w_2 , and these vertices are in $\Gamma(b)$, so we have a contradiction. \square

Lemma 4.1 shows that only cubic graphs can occur in the analogous result for negative Lehman graphs.

Proposition 4.3. *Let G be a cubic negative Lehman graph, and let \mathcal{P} be a partition of G into copies of $K_{2,2}$. If G is not the graph produced from $K_{4,4}$ by removing a perfect matching, then $c(G, \mathcal{P})$ is cubic.*

Proof. We again assume that there is a block $X \in \mathcal{P}$, and distinct black vertices, $b_1, b_2 \in X$, such that the out-neighbours of b_1 and b_2 are in the same block, Y . Let w_i be the out-neighbour of b_i , for $i = 1, 2$. Let n be the number of black vertices in G . Then $3s = n - 1$, where $s = |\Gamma(b)|$ for any black vertex b . If $s > 1$, then we choose a set $\Gamma(b)$ such that $w_1 \in \Gamma(b)$ and $b \neq b_2$. Then $X \cap \Gamma(b) = \emptyset$, for otherwise b_1 is adjacent with two vertices in $\Gamma(b)$. Because $b_2 \neq b$, we see that b_2 is adjacent with exactly one vertex in $\Gamma(b)$, so w_2 is in $\Gamma(b)$. Then any black vertex in Y is adjacent with two vertices in $\Gamma(b)$. This contradiction means that $s = 1$ and hence $n = 4$. Thus G is a cubic bipartite graph with eight vertices. It immediately follows that G is isomorphic to the graph produced from $K_{4,4}$ by removing a perfect matching. \square

Our next two lemmas, one for the positive case, and one for the negative case, show that biclique compression preserves the property of being a Lehman graph, but reverses the sign.

Lemma 4.4. *Let G be an r -regular Lehman graph with $k = 1$ and $r \geq 3$, and let \mathcal{P} be a partition of G into copies of $K_{r-1, r-1}$. Then $c(G, \mathcal{P})$ is an r -regular negative Lehman graph.*

Proof. Let b_X be an arbitrary black vertex in $c(G, \mathcal{P})$. We will prove the existence of a mate for b_X , namely a set of white vertices containing no neighbours of b_X and exactly one neighbour of every other black vertex.

Let b be a black vertex of G in the block X , and w be the out-neighbour of b which (by definition) is contained in a block Y distinct from X . Let $\Gamma_G(w)$ be the mate of w in G ; this is well-defined by Corollary 2.2. So $\Gamma_G(w)$ is a set of black vertices containing two neighbours of w and exactly one neighbour of every other white vertex in G . Note that no block of \mathcal{P} contains more than one vertex of $\Gamma_G(w)$, for otherwise that block would contain a white vertex, not equal to w , that is dominated by two vertices in

$\Gamma_G(w)$. In particular, Y does not contain two vertices in $\Gamma_G(w)$. Therefore b is in $\Gamma_G(w)$, along with exactly one vertex in Y .

We now describe a set of vertices Γ in $c(G, \mathcal{P})$, and then show that it is a mate for b_X . The set Γ is defined as follows:

$$\Gamma = \{w_Z : Z \in \mathcal{P}, Z \cap \Gamma_G(w) = \emptyset\}.$$

In other words, Γ contains the white vertices of $c(G, \mathcal{P})$ corresponding to the blocks that contain no vertices of $\Gamma_G(w)$.

Now let Z be a block of \mathcal{P} and suppose that its black vertices are b_1, b_2, \dots, b_{r-1} with out-neighbours w_1, w_2, \dots, w_{r-1} that lie in blocks Z_1, Z_2, \dots, Z_{r-1} respectively (see Figure 13). The key observation is that unless $\{b_i, w_i\} = \{b, w\}$,

$$(4) \quad Z_i \cap \Gamma_G(w) = \emptyset \text{ if and only if } b_i \in \Gamma_G(w).$$

This follows from the fact that each white vertex other than w is dominated by a unique black vertex in $\Gamma_G(w)$. Assume that $\{b_i, w_i\} \neq \{b, w\}$. In this case, if $b_i \in \Gamma_G(w)$ then w_i is dominated by b_i so none of the black vertices in Z_i are in $\Gamma_G(w)$. On the other hand, if $b_i \notin \Gamma_G(w)$ then the vertex dominating w_i must lie in Z_i and so Z_i does contain a vertex of $\Gamma_G(w)$. Note that when $\{b_i, w_i\} = \{b, w\}$, we have $Z = X$ and $Z_i = Y$. In this case, Z_i contains an element of $\Gamma_G(w)$, since Y contains a single vertex of $\Gamma_G(w)$, even though $b = b_i$ is also in $\Gamma_G(w)$.

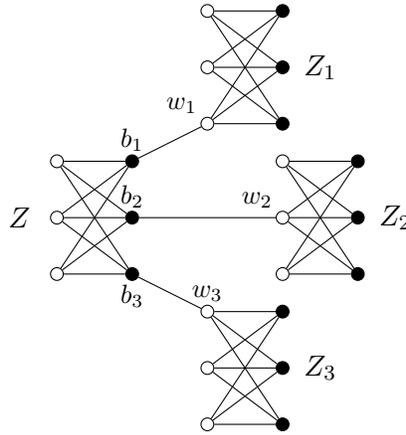


FIGURE 13. Configuration in biclique compression

In the compressed graph $c(G, \mathcal{P})$, the neighbours of b_Z are $w_{Z_1}, w_{Z_2}, \dots, w_{Z_{r-1}}$ along with w_Z , so we must count how many of these vertices are in Γ . First consider the case where $Z \neq X$. In this case, (4) implies that if $Z \cap \Gamma_G(w) = \emptyset$, then $w_Z \in \Gamma$, but none of $w_{Z_1}, w_{Z_2}, \dots, w_{Z_{r-1}}$ are in Γ and so b_Z is adjacent to exactly one vertex in Γ . On the other hand, if $Z \cap \Gamma_G(w) = \{b_i\}$, then w_Z is not in Γ . On the other hand, Z_i contains

no vertex in $\Gamma_G(w)$ by (4). Therefore w_{Z_i} is in Γ . Because Z contains exactly one vertex in $\Gamma_G(w)$, no vertex b_j is in $\Gamma_G(w)$ when $j \neq i$. Therefore $Z_j \cap \Gamma_G(w)$ is non-empty by (4), so w_{Z_j} is not in Γ . Therefore exactly one neighbour of b_Z is in Γ .

Finally we consider the case that $Z = X$. We assume $b_i = b$, in which case $Z_i = Y$. As both X and Y contain a vertex of $\Gamma_G(w)$, it follows that $w_X, w_Y \notin \Gamma$. Since Z contains a unique vertex of $\Gamma_G(w)$, we see that b_j is not in $\Gamma_G(w)$ when $j \neq i$. Therefore (4) implies that Z_j contains an vertex of $\Gamma_G(w)$. This means that none of the white vertices $w_{Z_1}, w_{Z_2}, \dots, w_{Z_{r-1}}$ are in Γ . Therefore Γ dominates every black vertex of $c(G, \mathcal{P})$ other than b_X exactly once each and does not dominate b_X at all, and so is a mate for b_X . As every black vertex of $c(G, \mathcal{P})$ has a mate, and as each mate fails to dominate a unique vertex, it follows that $c(G, \mathcal{P})$ is a negative Lehman graph. \square

Lemma 4.5. *Let G be a cubic negative Lehman graph, and let \mathcal{P} be a partition of G into copies of $K_{2,2}$. Then $c(G, \mathcal{P})$ is a cubic Lehman graph with $k = 1$.*

Proof. This proof is almost identical to that of Lemma 4.4. Once again, we select a block X , and then construct a set of vertices that will be a mate for b_X . As before, fix a vertex $b \in X$, and consider its out-neighbour w in a block Y . Then take the mate $\Gamma_G(w)$. Note that no block contains more than one vertex of $\Gamma_G(w)$. Define a set Γ of vertices of $c(G, \mathcal{P})$ by

$$\Gamma = \{w_Z : Z \in \mathcal{P}, Z \cap \Gamma_G(w) = \emptyset\}.$$

As $\Gamma_G(w)$ does not contain any neighbours of w , it does not contain b nor any of the vertices of Y . We need an analogue of (4), so suppose that Z is a block containing black vertices b_1, b_2 with out-neighbours, w_1 and w_2 respectively, that lie in blocks Z_1 and Z_2 respectively. Then unless $\{b_i, w_i\} = \{b, w\}$,

$$(5) \quad Z_i \cap \Gamma_G(w) = \emptyset \text{ if and only if } b_i \in \Gamma_G(w).$$

In the exceptional case, when $Z = X$ and $Z_i = Y$, neither X nor Y contain any vertices of $\Gamma_G(w)$.

Now we determine which neighbours of b_Z are in Γ , considering first the case that $Z \neq X$. If $Z \cap \Gamma_G(w) = \emptyset$, then $w_Z \in \Gamma$, but neither of $w_{Z_1}, w_{Z_2} \in \Gamma$, and so b_Z has a unique neighbour. If $Z \cap \Gamma_G(w) = \{b_i\}$ for some i , then w_{Z_i} is the unique neighbour of b_Z in Γ .

Finally, both X and Y contain no vertices of $\Gamma_G(w)$ and hence b_X is adjacent to both w_X and w_Y , thereby being twice dominated as required. \square

4.2. Biclique expansion. Next, we show how to reverse the construction discussed in the previous results in this section, that is, we determine when it is possible to replace each edge in a perfect matching with a biclique while preserving the property of being a Lehman graph.

Let G be an r -regular bipartite graph with black vertices b_1, \dots, b_n , and white vertices w_1, \dots, w_n . Assume that $\mathcal{M} = \{b_i w_i : 1 \leq i \leq n\}$ is a perfect matching of G . Now we define the bipartite graph $e(G, \mathcal{M})$, which will be the *biclique-expansion* of G . For each black vertex b_i of G there is a set B_i of $r - 1$ black vertices, and for each white vertex w_j , a set W_j of $r - 1$ white vertices. For each i , join every vertex of B_i to every vertex of W_i so that the subgraphs induced by $B_i \cup W_i$ are all bicliques. Now, for every edge not in \mathcal{M} , say $\{b_i, w_j\}$ where $i \neq j$, we add a single edge between B_i and W_j in such a way that $e(G, \mathcal{M})$ is r -regular. To see that this can always be done, consider the following procedure: for each black vertex b , arbitrarily order the edges of G that are incident with b , but not in \mathcal{M} . Similarly, arbitrarily order the non-matching edges incident with each white vertex. Now consider an edge $e = \{b, w\}$ which is a non-matching edge of G , with corresponding blocks B, W . If e is the i th edge in the ordering for b and the j th edge in the ordering for w , then join the i th vertex of B to the j th vertex of W . Then $e(G, \mathcal{M})$ is r -regular as the $r - 1$ edges of G incident with a given vertex b correspond to $r - 1$ edges of $e(G, \mathcal{M})$ each using a different vertex of B . Moreover, as the vertices of each colour within a biclique can be permuted arbitrarily without altering the isomorphism class of the whole graph, every choice of the edge-ordering at each vertex gives an isomorphic graph.

This construction is illustrated in Figure 14. The orderings of the edges at each vertex are given in the natural left-to-right order; for example, the two non-matching edges incident with b_2 are ordered $(\{b_2, w_1\}, \{b_2, w_4\})$, while the two non-matching edges incident with w_4 are ordered $(\{b_2, w_4\}, \{b_3, w_4\})$. So the edge between B_2 and W_4 connects the second vertex of B_2 to the first vertex of W_4 .

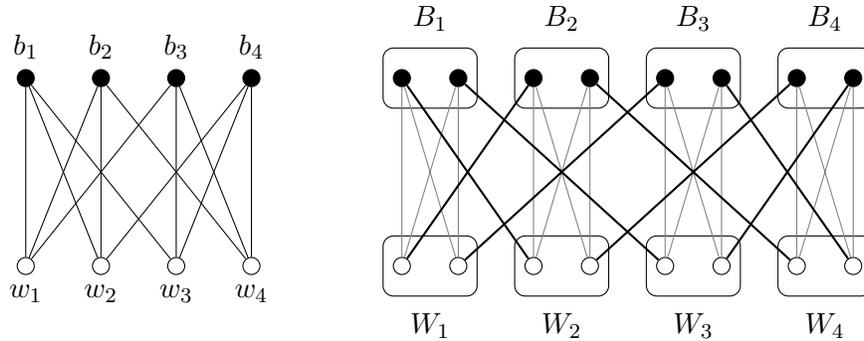


FIGURE 14. The r -regular bipartite graph, G , and $e(G, \mathcal{M})$.

Lemma 4.6. *Let \mathcal{M} be a perfect matching of the r -regular negative Lehman graph G . Then $e(G, \mathcal{M})$ is an r -regular Lehman graph with $k = 1$.*

Proof. Assume that G has black vertices b_1, b_2, \dots, b_n and white vertices w_1, w_2, \dots, w_n and that $\mathcal{M} = \{b_i w_i : 1 \leq i \leq n\}$. As above, for each i , let

B_i denote the set of $(r - 1)$ black vertices associated with b_i and W_i the set of white vertices associated with w_i . This is illustrated in Figure 15, where the complete connection between B_i and W_i is represented by a zigzag line.

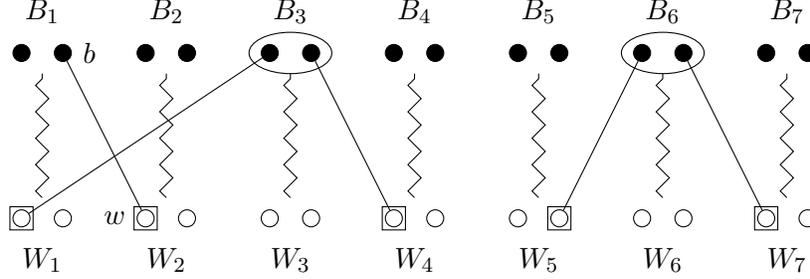


FIGURE 15. Finding the mate of b in $e(G, \mathcal{M})$ when G is negative

Now let b be an arbitrary black vertex of $e(G, \mathcal{M})$. We will show that there is a set Γ of white vertices of $e(G, \mathcal{M})$ that is a mate for b . Let w be the out-neighbour of b and assume that $b \in B_i$ and $w \in W_j$, for some $i \neq j$. In Figure 15 we have taken $i = 1$ and $j = 2$ but this is purely for illustrative purposes, and the argument only requires that $i \neq j$.

The vertex w_j in G has a mate $\Gamma_G(w_j)$, which is a set of black vertices of G that dominates every white vertex exactly once, except for w_j , which is not dominated at all. Now from this mate, define a set Γ of white vertices of $e(G, \mathcal{M})$ by taking the out-neighbours of the vertices in any set B_ℓ such that $b_\ell \in \Gamma_G(w_j)$ and adding the vertex w . In Figure 15, the marked sets B_3 and B_6 correspond to the vertices in $\Gamma_G(w_j)$, and so it is their out-neighbours, together with w , that form the purported mate of b . Note that no white vertex of G is dominated twice by $\Gamma_G(w_j)$, which means that no set W_ℓ contains more than one vertex of Γ .

There are three types of black set, namely the set B_i containing b , the sets corresponding to the vertices in $\Gamma_G(w_j)$ (that is, B_3 and B_6 in Figure 15) and the remaining sets. Note that $w_i \neq w_j$, so w_i is adjacent to exactly one vertex in $\Gamma_G(w_j)$, say b_ℓ . Furthermore, $\ell \neq i$, for otherwise w_j is dominated by a vertex in $\Gamma_G(w_j)$, which is impossible. Therefore W_i contains an out-neighbour of a vertex in B_ℓ , so W_i contains a vertex in Γ . All of the black vertices in B_i are dominated by the vertex in $W_i \cap \Gamma$. In addition, b (alone) is dominated a second time by w . Let ℓ be different from i . Assume that b_ℓ is in $\Gamma_G(w_j)$. If W_ℓ contains a vertex in Γ , then that vertex is an out-neighbour of a set that corresponds to a member of $\Gamma_G(w_j)$. But this would mean that w_ℓ is dominated by two vertices in $\Gamma_G(w_j)$, an impossibility. Therefore $W_\ell \cap \Gamma = \emptyset$, but each vertex of B_ℓ is dominated by its unique out-neighbour. On the other hand, if $b_\ell \notin \Gamma_G(w_j)$, then Γ does contain exactly one vertex of W_ℓ , but does not contain the out-neighbour of any of the vertices in B_ℓ , so every vertex of B_ℓ is dominated exactly once by the sole vertex in $W_i \cap \Gamma$.

Therefore, every black vertex of $e(G, \mathcal{M})$ is dominated once except for b , which is dominated twice, and so Γ is a mate for b . As a mate can be found for every black vertex of $e(G, \mathcal{M})$, it follows that it is a Lehman graph with $k = 1$. \square

When we expand a Lehman graph satisfying $k = 1$, we must restrict to the cubic case, by Proposition 4.1. In addition, not every perfect matching can be expanded, but only the one formed from the edges not in the auxiliary graph, which we previously termed the *rungs*.

Lemma 4.7. *Let G be a cubic Lehman graph with $k = 1$, and let \mathcal{R} be the perfect matching of rungs. Then $e(G, \mathcal{R})$ is a cubic negative Lehman graph.*

Proof. We use the notation from Lemma 4.6, and use Figure 16 as an analogous figure to Figure 15. As before, let b be an arbitrary black vertex lying in a set B_i and let w be its out-neighbour lying in block W_j . In Figure 16, $b \in B_2$ while $w \in W_3$, but the argument only requires that $i \neq j$.

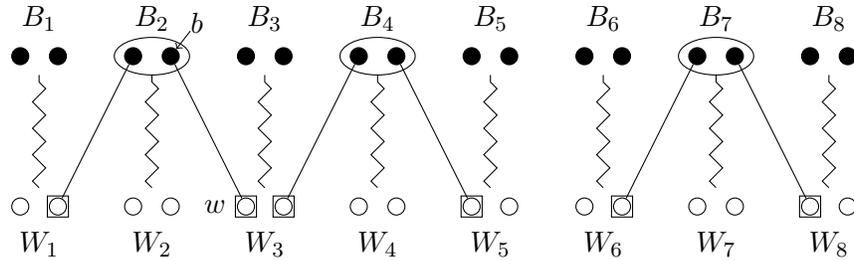


FIGURE 16. Finding the mate of b in $e(G, \mathcal{M})$ when G is positive

As G is a Lehman graph with $k = 1$, the vertex w_j has a mate $\Gamma_G(w_j)$ which is a set of black vertices dominating w_j twice and every other black vertex once. As $\{b_j, w_j\}$ is a rung of G , it follows that $b_j \notin \Gamma_G(w_j)$ and hence $b_i \in \Gamma_G(w_j)$. From this mate, we define a set Γ of white vertices of $e(G, \mathcal{M})$ by taking the out-neighbours of the vertices in the sets B_ℓ provided $b_\ell \in \Gamma_G(w_j)$ and then *removing* the vertex w .

Then arguments essentially identical to those in Lemma 4.6 apply unchanged, except for the special status of b and w . In summary, for $\ell \neq i$, the vertices of B_ℓ are either all dominated by a single vertex in W_ℓ , or each vertex in B_ℓ is dominated by its out-neighbour. Which case occurs depends on whether $b_\ell \in \Gamma_G(w_j)$ or not. As $b_i \in \Gamma_G(w_j)$ this general rule indicates that $b \in B_i$ should be dominated by its out-neighbour, which is w , but as this has been explicitly excluded from Γ , the vertex b is the unique undominated black vertex, as required for the mate of b in a negative Lehman graph. \square

We note, without proof, that if \mathcal{N} is any perfect matching of G other than the rungs, then although $e(G, \mathcal{N})$ is a well-defined r -regular graph, it is not a Lehman graph.

Let G be the graph obtained from $K_{r+1,r+1}$ by deleting a perfect matching. Then the bipartite adjacency matrix A of this graph is equal to $J - I_{r+1}$. If $B = I_{r+1}$, then $AB^T = J - I_{r+1}$ and so G is a negative Lehman graph of type $(r+1, r, 1)$. Up to isomorphism, G has a unique perfect matching \mathcal{M} and the expanded graph $e(G, \mathcal{M})$ is a Lehman graph of type $(r^2 - 1, r, r)$. The clutter matrices of these graphs are those found by Wang [13], who also showed that these matrices are minimally nonideal.

5. A CATALOGUE OF CUBIC LEHMAN MATRICES

In this section we describe the computation of a catalogue of cubic Lehman matrices with $k = \pm 1$ thereby partially extending, verifying, and in one instance correcting, the catalogue of Lütolf & Margot. Here we recall that our Lehman matrices are square but may have $k = -1$, while their Lehman matrices need not be square, but must have $k > 0$. In addition, they were primarily focussed on (not-necessarily square) mni matrices and so the two catalogues overlap, but are not directly comparable.

We used Gunnar Brinkmann's cubic graph generator `minibaum` [2] to generate cubic bipartite graphs on up to 40 vertices from which to extract the Lehman graphs. Although this is a huge computation, it is possible to prune the generation tree to some extent. For example, if two vertices of degree three in a partially-constructed graph have the same neighbourhood, then any cubic graph constructed by adding additional vertices will have a singular bipartite adjacency matrix, so there is no point in further extending that graph. It is certainly possible to do more sophisticated pruning, but there is a complicated trade-off between computer time, programming time, and the chance of introducing subtle bugs. In the end we opted to modify `minibaum` by the smallest amount required to make the computation feasible, in the end spending about two months on four 12-core computers.

There are two natural notions of equivalence for Lehman graphs/matrices one of which is a refinement of the other. In the graph context, we may view two Lehman graphs as being equivalent if and only if there is a *colour-preserving* isomorphism between them. However it is equally natural to just take graph isomorphism (ignoring the vertex colours) as the appropriate concept of equivalence, which we denote *colour-blind isomorphism*. In matrix terms, this corresponds to viewing two matrices as equivalent if one can be obtained from the other by using only row- and column-permutations, or whether matrix transposition is also permitted. The relationship between the two notions of equivalence is straightforward. Each Lehman graph (counted up to graph isomorphism) contributes either 1 or 2 to the count of Lehman graphs up to colour-preserving graph isomorphism, depending on whether it has a colour-reversing automorphism or not, respectively. In Tables 1 and 2 the columns $\ell(n)$ and $\ell'(n)$ give the numbers of Lehman graphs up to colour-blind and colour-preserving isomorphism respectively.

(n, r, s)	$2n$	$\ell(n)$	$\ell'(n)$
(5, 3, 2)	10	1	1
(8, 3, 3)	16	2	2
(11, 3, 4)	22	4	4
(14, 3, 5)	28	17	18
(17, 3, 6)	34	71	98
(20, 3, 7)	40	491	785

TABLE 1. Cubic Lehman matrices (with $k = 1$) of order at most 20

(n, r, s)	$2n$	$\ell(n)$	$\ell'(n)$
(4, 3, 1)	8	1	1
(7, 3, 2)	14	1	1
(10, 3, 3)	20	2	2
(13, 3, 4)	26	5	5
(16, 3, 5)	32	19	21
(19, 3, 6)	38	105	154

TABLE 2. Negative cubic Lehman matrices of order at most 19

The numbers given in Table 1 differ in only one place from the corresponding numbers found by Lütolf & Margot [10] — we find 98 Lehman graphs (up to colour-preserving isomorphism) of type (17, 3, 6) compared to the 97 that they found. By downloading the files associated with their paper, we have confirmed that this is a genuine omission and that our catalogues correspond in all other respects. This omitted graph is shown in Figure 4.

5.1. Cubic mni matrices. Recall that a clutter matrix is minimally non-ideal (mni) if it is not ideal, but all of its proper minors are ideal. A (not-necessarily square) mni matrix A consists of a (necessarily square) Lehman submatrix containing all the rows of minimum weight, say r , along with zero or more additional rows of strictly greater weight. It is easy to see that the polyhedron $Q(A)$ has a vertex at $\frac{1}{r}\mathbf{1}$ (the point with all coordinates equal to $\frac{1}{r}$) and [9] showed that if A is mni then this is the *unique* fractional vertex of $Q(A)$.

There are two common ways to represent a polyhedron computationally — the H -representation is a list of inequalities defining half-spaces whose intersection is the polyhedron, and the V -representation is a list of the vertices. In our situation, if A is an $m \times n$ clutter matrix, then $Q(A)$ is defined by $m + n$ inequalities. The first m inequalities, one per row of A , are all of the form

$$a_{i,0}x_0 + a_{i,1}x_1 + \cdots + a_{i,n}x_n \geq 1,$$

Parameters	Number
(5, 3, 2)	1
(8, 3, 3)	2
(11, 3, 4)	4
(14, 3, 5)	9
(17, 3, 6)	4
(20, 3, 7)	0

TABLE 3. Numbers of cubic mni Lehman matrices

while the remaining n inequalities, one per column of A , are simple non-negativity constraints of the form $x_j \geq 0$. Software is readily available (e.g. in SageMath) to convert the H -representation of a polyhedron to the V -representation (and vice versa if desired). For the sizes we are considering (matrices with around 20 rows and columns), the process takes no more than a second or so, and as the coordinates of the vertices of $Q(A)$ are rational, it is easy to check how many are fractional.

In this fashion, we can determine which of the Lehman matrices we have constructed are mni themselves (i.e. with no additional rows), and these numbers are shown in Table 3 (all these are equivalent to their transpose). This data is consistent with the view that only a small cubic Lehman graph can be mni, and that in general more constraints tend to create more fractional vertices. This seems convincing enough that we are willing to make the following conjecture.

Conjecture 5.1. *There are no $n \times n$ cubic mni matrices for $n > 17$.*

As difficult as it seems to understand square mni matrices, even just cubic ones, the situation is far worse when considering non-square mni matrices. Adding a row to a clutter matrix alters the polyhedron by intersecting it with a new halfspace, thereby both cutting off some existing vertices and adding some new ones. If the new halfspace cuts off more fractional vertices than it creates then (in some sense) the matrix is getting “closer” to being mni. Lütolf & Margot used a heuristic based on this general idea to add collections of rows to the (17, 3, 6) Lehman matrices that they had constructed, and succeeded in extending them to an mni matrix about 30% of the time. We have not attempted to extend this part of their project.

6. PROJECTIVE PLANES

The general problem of deciding when a Lehman matrix can be extended to an mni matrix by adding rows seems very difficult. In this section, we consider a very special sub-case of this problem, namely when the Lehman matrix in question is the point-line incidence matrix of a non-degenerate projective plane.

To do this, we need some more notation and background results about clutters and mni matrices. A *transversal* of the clutter $\mathcal{C} = (V, E)$ is a

subset of V having non-empty intersection with every element of E . The *blocker* of \mathcal{C} , written $b(\mathcal{C})$, is the clutter having V as its vertex set, and the set of all minimal transversals of \mathcal{C} as hyperedges. Edwards and Fulkerson made the observation that $b(b(\mathcal{C})) = \mathcal{C}$ [7]. The blocker involution exchanges deletion and contraction: $b(\mathcal{C} \setminus v) = b(\mathcal{C})/v$ for all $v \in V$. The blocker of an ideal clutter is also ideal [8] (see also [4, Theorem 1.17]). Therefore the blocker of an mni clutter is also mni. Let A be an mni clutter matrix. We say that the *core* of A , written $\text{core}(A)$, is the submatrix consisting of the minimum weight rows of A . For an integer $t \geq 2$, let J_t be the clutter $(\{0, 1, \dots, t\}, \{\{1, \dots, t\}, \{0, 1\}, \{0, 2\}, \dots, \{0, t\}\})$. Then J_t is mni [4, Exercise 4.2]. We call any clutter isomorphic to J_t a *degenerate projective plane*. We can now state the fundamental theorem of minimally nonideal matrices, which is due to Lehman ([9], see also [4, Theorem 4.3 and Corollary 4.5]).

Theorem 6.1 (Lehman). *Let \mathcal{C} be a minimally nonideal clutter. Let A be the clutter matrix corresponding to \mathcal{C} , and let B be the matrix corresponding to $b(\mathcal{C})$. If \mathcal{C} is not a degenerate projective plane, then we can permute rows as necessary so that $(\text{core}(A), \text{core}(B))$ is a Lehman pair.*

Any non-degenerate projective plane can be considered as a clutter, where the vertex set is the set of points, and the hyperedges are the lines. These clutters are heavily-studied and highly-structured, so we might hope that it would be possible to classify mni matrices whose core is such a clutter. We make the following conjecture:

Conjecture 6.2. *If A is a minimally nonideal matrix whose core is a non-degenerate projective plane, then A is square and equal to the point-line incidence matrix of the Fano plane.*

Novick [11] showed that the Fano plane is the only non-degenerate projective plane whose point-line incidence matrix is mni, thus proving Conjecture 6.2 under the assumption that A is square. Her proof was not phrased in geometric terms, so we give an alternative proof that rehearses some of the terminology and ideas we will use later. A *triangle* in a projective plane is the union of three lines that do not share a common point. The *corners* of the triangle are the points in two of the lines.

Proposition 6.3 (Novick [11]). *Let A be the point-line incidence matrix of a non-degenerate projective plane \mathcal{P} . Then A is an mni matrix if and only if \mathcal{P} is the Fano plane.*

Proof. One direction of the proof is well known. Suppose that \mathcal{P} is a projective plane of order $k > 2$, and let T be a triangle of \mathcal{P} . Form a minor of \mathcal{P} (now viewed as a clutter) by first deleting all points not in T , and then contracting all the points in T except for the corners. As each line of \mathcal{P} contains at least four points, the only lines contained in T are the three lines of the triangle. This means that after deleting the points not in T , we obtain

a clutter with exactly three hyperedges. After subsequently contracting the non-corner points, we have a minor isomorphic to the degenerate projective plane J_2 . As \mathcal{P} has a proper minor that is mni, it is not mni itself. \square

Our next results show that no counterexample to Conjecture 6.2 can have a core that is the point-line incidence matrix of either the Fano plane $\text{PG}(2, 2)$ or the ternary plane $\text{PG}(2, 3)$. (In [10], Lütolf & Margot denote $\text{PG}(2, 3)$ as $L_{13}^4(1)$. In a footnote to that paper, they assert without proof the result stated in Theorem 6.5.)

Theorem 6.4. *If A is a minimally nonideal clutter matrix whose core is the point-line incidence matrix of the Fano plane $\text{PG}(2, 2)$, then A is square.*

Theorem 6.5. *There is no minimally nonideal clutter matrix whose core is the point-line incidence matrix of $\text{PG}(2, 3)$.*

The proofs of these results rely on the following lemmas. We are hopeful that the geometric approach underlying these lemmas may be further developed. A *blocking set* of a projective plane \mathcal{P} is a minimal set of points that *meets* every line of \mathcal{P} , but does not completely *contain* any of the lines (see Bruen [3]).

Lemma 6.6. *Let A be a minimally nonideal clutter matrix for a clutter \mathcal{C} whose core is a non-degenerate projective plane \mathcal{P} . Then*

- (i) *The blocker $b(\mathcal{C})$ also has \mathcal{P} as its core.*
- (ii) *Every hyperedge of \mathcal{C} is either a line of \mathcal{P} , or contains a blocking set, but not a line, of \mathcal{P} .*

Proof. Let the order of \mathcal{P} be k . From the properties of projective planes, we know that $\text{core}(A)\text{core}(A)^T = J + kI$. Furthermore, Theorem 6.1 says that $\text{core}(A)\text{core}(b(A))^T = J + k'I$, for some positive integer k' . Assume that $\text{core}(b(A)) \neq \text{core}(A)$, so that $k' \neq k$. The dot product of any row in $\text{core}(A)$ with the corresponding column of $\text{core}(b(A))^T$ is $k'+1$, but the row of $\text{core}(A)$ has weight $k+1$. Therefore $k' < k$. Theorem 2.1 says that $(k+1)s' = n+k'$, where each row or column sum of $\text{core}(b(A))$ is s' . The same theorem says that $(k+1)^2 = n+k$. This implies that $(k+1)(k+1-s') = k-k'$. But the right side of this equation is positive and less than k , and if the left side is positive, it is at least $k+1$. This contradiction proves (i). As $b(\mathcal{C})$ contains the lines of \mathcal{P} , and $b(b(\mathcal{C})) = \mathcal{C}$, it follows that every hyperedge of \mathcal{C} is a transversal to \mathcal{P} . Therefore each hyperedge of \mathcal{C} is a line of \mathcal{P} or a blocking set of \mathcal{P} . As \mathcal{C} is a clutter, no hyperedge of \mathcal{C} can properly contain a line of \mathcal{P} . \square

For the remainder of this section, we fix some notation as follows. Throughout, A is a minimally nonideal clutter matrix for a clutter \mathcal{C} whose core is a projective plane \mathcal{P} . Let C denote the vertices of \mathcal{C} (and hence \mathcal{P}), although we will usually call them *points* in our geometric arguments. Through an abuse of notation, we also use \mathcal{C} to refer to the set of hyperedges in \mathcal{C} . We frequently argue with respect to an arbitrary triangle T

which, unless otherwise stated, is the union of three lines L_x , L_y and L_z that pairwise meet in three distinct *corner points* $x = L_y \cap L_z$, $y = L_x \cap L_z$ and $z = L_x \cap L_y$. The non-corner points of the three lines are partitioned into the sets $X = L_x - \{y, z\}$, $Y = L_y - \{x, z\}$, and $Z = L_z - \{x, y\}$. (For convenience, this notation is illustrated in Figure 17.)

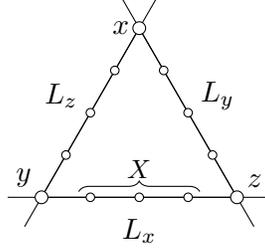


FIGURE 17. A triangle in a projective plane

Lemma 6.7. *Let T be a triangle of \mathcal{P} , and let $R \in \mathcal{C} - \mathcal{P}$ be such that $R \subseteq T$. Then one of the following statements holds.*

- (i) $R = X \cup Y \cup Z$.
- (ii) R has the form $\{x\} \cup Y \cup Z \cup X'$, $\{y\} \cup X \cup Z \cup Y'$, or $\{z\} \cup X \cup Y \cup Z'$, where X' , Y' , and Z' are, respectively, non-empty subsets of X , Y , and Z .
- (iii) $R = \{x, y, z\} \cup X' \cup Y' \cup Z'$, where X' , Y' , and Z' are, respectively, proper subsets of X , Y , and Z .

Proof. This argument falls into cases according to how many of the corners of the triangle are contained in R .

If R contains all three corners of T then because R does not contain a line of \mathcal{P} , statement (iii) holds.

Now suppose that R avoids at least one corner, say $x \notin R$. If R avoids some point $x' \in X$, then R does not intersect the line through x and x' , a contradiction to Lemma 6.6 (ii). Thus $X \subseteq R$. Similarly, if R avoids y it contains Y and if it avoids z it contains Z . Thus, if $x, y, z \notin R$, statement (i) holds.

Now R cannot contain exactly two of x , y , and z , because if it contains, say, $\{x, y\}$ and avoids z , then $Z \cup \{x, y\} = L_z$ is contained in R , contradicting Lemma 6.6. Up to symmetry, the last case we must consider is when $x \in R$ and $y, z \notin R$. Then $Y \cup Z \subseteq R$. Moreover, R must contain a non-empty set of points in X , or it avoids L_x . Thus statement (ii) holds. \square

Let T be a triangle in \mathcal{P} , and let $R \subseteq T$. If R has the form indicated in (i), (ii), or (iii) in Lemma 6.7, then we refer to R as, respectively, a *0-corner*, *1-corner*, or *3-corner* of T . If R is a 1-corner containing x , then we call R an *x -based 1-corner* (or a *1-corner based at x*). The terms *y -based* and *z -based* are defined similarly.

Lemma 6.8. *Let T be a triangle of \mathcal{P} . Then \mathcal{C} contains the 0-corner of T .*

Proof. First suppose that \mathcal{C} does not contain any 0-corner or 1-corner of T . As in the proof of Lemma 6.3, let \mathcal{H} be the clutter obtained by deleting all points outside T , and contracting $X \cup Y \cup Z$. Then the hyperedges of \mathcal{H} are the minimal sets of the form $R - (X \cup Y \cup Z)$, for some $R \in \mathcal{C}$ contained in T . By Lemma 6.7, the only hyperedges of \mathcal{C} contained in T are the lines of T and possibly some 3-corners. The lines of T give rise to the sets $\{x, y\}$, $\{y, z\}$, and $\{x, z\}$, while every 3-corner becomes $\{x, y, z\}$ and therefore cannot be minimal. It follows that \mathcal{H} is isomorphic to J_2 . Since \mathcal{C} contains a proper J_2 -minor, it cannot be mni, and we have a contradiction. Therefore \mathcal{C} contains a 0-corner or a 1-corner of T .

Now suppose that \mathcal{C} does not contain the 0-corner of T , in which case \mathcal{C} at least one 1-corner of T . Without loss of generality, we can assume that \mathcal{C} contains one or more x -based 1-corners of T , say R_1, \dots, R_d . Now let $X_i = X \cap R_i$ for $i = 1, \dots, d$, and let W be a minimal subset of X such that $W \cap X_i \neq \emptyset$ for $i = 1, \dots, d$. Since W is minimal, for every $w \in W$ there exists an $i \in \{1, \dots, d\}$ such that $X_i \cap W = \{w\}$.

Now consider two cases, depending on whether all the 1-corners of T contained in \mathcal{C} are based at x or not. In both cases we show that \mathcal{C} has a proper minor isomorphic to a degenerate projective plane, contradicting the fact that A is minimally nonideal.

First suppose that all the 1-corners of T are based at x . Define the minor $\mathcal{H} = \mathcal{C} \setminus ((C - T) / (Y \cup Z \cup (X - W)))$. The sets in \mathcal{H} are the minimal sets of the form $R - (Y \cup Z \cup (X - W))$ for some $R \in \mathcal{C}$ such that $R \subseteq T$. The only members of \mathcal{C} contained in T are the lines of T (which produce the sets $\{x, y\}$, $\{x, z\}$ and W), the x -based 1-corners and possibly some 3-corners (which all contain $\{x, y\}$ and will therefore not be minimal). The 1-corners that have exactly one vertex in W produce sets $\{x, w\}$ for every $w \in W$; the other 1-corners are not minimal. Thus the sets in \mathcal{H} are $\{x, y\}$, $\{x, z\}$, and $\{\{x, w\}\}_{w \in W}$. Thus \mathcal{H} is isomorphic to $J_{|W|+2}$.

Now suppose that \mathcal{C} contains a y -based 1-corner, as well as an x -based 1-corner. Define $\mathcal{H} = \mathcal{C} \setminus ((C - T) \cup \{z\}) / (Y \cup Z \cup (X - W))$. The sets in \mathcal{H} arise from members of \mathcal{C} contained in $T - z$. These are L_z and the 1-corners based at x and y . The line L_z becomes $\{x, y\}$, all the y -based 1-corners become the set $\{y\} \cup W$ and, as before, the x -based 1-corners produce $\{\{x, w\}\}_{w \in W}$. Therefore \mathcal{H} is isomorphic to $J_{|W|+1}$, and the proof is complete. \square

Lemma 6.9. *Assume that the order of \mathcal{P} is greater than two. Then \mathcal{C} is not the clutter obtained from \mathcal{P} by adding all the 0-corners of triangles of \mathcal{P} .*

Proof. Assume the lemma fails, so that \mathcal{C} contains the lines of \mathcal{P} , the 0-corners of triangles, and no other hyperedges. Consider three copunctual lines L_1, L_2 , and L_3 , through the point x , and a fourth line, L_4 , such that $x \notin L_4$. Let X be union of these four lines. For $i = 1, 2, 3$, let y_i be the point of intersection of L_i with L_4 . This configuration contains three triangles

$T_1 = L_2 \cup L_3 \cup L_4$, $T_2 = L_1 \cup L_3 \cup L_4$, and $T_3 = L_1 \cup L_2 \cup L_4$. Let R_1 , R_2 , and R_3 be the 0-corners of T_1 , T_2 , and T_3 respectively. For $i = 1, 2, 3$ pick a point $a_i \in L_i - \{x, y_i\}$. The clutter obtained from \mathcal{C} by deleting all points not in X and contracting all points in X other than $\{x, y_1, y_2, y_3, a_1, a_2, a_3\}$ is the Fano plane. Therefore \mathcal{C} has a proper minor isomorphic to an mni clutter, implying that \mathcal{C} is not mni, which contradicts our assumption. \square

Now we can prove Theorems 6.4 and 6.5.

Proof of Theorem 6.4. By Lemma 6.6, any set in $\mathcal{C} - \mathcal{P}$ contains a blocking set of the Fano plane. As the Fano plane has no blocking sets (see Bruen [3]), it follows that $\mathcal{C} - \mathcal{P}$ is empty and A is the point-line incidence matrix of the Fano plane. \square

Proof of Theorem 6.5. In this case, \mathcal{C} contains at least the lines and the 0-corners of the ternary plane $\text{PG}(2, 3)$. The only blocking sets of $\text{PG}(2, 3)$ are the 0-corners of the triangles (Di Paola [6]), and as \mathcal{C} is a clutter, it now follows that it contains exactly the lines and the 0-corner of every triangle and no other sets, contradicting Lemma 6.9. \square

7. OPEN PROBLEMS

In the cubic Lehman graphs described in this paper, 4-cycles play a major role, either as part of a ladder segment or as part of a biclique partition. While there is nothing in the definition of Lehman graph that immediately implies the existence of 4-cycles, it seems difficult to find Lehman graphs without them.

Question 7.1. *Are there any cubic Lehman graphs (with $k = 1$) of girth at least 6?*

The restriction of this question to $k = 1$ is necessary because the Heawood graph and the Desargues graph, which are Lehman graphs of type $(7, 3, 3)$ and $(10, 3, 4)$ respectively, both have girth 6 and $k = 2$.

Given any cubic bipartite graph with $2n$ vertices, we can count the number of vertices of one colour that have a valid mate, knowing that the graph is a Lehman graph if and only if this number is n . In a Lehman graph every vertex, black or white, has a mate, but if the graph is not Lehman then there may be a different number of black vertices with mates than white ones with mates. None of the cubic bipartite graphs on $17 + 17$ vertices with girth 6 have more than six vertices with mates, far short of the 17 required for a Lehman graph. So if we regard the number of vertices with mates as a measure of how ‘‘Lehman-like’’ a graph is, then these small girth 6 graphs are not all close to being a Lehman graph.

Question 7.2. *Are there any cubic mni Lehman matrices (with $k = 1$) of order greater than 17×17 ?*

Table 3 shows that as s increases, the number of $(3s - 1, 3, s)$ Lehman matrices that are mni first increases, reaching a maximum at $s = 5$, and then decreases, actually reaching zero when $s = 7$. Given that the average number of fractional points in the polytopes $Q(A)$ increases rapidly as the order of A increases, it would not be surprising if square cubic mni matrices only occurred for small orders. Indeed we conjecture that this is so (Conjecture 5.1).

Question 7.3. *Are there any mni matrices whose core is the point-line incidence matrix of a non-degenerate projective plane of order greater than two?*

If an mni matrix has a non-degenerate projective plane as a core, then the hyperedges all have geometric interpretations as lines or blocking sets, enabling the use of geometric arguments. It would be interesting if such arguments can be pushed further to eliminate more non-degenerate projective planes as potential cores of mni matrices.

Question 7.4. *Are there more infinite families of Lehman matrices with $k > 1$?*

The only known infinite family of Lehman matrices with $k > 1$ is the family of point-line incidence matrices of non-degenerate projective planes, which are extremal structures in many ways. We can make a heuristic argument that we *expect* Lehman matrices with $k > 1$ to be far rarer, perhaps vanishingly rare. If (A, B) is a Lehman pair of type (n, r, s) with $k = rs - n$, then $\det(A)\det(B) = k^{n-1}(n+k) = k^{n-1}rs$. If $k = 1$, then $|\det(A)| = r$ and $|\det(B)| = s$. These are the smallest possible non-zero values for the absolute value of the determinant of an r -regular (resp. s -regular) matrix, and so there are many such r -regular matrices each of which is a candidate to be a Lehman matrix. However if $k > 1$, then there is an extra factor of k^{n-1} in $\det(AB)$, which must be allocated between the determinants of A and B . With far fewer potential Lehman matrices, we are not surprised that the known examples are either small, very highly-structured or both.

Question 7.5. *Are there more infinite families of mni Lehman matrices?*

For any odd n , the circulant matrix with first row $(1, 1, 0, \dots, 0)$, and its blocker are mni Lehman matrices. Apart from these, Wang's [13] ingenious construction provides the only known infinite family of mni Lehman matrices where both $r, s > 2$. On one hand, the very existence of such a family makes it seem plausible that there are more, but on the other hand, our structural results make it clear that this family really is very special. It would be interesting to find *any* new square mni Lehman matrix.

Question 7.6. *Can similar construction results be developed for higher valency Lehman graphs?*

With respect to this question, it would be interesting to know all the $(15, 4, 4)$ -Lehman graphs. Due to the sheer numbers of bipartite quartic graphs on 30 vertices, any exhaustive computational approach is likely to require significantly stronger techniques for early pruning of the search, or much stronger constraints on the graph structure. With a heuristic local search based on finding edge-exchanges that increase the number of vertices-with-mates, we have found 58 Lehman graphs of type $(15, 4, 4)$ to date. Although we have no good reason to believe that we have covered even a minuscule fraction of the search space, the fact that our searches repeatedly find the same 58 matrices starting from numerous randomly-chosen bipartite quartic graphs supports the view that this list may be few others. None of the 58 matrices are mni.

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