

MAXIMUM SIZE BINARY MATROIDS WITH NO $AG(3, 2)$ -MINOR ARE GRAPHIC

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ABSTRACT. We prove that the maximum size of a simple binary matroid of rank $r \geq 5$ with no $AG(3, 2)$ -minor is $\binom{r+1}{2}$ and characterise those matroids achieving this bound. When $r \geq 6$, the graphic matroid $M(K_{r+1})$ is the unique matroid meeting the bound, but there are a handful of smaller examples. In addition, we determine the size function for non-regular simple binary matroids with no $AG(3, 2)$ -minor and characterise the matroids of maximum size for each rank.

1. INTRODUCTION

In his survey paper “Extremal Matroid Theory”, Kung [4] describes the subject as being primarily concerned with instances of the following problem:

FUNDAMENTAL PROBLEM *Let \mathcal{M} be a class of matroids satisfying given properties. Determine the size function*

$$h(\mathcal{M}; r) = \max\{|E(M)| : M \in \mathcal{M} \text{ and } \text{rank}(M) = r\}$$

and characterise the matroids of maximum size for each rank.

For example, for the class \mathcal{R} of *simple regular matroids*, an early theorem of Heller [2] implies that

$$h(\mathcal{R}; r) = \binom{r+1}{2},$$

and from this, it quickly follows that the maximum size regular matroids are graphic.

There are various classes of matroids, such as “all simple $GF(q)$ -representable matroids”, for which the fundamental problem is essentially trivial, but other than these cases there are relatively few classes of non-graphic matroids¹ for which the fundamental problem is completely solved. In view of the importance of *minors* in structural matroid theory, most of the exact results known are for various minor-closed classes of matroids, and mostly for matroids representable over a fixed finite field. Here the Growth Rate Theorem of Geelen, Kung and Whittle [1] implies that any proper minor-closed class of simple $GF(q)$ -representable matroids that does not include all $GF(q')$ -representable matroids for some $q' \mid q$ has either a *linear* or *quadratic* size function. The former occurs if any graphic matroid is excluded from the class, and the latter otherwise. However, the Growth Rate Theorem is a qualitative theorem and, while it gives the “shape” of the size function, it does not provide an exact solution. The most important fixed finite field is the *binary field*, but even for binary matroids only a few exact results are known, and these are mostly consequences of exact structural descriptions of the relevant classes of matroids. A selection of these results are given in Table 1, where $\mathcal{EX}(M_1, M_2, \dots)$ denotes the class of simple binary matroids with no M_1 -minor, M_2 -minor etc. Although a few more such exact characterisations are known, these are mostly for even smaller classes of matroids defined by excluding even more small matroids.

In this paper we consider the class $\mathcal{EX}(AG(3, 2))$, being the class of simple binary matroids with no $AG(3, 2)$ -minor and prove the following main result:

¹These questions for subclasses of graphic matroids are the subject of extremal graph theory, and not further addressed here.

Class	Size function $h(r)$	Maximum-sized matroids
Cographic	$3(r-1)$	Bond matroids of 3-connected cubic multigraphs (see [3]).
$\mathcal{E}\mathcal{X}(F_7, F_7^*)$	$\binom{r+1}{2}$	$M(K_{r+1})$ ([2]).
$\mathcal{E}\mathcal{X}(W_4)$	$\begin{cases} 3r-2, & r \text{ odd,} \\ 3r-3, & r \text{ even.} \end{cases}$	Parallel connections of copies of F_7 and the binary spike Z_4 found in Oxley [6] ([7]).
$\mathcal{E}\mathcal{X}(K_{3,3})$	$\begin{cases} 14r/3-7, & r \equiv 0 \pmod{3}, \\ 14r/3-11/3, & r \equiv 1 \pmod{3}, \\ 14r/3-19/3, & r \equiv 2 \pmod{3}. \end{cases}$	Parallel connections of copies of $PG(1,2)$, $PG(2,2)$ and $PG(3,2)$ ([5]).
Conjecture for $\mathcal{E}\mathcal{X}(K_5)$	$\begin{cases} 9r/2-13/2, & r \text{ odd,} \\ 9r/2-6, & r \text{ even.} \end{cases}$	Parallel connections along a line of copies of C_{12} and $PG(2,2)$.

TABLE 1. Some known and conjectured size functions for classes of binary matroids

1.1. Theorem. *Let M be a simple binary matroid of rank $r \geq 5$ with no $AG(3,2)$ -minor. Then $|M| \leq \binom{r+1}{2}$. Moreover, if $r \geq 6$ and $|M| = \binom{r+1}{2}$ then M is the graphic matroid of K_{r+1} .*

Note that this theorem contradicts Theorem 6.8 of Kung [4] where it is claimed that $\mathcal{E}\mathcal{X}(AG(3,2))$ has a size function strictly larger than that of graphic matroids. This was an error caused by incorrectly extrapolating to higher ranks the construction of the matroid $M(K_5)^+$ which is the unique rank-4 extension of the graphic matroid $M(K_5)$, and the unique maximum-sized rank-4 matroid in $\mathcal{E}\mathcal{X}(AG(3,2))$.

We were initially motivated to study the class $\mathcal{E}\mathcal{X}(AG(3,2))$ due to its connections to Seymour's 1-flowing conjecture, which asserts that a binary matroid is 1-flowing if and only if it has no $AG(3,2)$ -, T_{11} - or T_{11}^* -minor (see Seymour [9] for details). Regular matroids satisfy the 1-flowing conjecture and so the non-regular members of this class are important for the resolution of this conjecture. Our second main result is the determination of the maximum sized *non-regular* matroids in $\mathcal{E}\mathcal{X}(AG(3,2))$.

1.2. Theorem. *Let M be a simple non-regular binary matroid of rank $r \geq 6$ with no $AG(3,2)$ -minor. Then $|M| \leq \binom{r}{2} + 4$. Moreover, if $|M| = \binom{r}{2} + 4$ and $r \geq 7$, then M is the generalized parallel connection along a line of $M(K_r)$ and F_7 .*

It is quite common in such results that there are various maximum-sized ‘‘sporadic’’ matroids at lower ranks, and these can obscure the emergence of the asymptotic behaviour, and cause difficulties in framing an inductive proof. We deal with this ‘‘low-level junk’’ through computer analysis of the lower rank cases, a development that Kung recognised would be necessary when he observed that ‘‘because many small examples need to be examined, some computer assistance will be needed’’. We place particular emphasis on the correctness of these computations as they form the foundation of the proof, and ensure that they are performed at least twice independently.

The proof of Theorem 1.1 is in Section 4, while Theorem 1.2 is proved in Section 5. Section 2 describes the computational results for the low rank cases that provide the base cases for the subsequent inductive proofs, while Section 3 presents fundamental results on even cycle matroids and graft matroids which are needed for the proofs of the main results.

Rank	Max size	Number	Max size non-regular	Number
3	7	1	7	1
4	11	1	11	1
5	15	4	15	3
6	21	1	19	13
7	28	1	25	1
8	36	1	32	1

 TABLE 2. Numbers of maximum-sized matroids in $\mathcal{EX}(AG(3,2))$ for rank up to 8

2. LOW RANK

To get a clean base for an inductive proof, a case-analysis of the low-rank situation is necessary, and this is best done with the aid of a computer. To increase confidence in this part of the proof, we ensure that all computations are done twice using independent software packages and approaches.

For binary matroids of any moderate fixed rank r , it is possible to determine the cycle index of the group $PGL(r, 2)$, and use Pólya's Enumeration Theorem to calculate the number of orbits of this group on subsets of $PG(r - 1, 2)$ of each size, and thereby determine the exact numbers of binary matroids of rank up to r . Although the total numbers of binary matroids grow extremely rapidly with rank, it is straightforward to produce a complete list of all 475499108 (including the empty matroid) binary matroids of rank up to 6. It is easy to verify that the matroids produced are pairwise non-isomorphic, and as the numbers of matroids of every rank and size agree with the theoretical numbers, we are very confident that this list is both correct and complete. Working with the much smaller subset consisting of the binary matroids of size up to 22, it is relatively straightforward to calculate the minors of each matroid that arise from a single deletion or contraction (and simplification). Then information about which matroids in this list have $AG(3, 2)$ as a minor can be propagated throughout the catalogue, finally yielding a catalogue of all the matroids of rank 6 and size up to 22.

A complementary approach is to construct a catalogue of *only* the matroids in $\mathcal{EX}(AG(3, 2))$ by a 'bootstrap' approach that grows the catalogue according to size. At each stage of the process, a binary matroid of size k is extended to one of size $k + 1$ by the addition of a point in the ambient vector space. Rather than testing directly whether the new matroid has an $AG(3, 2)$ -minor, each of its single element deletions and contractions are tested for membership in the catalogue constructed so far. By starting with the list of binary matroids of some fixed rank, and size at most 8, but excluding $AG(3, 2)$ itself, this approach generates only those matroids with no $AG(3, 2)$ -minor. This approach was implemented twice, once in the computer algebra system GAP and once in C. The former uses only very well-tested standard functions for isomorph rejection and is very easy to verify, but manages only up to rank 7, while the latter is much faster and could manage rank 8, but for which external validation is harder.

In any event, all approaches produced entirely consistent results for the low-rank case analysis, and these results are shown in Table 2.

This table shows that already at rank 6, the unique maximum sized matroid in $\mathcal{EX}(AG(3, 2))$ is $M(K_6)$, and this provides the base case for the inductive proof given in Section 4. For lower ranks, the extremal matroids are the Fano plane for rank three, $M(K_5)^+$ for rank four and at rank 5 there are three coextensions of $M(K_5)^+$ along with $M(K_6)$. The three non-regular examples are shown in Appendix A.

For ranks 3, 4 and 5, the maximum sized matroids in $\mathcal{EX}(AG(3, 2))$ already include non-regular ones as described previously. By rank 7, the parallel connection along a line of $M(K_6)$ and F_7 is the unique maximum sized non-regular matroid and this forms the base case for the inductive proof

given in Section 5. At rank 6, there are 12 additional examples, all of which are coextensions of the three non-regular maximum sized matroids of rank 5 and size 15.

3. EVEN CYCLE MATROIDS AND GRAFT MATROIDS

In this section we introduce even cycle matroids and graft matroids. Even cycle matroids are used for the proofs of both the main theorems, while graft matroids are only used in the proof of Theorem 1.2. The results about even cycle matroids are mostly folklore, but a complete presentation of these results can be found in Pivotto [8].

A *signed graph* is a pair (G, Σ) , where G is a graph and Σ is a set of edges of G . Let A be the vertex-edge incidence matrix of G and S the row incidence vector of edges in Σ ; let A' be obtained from A by adding row S . Then the binary matroid represented by A' is the *even cycle matroid* represented by (G, Σ) , denoted $\text{ecycle}(G, \Sigma)$. Since the rows of A span the cuts of G , replacing Σ with $\Sigma \triangle C$, for a cut C of G , does not change the matroid. We call any such set $\Sigma \triangle C$ a *signature* of (G, Σ) and the operation of replacing Σ by $\Sigma \triangle C$ is called a *resigning*.

Suppose that M is a binary matroid and M/x is graphic for some element x . Then any matrix representation of M may be row-reduced to the following form

$$A = \left[\begin{array}{c|c} x & \\ \hline 0 & \\ \vdots & A' \\ 0 & \\ \hline 1 & S \end{array} \right],$$

where $M(A')$ is a graphic matroid. Let G be a graph representing $M(A')$ with the addition of a loop labelled x ; let Σ be the set of edges in G which correspond to columns of A which have a 1 in the last row. Then A represents $\text{ecycle}(G, \Sigma)$, i.e. M is the even cycle matroid of (G, Σ) .

A set of edges F in a signed graph (G, Σ) is Σ -*even* if $|F \cap \Sigma|$ is even (and is Σ -*odd* otherwise). If every cycle in G is Σ -even, then the empty set is a signature of (G, Σ) and $\text{ecycle}(G, \Sigma)$ is a graphic matroid. Let X be the set of Σ -odd loops of G and suppose that every Σ -odd cycle of G not in X uses a specific vertex w ; then every non-loop cycle of G that doesn't use w is Σ -even, so for some signature Σ' , we have that $\Sigma' - X \subseteq \delta_G(w)$. Construct a graph H from G by splitting the vertex w into two vertices w' and w'' so that the edges in $\delta(w) \cap \Sigma'$ are incident with w' , the edges in $\delta(w) - \Sigma'$ are incident with w'' and the edges in X become edges between w' and w'' . With this construction $\text{ecycle}(G, \Sigma)$ is the graphic matroid of H . We will use this construction both in the proof of Theorem 1.1 and Theorem 1.2.

Minor operations on even cycle matroids correspond to minor operations on signed graphs in the following way. Let (G, Σ) be a signed graph and e an edge of G . Then we define $(G, \Sigma) \setminus e$ as $(G \setminus e, \Sigma - \{e\})$. Contraction is defined as follows: if e is a loop of G which is in Σ , then $(G, \Sigma)/e = (G \setminus e, \emptyset)$; otherwise we pick any signature Σ' which does not contain e and define $(G, \Sigma)/e = (G/e, \Sigma')$. The choice of Σ' is not unique, but this is irrelevant for the matroid. With these definitions we have that $\text{ecycle}(G, \Sigma) \setminus e = \text{ecycle}((G, \Sigma) \setminus e)$ and $\text{ecycle}(G, \Sigma)/e = \text{ecycle}((G, \Sigma)/e)$.

A *digon* is a pair of parallel edges of (G, Σ) with different parities.

We will make use of the even cycle representation of F_7 , which is given in Figure 1.

Next we introduce graft matroids, which will be used in the proof of Theorem 1.2. A *graft* is a pair (G, T) where G is a graph and T a set of vertices of G of even size. The *graft matroid* arising from a graft (G, T) is the binary matroid with matrix representation obtained from the vertex-edge incidence matrix of G by adding a column that is the incidence vector of T . The element corresponding to the incidence vector of T is called the *graft element* of this matroid. A *T -join* of a graft (G, T) (or a *T -join* of G) is a set of edges J of G such that the vertices of odd degree in $G[J]$

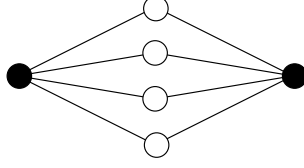


FIGURE 2. The graft element is represented by white vertices. Contracting the graft element produces $AG(3, 2)$.

Proof. Let $C = \{c_1, c_2, c_3, c_4\}$, and let $\{c_i, d_i\}$ be a parallel pair of M/x for every i , so that $\{x, c_i, d_i\}$ is a line of M . Let N be a minor of M/x such that C is a circuit of N . Consider a binary reduced representation of N such that c_1, c_2 , and c_3 all label rows, and extend this to a representation of M/x . Now we can construct a representation of M by adding a row labelled x . Restricting this representation to $C \cup \{x, d_1, d_2, d_3, d_4\}$ gives the following submatrix.

$$\begin{array}{c} \\ x \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{bmatrix} d_1 & d_2 & d_3 & c_4 & d_4 \\ 1 & 1 & 1 & \alpha & \alpha + 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

We add the first row to all subsequent rows, and then delete the column labelled by c_4 or d_4 that has 1 as its first entry. The resulting matrix represents $AG(3, 2)$, so we are done. \square

4.2. Lemma. *Let M be a 3-connected binary matroid that contains two distinct F_7 -restrictions. Then M contains an $AG(3, 2)$ -minor.*

Proof. Assume that the lemma fails for M , where M is as small as possible. Let R_1 and R_2 be the two F_7 -restrictions. The hypotheses imply that $r(R_1 \cup R_2) > 3$, since M is binary and has no parallel pairs. If $r(R_1 \cup R_2) = 4$, then submodularity implies that $cl(R_1) \cap cl(R_2)$ is a line. Now it is easy to verify that $(R_1 \cup R_2) \setminus (R_1 \cap R_2)$ is an $AG(3, 2)$ -restriction.

Therefore $r(R_1 \cup R_2) > 4$. Because M has no parallel pairs and R_1 and R_2 are projective geometries, $cl(R_1) = R_1$ and $cl(R_2) = R_2$. If $R_1 \cup R_2 = E(M)$, then $(R_1, R_2 \setminus R_1)$ is a 1- or a 2-separation of M . Therefore there is an element e in neither $cl(R_1)$ nor $cl(R_2)$. Hence R_1 and R_2 are F_7 -restrictions of $M \setminus e$ and M/e . Bixby's lemma says that either $si(M/e)$ or $co(M \setminus e)$ is 3-connected. Assume $si(M/e)$ is 3-connected. When we suppress parallel pairs we delete elements other than those in R_1 and R_2 , or, if a parallel pair contains $r_1 \in R_1$ and $r_2 \in R_2$, we delete r_2 and replace R_2 with $(R_2 - r_2) \cup r_1$. Note that $r_{M/e}(R_1 \cup R_2) > 3$, so not every element in R_1 is parallel with a point in R_2 . Therefore the resulting matroid is isomorphic to $si(M/e)$, and has distinct F_7 -restrictions, R_1 and R_2 . By the minimality of M , it has an $AG(3, 2)$ -minor. Therefore $co(M \setminus e)$ is 3-connected. If R_1 and R_2 are not F_7 -restrictions of $co(M \setminus e)$, then there must be a series pair of $M \setminus e$ contained in $R_1 \cup R_2$. But an F_7 -restriction cannot intersect a series pair, by virtue of the fact that a circuit and a cocircuit cannot intersect in a single element. Hence $co(M \setminus e)$ has an $AG(3, 2)$ -minor, and this contradiction completes the proof. \square

4.3. Lemma. *Let M be a binary matroid and ℓ_1, \dots, ℓ_k be lines of M all using an element x , where $k \geq 4$. Suppose that there is no F_7 -restriction of M using x . If $r(\ell_1 \cup \dots \cup \ell_k) \leq k$, then M contains an $AG(3, 2)$ -minor.*

Proof. Let N be the restriction of M to $\ell_1 \cup \dots \cup \ell_k$. For $i = 1, \dots, k$, let a_i and b_i be the points in ℓ_i other than x . Since N/x has rank less than k , we may relabel the points in ℓ_1, \dots, ℓ_k so that there is a circuit C contained in $\{a_1, \dots, a_k\}$, say $C = \{a_1, a_2, \dots, a_j\}$. If $j = 3$, then $\ell_1 \cup \ell_2 \cup \ell_3$ is an F_7 -restriction of M using x . Therefore $j \geq 4$; then $\{a_1, a_2, a_3, a_4\}$ is a circuit in $N/\{a_5, \dots, a_j\}$, so Lemma 4.1 implies that M has an $AG(3, 2)$ -minor. \square

Proof of Theorem 1.1. Assume by contradiction that the first part of the theorem fails. Let M be a counterexample of minimum rank; in other words, M has no $AG(3,2)$ -minor and $|M| > \binom{r+1}{2}$. Computations show that the theorem holds for matroids of rank 5, therefore $r \geq 6$ and for every minor N of M of rank $r - 1$ we have $|N| \leq \binom{r}{2}$.

We first show that M must be 3-connected. If not, M is the 1- or 2-sum of two matroids N_1 and N_2 of rank smaller than r . We consider only the case when M is the 2-sum of N_1 and N_2 , as the other case is similar. Let r_1 and r_2 denote the rank of N_1 and N_2 respectively. Then $r = r_1 + r_2 - 1$. M has no parallel pairs, so we may assume N_1 has at most one parallel pair and N_2 is simple. By the minimality of M , $|N_1| \leq \binom{r_1+1}{2} + 1$ and $|N_2| \leq \binom{r_2+1}{2}$, unless one or both of r_1, r_2 are equal to 3 or 4 (see Table 2). We let the reader verify this case independently and just consider the case that $r_1, r_2 \notin \{3, 4\}$. Therefore

$$\begin{aligned} |M| &= |N_1| + |N_2| - 2 \leq \binom{r_1+1}{2} + \binom{r_2+1}{2} - 1 \\ &= \frac{1}{2}(r_1^2 + r_1 + r_2^2 + r_2) - 1 \\ &= \binom{r+1}{2} + r_1 + r_2 - r_1 r_2 - 1 < \binom{r+1}{2}. \end{aligned}$$

This contradicts the assumption on the size of M . It follows that M is 3-connected.

Now consider any element x of M and let $N = si(M/x)$. Then the minimality of M implies that $|N| \leq \binom{r}{2}$ and

$$|M| - |N| \geq \binom{r+1}{2} + 1 - \binom{r}{2} = \frac{1}{2}(r^2 + r - r^2 + r) + 1 = r + 1.$$

Therefore in M/x there are at least r parallel pairs, so x is in at least r lines. Let ℓ_1, \dots, ℓ_r be such lines. The rank of $\ell_1 \cup \dots \cup \ell_r$ is at most the rank of M , which is r . Lemma 4.3 implies that there is an F_7 -restriction of M using x . The point x was chosen arbitrarily, hence we showed that every point of M is contained in an F_7 -restriction. The rank of M is at least 5, so M contains at least two distinct F_7 -restrictions. It follows by Lemma 4.2 that M has an $AG(3,2)$ -minor, a contradiction. This concludes the proof of the first part of the theorem.

Now assume that the second part of the statement does not hold. Let M be a counterexample of minimum rank, i.e. $|M| = \binom{r+1}{2}$ and M is not graphic. Computations show that the theorem holds for matroids of rank 6, therefore $r \geq 7$ and the statement holds for every minor N of M of rank $r - 1$.

By the same argument as the one above, M is 3-connected. By Lemma 4.2, there exists an element x in M which is not in an F_7 -restriction. Let $N = si(M/x)$; then the size of N is equal to $|M| - 1$ minus the number of lines through x . Because N is $AG(3,2)$ -free and simple, N has size at most $\binom{r}{2}$, so x is in at least $r - 1$ lines. Lemma 4.3 implies that x is in exactly $r - 1$ lines $\ell_1, \dots, \ell_{r-1}$ and $r(\ell_1 \cup \dots \cup \ell_{r-1}) = r$. In particular, $|N| = \binom{r}{2}$ and by the minimality of M , the matroid N is the graphic matroid of K_r .

For $i = 1, \dots, r - 1$, let a_i be the point in N that corresponds to the parallel pair $\ell_i - \{x\}$ in M/x . To simplify the exposition, we denote the graph K_r representing N by G and we identify every edge of G with the corresponding element in N . Let $A = \{a_1, \dots, a_{r-1}\}$. Since the rank of $\ell_1 \cup \dots \cup \ell_{r-1}$ is r , the set A has rank $r - 1$ in N . It follows that A is a spanning tree in G .

Since M is the lift of a graphic matroid, M is an even cycle matroid; we describe its representation as a signed graph (G', Σ) next. Construct the graph G' from G as follows: add a loop (corresponding to the element x) and then, for every $a_i \in A$, add an edge parallel to a_i in G ; Σ is comprised of x plus all the elements of M that are lifted by x (i.e. the elements whose fundamental circuit for the basis $A \cup \{x\}$ contains x). Then $M = \text{ecycle}(G', \Sigma)$ and a line of M is either a Σ -even triangle of G' or the union of a Σ -odd loop and a digon.

We claim that $G[A]$ is a star of G ; if not, it is possible to find an edge $e \in E(G) - A$ which is not in a triangle with two elements in A . Then the element e is in at most $r - 2$ Σ -even triangles in G' and e is not in any digon, so e is in at most $r - 2$ lines of M ; this implies that $|si(M/e)| > \binom{r}{2}$, a contradiction. It follows that $G[A]$ is a star $\delta(w)$ of G . Next we show that every triangle in $G \setminus A$ is Σ -even. Suppose this is not the case, and choose an edge e in such a triangle. Then e is in at most $r - 2$ Σ -even triangles in (G', Σ) and e is not in any digon; it follows that e is in at most $r - 2$ lines of M , again a contradiction. Therefore every cycle in $G \setminus A$ is Σ -even, since every cycle in $G \setminus A$ is the symmetric difference of triangles in $G \setminus A$. Since $(G' - w) \setminus x$ is equal to $G - A$, this implies that every odd cycle of (G', Σ) (except for x) uses the vertex w . By splitting w as described in Section 3, we construct a graphic representation of M , and we see that M is isomorphic to $M(K_{r+1})$. \square

5. PROOF OF THEOREM 1.2

Let M be a simple non-regular binary matroid of rank r with no $AG(3, 2)$ -minor. We need to show that $|M| \leq \binom{r}{2} + 4$ (if $r \geq 6$) and, for $r \geq 7$, the only matroid achieving the bound is the generalized parallel connection along a line of $M(K_r)$ and F_7 .

We prove the statement by induction on the rank. The base case (when the rank is 6 or 7) was proved computationally, as explained in Section 2. Therefore we may assume that $r \geq 8$ and the statement holds for matroids of smaller rank than M .

The proof is split into two subsections; in the first one we show the bound on the size of M , while in the second, more technical one, we characterize the matroids achieving this bound. To prove the bound on the size we proceed with a series of claims. First we show that M needs to be 3-connected. Next we consider an element x such that M/x is non-regular; in Claim 2 we show that x needs to be in many lines of M . The longest claim of this section, Claim 3, shows that if x is in no F_7 -restriction and $si(M/x)$ is the generalized parallel connection along a line of $M(K_{r-1})$ and F_7 then M is what we would expect, namely the generalized parallel connection along a line of $M(K_r)$ and F_7 . Claim 4 concludes the proof of the first half of the theorem, and Section 5.1.

In Section 5.2 we characterize the non-regular matroids achieving the bound. We use the claims in Section 5.1 to show that there must exist an element x in M which is in $r - 1$ lines that span $E(M)$. Next we analyze the elements that are not in these lines. We show that M must contain special elements that are not in the span of any two of the lines through x . We show that we must have exactly $r - 3$ special elements and each one must be in exactly $r - 2$ lines. Moreover we show that contracting any special element maintains non-regularity. Since each special element y is in $r - 2$ lines, the induction hypotheses imply that $si(M/y)$ is the generalized parallel connection of $M(K_{r-1})$ with F_7 . The fact that there are at least $r - 3$ special elements, together with Lemma 4.2, imply that at least one of them is not in an F_7 -restriction, and we conclude the proof by Claim 3.

5.1. Bound on the size. In this section we prove the first part of Theorem 1.2, i.e. we show that $|M| \leq \binom{r}{2} + 4$.

1. **Claim.** M is 3-connected.

Proof. If not, then M is the 1- or 2-sum of two matroids N_1 and N_2 of rank smaller than r . We consider only the case when M is the 2-sum of N_1 and N_2 , as the other case is similar. Neither N_1 nor N_2 has an $AG(3, 2)$ -minor. As M is non-regular, at least one of N_1 and N_2 is non-regular. We let N_1 be non-regular. Let r_1 and r_2 denote the rank of N_1 and N_2 respectively. Then $r = r_1 + r_2 - 1$ and $r_1, r_2 \geq 2$. As M has no parallel pairs, we may assume N_1 has at most one parallel pair and N_2 is simple. By the minimality of M , Theorem 1.1 and Table 2, $|N_1| \leq \binom{r_1}{2} + 6$ and $|N_2| \leq \max\{\binom{r_2+1}{2}, \binom{r_2}{2} + 4\}$. If $r_2 = 2$, $|N_2|$ is at most 3 and for $r_2 \geq 3$, $\max\{\binom{r_2+1}{2}, \binom{r_2}{2} + 4\} \leq \binom{r_2+1}{2} + 1$.

Therefore

$$\begin{aligned} |M| &= |N_1| + |N_2| - 2 \leq \binom{r_1}{2} + \binom{r_2 + 1}{2} + 5 \\ &= \frac{1}{2}(r_1^2 - r_1 + r_2^2 + r_2) + 5 \\ &= \binom{r}{2} + r_1 + 2r_2 - r_1r_2 + 4. \end{aligned}$$

As $|M| \geq \binom{r}{2} + 4$, it follows that $r_1 + 2r_2 - r_1r_2 \geq 0$. It is easily verified that this cannot occur, since $r_1 + r_2 \geq 8$ and $r_1 \geq 3$ (because N_1 is non-regular). It follows that M is 3-connected. \diamond

2. Claim. *If $|M| = \binom{r}{2} + 4 + p$ then each element x of M such that M/x is non-regular is in at least $r - 2 + p$ lines.*

Proof. Let $N = si(M/x)$. Then the induction hypotheses imply that $|N| \leq \binom{r-1}{2} + 4$ and

$$|M| - |N| \geq \binom{r}{2} + 4 + p - \binom{r-1}{2} - 4 = \frac{1}{2}(r^2 - r - r^2 + 3r - 2) + p = r - 1 + p.$$

Therefore x is in at least $r - 2 + p$ lines. \diamond

3. Claim. *Suppose that M contains an element x such that x is in no F_7 -restriction and $si(M/x)$ is the parallel connection of $M(K_{r-1})$ and F_7 . Then M is the parallel connection of $M(K_r)$ and F_7 .*

Proof. Consider such an element x and let $N = si(M/x)$. Claim 2 implies that x is in $k \geq r - 2$ lines. Let ℓ_1, \dots, ℓ_k be the lines through x . Lemma 4.3 and the fact that there is no F_7 -restriction using x implies that $r(\ell_1 \cup \dots \cup \ell_k) = k + 1$. Since N is the generalized parallel connection of $M(K_{r-1})$ with F_7 , N is a graft matroid as described in Section 3. More precisely, N is the graft matroid of (G, T) , where G is obtained from K_{r-1} by adding a vertex v adjacent to three vertices of the K_{r-1} and T is comprised of v and its neighbours. Let the three neighbours of v be v_1, v_2, v_3 and let $y_i = vv_i$ for $i = 1, 2, 3$; denote by z_1 the edge forming a triangle with y_2 and y_3 and define z_2 and z_3 similarly. Let $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2, z_3\}$ and t be the graft element of N . Note that $Y \cup Z \cup \{t\}$ is an F_7 -restriction of N and the only lines containing t in N are of the form $\{t, y_i, z_i\}$, for $i = 1, 2, 3$.

For $i = 1, \dots, k$, let a_i be the point in N that corresponds to the parallel pair $\ell_i - \{x\}$ in M/x . Let $A = \{a_1, \dots, a_k\}$. Since $r(\ell_1 \cup \dots \cup \ell_k) = k + 1$ and $k \geq r - 2$ we have that A is an independent set of N of size $r - 1$ or $r - 2$. Since $Y \cup \{t\}$ is a circuit of N , there is an element in this set not contained in A . It is easy to verify that $N \setminus y_1$ is graphic, and that in fact, $N \setminus y_1$ is equal to the matroid obtained from $M(G)$ by relabelling y_1 as t and swapping the labels of y_2 and y_3 . Because $\{y_1, y_2, y_3, t\}$ is a circuit of N , we can represent N by relabelling G in this way, and then adding y_1 as a graft element corresponding to the set $\{v, v_1, v_2, v_3\}$. A similar symmetry applies to y_2 and y_3 . Because of these symmetries, we lose no generality in assuming that t is not in A .

Now $M \setminus t$ is the lift of a graphic matroid (namely $M(G)$); that is, $M \setminus t$ is an even cycle matroid. Next we describe its representation as a signed graph (G', Σ) . Construct the graph G' from G as follows: add a loop (corresponding to the element x) and then, for every $a_i \in A$, add an edge parallel to a_i in G . Fix a basis B of $M \setminus t$ containing A and x ; then Σ is comprised of x plus all the elements of M whose fundamental circuit with B contains x . With this construction $M \setminus t = \text{ecycle}(G', \Sigma)$, since $G' \setminus x$ is a graphic representation of $M \setminus t/x$ (see Section 3).

To complete our proof we will prove that the following three conditions hold:

- (a) $G[A]$ is a star $\delta_G(w)$;
- (b) every triangle of $G' - w$ is Σ -even;
- (c) $Y \cap A = \emptyset$.

First we show that these conditions imply the result. Assume that (a), (b), and (c) hold. Condition (b) implies that every cycle of $(G' - w) \setminus x$ is Σ -even. Therefore there exists a signature Σ' of (G', Σ) such that $\Sigma' - \{x\}$ is contained in $\delta_{G'}(w)$. For every $i \in [k]$, let b_i be the edge of G' parallel to a_i ; since $\{a_i, b_i\}$ is a Σ -odd cycle of G' , it is also a Σ' -odd cycle. Therefore Σ' contains x and exactly one of a_i, b_i for every i . Let H be the unsigned graph obtained from G' by splitting w into vertices r and s , where x is the edge joining r and s , and all edges in $\Sigma' - \{x\}$ are incident with r , while each edge in $\delta_{G'}(w) - \Sigma'$ is incident with s . Note that condition (c) implies v is incident with exactly three edges in H , namely y_1, y_2 , and y_3 and $H - v$ is isomorphic to K_r . Condition (b) implies that $M(H) = M \setminus t$, as discussed in Section 3. Thus M is a graft of the form (H, S) , where t is the graft element corresponding to S . Since $M/x \setminus \{b_1, \dots, b_k\}$ is the graft matroid of $(G, \{v, v_1, v_2, v_3\})$, we can assume that either (i) $S = \{v, v_1, v_2, v_3\}$, or (ii) r or s is in $\{v_1, v_2, v_3\}$, and $S = \{v, v_1, v_2, v_3\} \Delta \{r, s\}$, or (iii) neither r nor s is in $\{v_1, v_2, v_3\}$, and $S = \{v, v_1, v_2, v_3, r, s\}$. It is easily verified that in case (ii), t is in exactly two lines in M , and that in case (iii), t is in no lines at all. Therefore if (ii) or (iii) holds, then $|M/t| \geq |M| - 3 \geq \binom{r}{2} + 1$, a contradiction to Theorem 1.1. Hence case (i) holds, and M is the generalized parallel connection of $M(K_r)$ with F_7 . We conclude that to complete the proof of the claim we only need to show that conditions (a), (b) and (c) hold.

We start by showing that $G[A - Y]$ is a star. Since A is an independent set in N , $G[A]$ is a forest. Let $E' = E(G) - (Y \cup Z \cup A)$. Since G' is obtained from G by adding edges, we may see E' as a subset of edges of G' . For every edge $e \in E'$, there is an F_7 -restriction, $\{t\} \cup Y \cup Z$, in N/e , so M/e is non-regular. It follows, by Claim 2, that every $e \in E'$ is in at least $r - 2$ lines in M . No element in E' is in a line with t in N , since the only lines containing t are given by T -joins of size two. This implies there is no line of M containing t and an element of E' . Moreover, since no elements of A are in E' , no elements of E' are in a digon in G' . Assume that $e \in E'$ is not in a triangle with two edges in A . Observe that e is not in a triangle of G' with v , as $e \notin Y \cup Z$. Since $G' - v$ has $r - 1$ vertices, e is in at most $r - 3$ Σ -even triangles in (G', Σ) . It follows that e is in at most $r - 3$ lines in M , a contradiction. Therefore every $e \in E'$ is in a triangle with two edges in A . From this we deduce that $A - Y$ is a tree spanning $G - v$ and, since $G - v$ is isomorphic to K_{r-1} and $r - 1 \geq 7$, there exists a leaf u of $G[A - Y]$ not in $\{v, v_1, v_2, v_3\}$. For every other leaf u' of $G[A - Y]$, the edge uu' is in E' , hence the uu' -path in $G[A - Y]$ must have length two. It follows that $G[A - Y]$ is a star $\delta_G(w)$ spanning $G - v$.

Since every edge in E' is in at least $r - 2$ lines of M , every triangle of G' containing an edge of E' and not using the vertex w is Σ -even. This implies that if $w \notin \{v_1, v_2, v_3\}$ the triangle $\{z_1, z_2, z_3\}$ is also Σ -even, by the following argument. Consider a vertex $u \in V(G') - \{v, v_1, v_2, v_3, w\}$ and let H be the induced K_4 of G' formed by $\{u, v_1, v_2, v_3\}$. Then every triangle of H other than Z is Σ -even. Since Z is the symmetric difference of the other three triangles in H , Z is also Σ -even.

Since $|A| \leq r - 1$, the fact that $A - Y$ spans $G - v$ implies in particular that $|A \cap Y| \leq 1$. If (c) fails we may assume that $y_1 \in A$. If y_2 is in at most 2 lines, then $|M/y_2| \geq |M| - 3 \geq \binom{r}{2} + 1$, and we have a contradiction. Therefore y_2 , and by symmetry y_3 , are in at least three lines of M . This means the triangles $\{z_1, y_2, y_3\}$, $\{z_2, y_1, y_3\}$, and $\{z_3, y_1, y_2\}$ are Σ -even. Therefore we may assume (up to resigning) that no edge of $E(G) - A$ is in Σ .

To conclude the proof of (c) (and consequently the proof of (a)), we need to consider two cases: either $w_1 = v_1$ or not (recall that we are assuming that $y_1 \in A$ and $y_2, y_3 \notin A$). First we consider the case that $w \neq v_1$. By symmetry between v_2 and v_3 , we may assume that $w \neq v_2$; let (H, Γ) be the subgraph of (G', Σ) induced by $\{v, v_1, v_2, w\}$ together with the loop x . Then removing parallel edges of the same parity from $(H, \Gamma)/y_2$ we obtain the signed graph representation of F_7 (see Figure 1). Therefore M/y_2 is non-regular. It follows that y_2 must be in at least $r - 2 \geq 6$ lines in M . However y_2 is in only two triangles in G' , so y_2 is in at most 4 lines in M , a contradiction. It follows that this case does not occur, i.e. $w = v_1$. This (together with the previous paragraph) implies that every Σ -odd cycle of G' (other than x) uses the vertex v_1 . Hence $M \setminus t$ is the graphic matroid of the

graph H which is obtained from G' by splitting v_1 . It follows that M is the graft matroid of (H, S) , for the appropriate choice of S . By the configuration of the edges in A , H is obtained from K_r by adding a vertex v adjacent to four vertices of the K_r ; the edges incident with v are y_1, y_2, y_3, x . Since $Y \cup \{t\}$ is a circuit of N , one of $Y \cup \{t\}$ or $Y \cup \{x, t\}$ is a circuit of M . First suppose that $Y \cup \{t\}$ is a circuit of M ; in this case Y is an S -join, i.e. S is the set of vertices of odd degree in $H[Y]$. Therefore $(H, S)/x = (H', S')$, where H' is isomorphic to K_r and $|S'| = 4$. Since $r \geq 6$, this graft contains the graft in Figure 2 as a minor; since the graft matroid of this graft contains $AG(3,2)$ as a minor (see Section 3), this case cannot occur. The case where $Y \cup \{x, t\}$ is similar, only now S is the set of neighbours of v and (H', S') is obtained by deleting v . This concludes the proof of conditions (a) and (c).

To prove (b) it only remains to show that every triangle of $G' - w$ containing edges in Y is Σ -even. If not, we may assume that $\{y_1, y_2, z_3\}$ is Σ -odd; since (c) holds, the only lines of M through y_1 are $\{y_1, z_1, t\}$ and, possibly, $\{y_1, z_2, y_3\}$, so y_1 is in only two lines of M , a contradiction. \diamond

4. **Claim.** $|M| = \binom{r}{2} + 4$.

Proof. Suppose otherwise, i.e. $|M| \geq \binom{r}{2} + 5$. Let x be any element of M such that M/x is non-regular; Claim 2 implies that x is in at least $r - 1$ lines. If x is in r (or more) lines, then Lemma 4.3 implies that there is an F_7 -restriction of M using x . It follows that for every element x of M such that M/x is non-regular, either x is in exactly $r - 1$ lines or x is in an F_7 -restriction. The matroid M is non-regular, 3-connected and is not isomorphic to F_7^* ; the dual of Proposition 12.2.3 in [6] implies that M has an F_7 -minor. As $r \geq 8$, there is an independent set $I = \{x_1, x_2, x_3, x_4, x_5\}$ of M such that M/I has an F_7 -minor (hence is non-regular). Each element in I is either in an F_7 -restriction or is in exactly $r - 1$ lines. If every element in I is in an F_7 -restriction of M , then M contains at least two distinct F_7 -restrictions (because the rank of F_7 is three) and by Lemma 4.2 $AG(3,2)$ is a minor of M . It follows that there exists an element x of M such that M/x is non-regular and x is in exactly $r - 1$ lines. Therefore $|si(M/x)| \geq \binom{r}{2} + 5 - r = \binom{r-1}{2} + 4$ and the inductive hypotheses imply that $si(M/x)$ is the generalized parallel connection of $M(K_{r-1})$ with F_7 ; Claim 3 implies that M is the generalized parallel connection of $M(K_r)$ with F_7 . However this is not possible, since we are assuming that $|M| \geq \binom{r}{2} + 5$. We conclude that this case does not occur, i.e. $|M| = \binom{r}{2} + 4$. \diamond

5.2. **The characterization.** We conclude the proof of Theorem 1.2 by characterizing the non-regular matroids of maximum size. More specifically, we prove that M is the generalized parallel connection along a line of $M(K_r)$ and F_7 .

Consider an element x of M such that M/x is non-regular. Let $N = si(M/x)$. Claim 2 implies that x is in at least $r - 2$ lines. We may again choose x so that there is no F_7 -restriction of M using x . Therefore (by Lemma 4.3) x is in either $r - 2$ or $r - 1$ lines. If x is in exactly $r - 2$ lines, then $|N| = \binom{r-1}{2} + 4$, so N is the generalized parallel connection of $M(K_{r-1})$ with F_7 and by Claim 3 we conclude that M is the generalized parallel connection of $M(K_r)$ with F_7 .

It remains to consider the case when x is in exactly $r - 1$ lines $\ell_1, \dots, \ell_{r-1}$, where $r(\ell_1 \cup \dots \cup \ell_{r-1}) = r$. Let $L = \ell_1 \cup \dots \cup \ell_{r-1}$. Then $M = cl(L)$. For $i = 1, \dots, r - 1$, let a_i and b_i be the points in ℓ_i other than x . Suppose that every element of $M - L$ is on a line containing two points from L . Then in the standard binary matrix representation of N with respect to the basis $\{a_1, \dots, a_{r-1}\}$, every column has at most two non-zero entries; therefore N is a graphic matroid. However, x was chosen so that N is non-regular. It follows that there exists an element y such that y is not on a line with two elements from L . We call such a y a *special element* of M . In the remainder of the proof we will analyse the special elements of M : we show that each special element y is in exactly $r - 2$ lines of M and that M/y is non-regular. Moreover, we will show that we have enough special elements to be able to choose one of them to not be in an F_7 -restriction; this concludes the proof by induction and by Claim 3.

Let y be any special element of M and let C be the fundamental circuit of y with respect to the basis $\{x, a_1, \dots, a_{r-1}\}$ of M . If x and a_i are in C , then we may consider, instead of C , the circuit $C \triangle \ell_i$ and swap the labels of a_i and b_i : by doing so we may assume that $x \notin C$. Hence we may assume that $C = \{y, a_1, \dots, a_k\}$, with $k \geq 3$. If $k \geq 4$ then in $M/\{y, a_5, \dots, a_k\}$ the set $(\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4) - \{x\}$ is an $AG(3, 2)$ -restriction. Therefore $k = 3$ and y is in a circuit with a_1, a_2, a_3 . In particular, M/y contains an F_7 -restriction (with lines ℓ_1, ℓ_2, ℓ_3). It follows that for every special element y of M the following properties hold:

- (S1) y is in a circuit $C = \{y, c_i, c_j, c_k\}$ for distinct i, j, k , where $c_i \in \{a_i, b_i\}$, $c_j \in \{a_j, b_j\}$ and $c_k \in \{a_k, b_k\}$.
- (S2) M/y is non-regular.

Next we show that

- (S3) if y is in a line $\{y, c, d\}$, then one of c or d is either in L or is special.

Suppose that y is in a line $\{y, c, d\}$, where $c, d \notin L$. To simplify the exposition, suppose that $\{y, a_1, a_2, a_3\}$ is a circuit as in (S1). If c is not special, then c is in some line $\{c, c_1, c_2\}$, where $c_1, c_2 \in L$ and c_1, c_2 are in distinct lines with x . It follows that $\{y, c, d\} \triangle \{y, a_1, a_2, a_3\} \triangle \{c, c_1, c_2\} = \{d\} \cup (\{a_1, a_2, a_3\} \triangle \{c_1, c_2\}) = D$ is a disjoint union of circuits of M ; since the set $\{a_1, a_2, a_3\} \triangle \{c_1, c_2\}$ does not contain any lines, D is actually a circuit of M . It remains to show that $\{c_1, c_2\} \not\subset \{a_1, a_2, a_3, b_1, b_2, b_3\}$, as then d is special and (S3) holds. Note that $\{y, a_1, a_2, a_3\} \triangle \ell_1 \triangle \ell_2 = \{y, b_1, b_2, a_3\}$ is also a circuit of M (and so are $\{y, a_1, b_2, b_3\}$ and $\{y, b_1, a_2, b_3\}$). Therefore, to show that $\{c_1, c_2\} \not\subset \{a_1, a_2, a_3, b_1, b_2, b_3\}$, we only need to consider three cases:

- $\{c_1, c_2\} = \{a_1, a_2\}$: in this case $D = \{d, a_3\}$, but M is a simple matroid.
- $\{c_1, c_2\} = \{a_1, b_1\}$: this implies that $c = x$, but c is not in L .
- $\{c_1, c_2\} = \{a_1, b_2\}$: this would imply that $D = \{d, a_2, b_2, a_3\}$, hence $D \triangle \ell_2 = \{d, x, a_3\}$ is a line of M , but this is not possible, since $d \notin L$.

The next property we show for a special element y is:

- (S4) y is in at most two lines containing an element in L .

To prove (S4), we require the following technical result, which forbids certain configurations of pairs of special points. We defer the proof of this result to the very end of the paper.

5.1. Lemma. *Let M be a 3-connected binary matroid. Suppose that an element x of M is in distinct lines $\ell_1, \dots, \ell_{r-1}$ such that $r(\ell_1 \cup \dots \cup \ell_{r-1}) = r$. Let c_i be an element in $\ell_i - \{x\}$ for every i . Suppose that there exist two elements y_1 and y_2 such that $\{y_1, c_1, c_2, c_3\}$ and one of $\{y_2, c_3, c_4, c_5\}$ or $\{y_2, c_4, c_5, c_6\}$ are circuits of M . Then M contains an $AG(3, 2)$ -minor.*

Now we can proceed to show that (S4) holds. To simplify the exposition suppose that $C = \{c_1, c_2, c_3, y\}$ is a circuit as in (S1), where $c_i \in \ell_i$ for $i = 1, 2, 3$. First we show that y is not in a line with a_j or with b_j for $j \geq 4$. Suppose that y is in a line $\ell = \{y, c_j, d\}$, where $c_j \in \ell_j$ and $j \geq 4$. The element y is special, therefore $d \notin L$. Therefore $\ell \triangle C = \{d, c_1, c_2, c_3, c_j\}$ is a circuit and in M/d the elements in $(\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_j) - \{x\}$ form an $AG(3, 2)$ -restriction. Next we show that y is not in two lines $\{y, c, a_j\}$ and $\{y, d, b_j\}$; if this is the case, then $\{c, d, x\}$ is a line, therefore $c, d \in L$, contradicting the fact that y is special.

It follows that if y is in three lines with elements in L , then we may assume these lines are $\{y, a_1, d_1\}$, $\{y, a_2, d_2\}$, $\{y, a_3, d_3\}$, where $d_1, d_2, d_3 \notin L$. By possibly taking the symmetric difference of C with two of the lines ℓ_1, ℓ_2, ℓ_3 , we may assume that $C = \{a_1, a_2, c_3, y\}$, where $c_3 \in \{a_3, b_3\}$.

If $c_3 = b_3$, then consider the matrix representation for M with respect to the basis $\{x, a_1, a_2, \dots, a_{r-1}\}$. Contracting a_4, \dots, a_{r-1} we obtain the following submatrix:

$$\begin{array}{cccccccccccc} x & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & y & d_1 & d_2 & d_3 \\ \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \right] \end{array}$$

Adding the second, third and fourth rows to the first and deleting the columns corresponding to a_1, a_2 and a_3 we obtain the following matrix, which is a representation of $AG(3,2)$.

$$\begin{array}{cccccccc} x & b_1 & b_2 & b_3 & y & d_1 & d_2 & d_3 \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \right] \end{array}$$

It follows that $c_3 = a_3$. By (S2), y is in at least $r-2$ lines. Moreover, at most three of these lines are with elements in L . By (S3) this implies that there exists at least another special element y' . Let C' be a circuit for y' as in (S1). By Lemma 5.1 (and possibly taking the symmetric difference of C' with two of the lines spanning y') we may assume that $a_1, a_2 \in C'$. If $b_3 \in C'$, then $C \triangle C' \triangle \ell_3 = \{x, y, y'\}$ is a line of M , contradicting the fact that all the lines through x are in L . Therefore we may assume that $C' = \{y', a_1, a_2, a_4\}$. It is routine to check that $\{x, y, a_3, b_1, b_2, b_4, d_1, d_2\}$ is an $AG(3,2)$ -restriction of M/y' . We conclude that (S4) holds.

Let $S = \{y_1, \dots, y_s\}$ be the set of all special elements of M . We already showed that S is non-empty. Because of property (S2), y_1 is in at least $r-2$ lines. By (S3) and (S4), there are at least $r-4 \geq 4$ special elements distinct from y_1 . Therefore $s \geq 5$. For every $t \in [s]$, let C_t be a circuit containing y_t as in (S1). Note that if $C_t = \{y_t, c_i, c_j, c_k\}$, where $c_i \in \ell_i$ and so on, then $C_t \triangle \ell_i \triangle \ell_j$ is also a circuit for y_i as in (S1); we will make repeated use of this simple observation.

The final property we require for special elements is the following.

(S5) We may assume that $C_i = \{a_1, a_2, a_{i+2}\}$ for every $i = 1, \dots, s$. In particular $s \leq r-3$.

By property (S1), every special element y is associated with a triple $\{i, j, k\}$, where y is in the span of $\ell_i \cup \ell_j \cup \ell_k$. For every special element y_t let the associated triple be T_t . First we show that for every distinct $i, j \in [s]$ the triples T_i and T_j share exactly two elements. By Lemma 5.1, $|T_i \cap T_j| \geq 2$. If this quantity is 3 then $\{x, y_1, y_2\}$ is a line of M , a contradiction. Therefore, for all distinct $i, j \in [s]$, $|T_i \cap T_j| = 2$. Since $s \geq 5$, this implies that there are indices $t_1, t_2 \in [r-1]$ such that $T_i \cap T_j = \{t_1, t_2\}$ for all distinct $i, j \in [s]$. By relabelling we may assume that $C_i = \{a_1, a_2, a_{i+2}\}$ for every $i = 1, \dots, s$, and property (S5) holds.

By (S2) and Claim 2, every special element is in at least $r-2$ lines. By (S3) and (S4), every y_i is in at most $(|S|-1)+2 = |S|+1$ lines. By (S5), $|S| \leq r-3$. Therefore $|S| = r-3$ and every special element is in exactly $r-2$ lines. Moreover S is an independent set: if not, it contains a circuit C and we may assume $C = \{y_1, \dots, y_k\}$. Now $C \triangle C_1 \triangle \dots \triangle C_k \subseteq \{a_1, \dots, a_{k+2}\}$ is dependent, a contradiction.

Since S is independent and $s \geq 4$, we may choose a special element y that is not in an F_7 -restriction, or we would obtain an $AG(3,2)$ -minor by Lemma 4.2. Since y is in exactly $r-2$ lines of M and M/y is non-regular (property (S2)), we conclude the proof of Theorem 1.2 by Claim 3.

Proof of Lemma 5.1. Assume that the lemma fails for M , where M is as small as possible. First we assume that $\{y_2, c_3, c_4, c_5\}$ is a circuit of M . It follows easily from the hypotheses that $\{x, c_1, c_2, c_3, c_4, c_5\}$ is independent in M , so $\{c_1, c_2, c_3, c_4, c_5\}$ is independent in M/x , and neither $\{c_1, c_2, c_3\}$ nor $\{c_3, c_4, c_5\}$ spans x in M . Hence $\{y_1, c_1, c_2, c_3\}$ and $\{y_2, c_3, c_4, c_5\}$ are circuits in M/x . Their symmetric difference, $C = \{y_1, y_2, c_1, c_2, c_4, c_5\}$ is a disjoint union of circuits in M/x . If C is not a

circuit of M/x , there is a circuit in C containing y_1 , but not y_2 . The symmetric difference of this circuit with $\{y_1, c_1, c_2, c_3\}$ is a circuit contained in $\{c_1, c_2, c_3, c_4, c_5\}$, which is a contradiction as this set is independent in M/x . Thus C is a circuit of M/x , so $\{c_1, c_2, c_4, c_5\}$ is a circuit in $M/\{x, y_1, y_2\}$. Now Lemma 4.1 implies the result.

Next we consider the case that $t = 6$. Let $L_1 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \{y_1\}$, $L_2 = \ell_4 \cup \ell_5 \cup \ell_6 \cup \{y_2\}$. Assume there is an element $e \notin cl(L_1 \cup L_2)$. Bixby's lemma says that $co(M \setminus e)$ or $si(M \setminus e)$ is 3-connected. In the latter case, $(M/e)|(L_1 \cup L_2) = M|(L_1 \cup L_2)$, and there are no parallel pairs of M/e contained in $L_1 \cup L_2$. Thus $si(M/e)$ provides us with a smaller counterexample to the lemma, and we have a contradiction. Therefore $co(M \setminus e)$ is 3-connected. Assume there is a triad of M that contains e and an element from $L_1 \cup L_2$. Such a triad must intersect one of the lines ℓ_1, \dots, ℓ_6 or one of $\{y_1, c_1, c_2, c_3\}$ or $\{y_2, c_4, c_5, c_6\}$ in a single element, a contradiction. Let S be a set such that $co(M \setminus e) \cong M \setminus e/S$. By the previous argument, $S \cap (L_1 \cup L_2) = \emptyset$. If $X \subseteq L_1 \cup L_2$, and $r(X \cup S) \neq r(X) + r(S)$, then there is a circuit contained in $L_1 \cup L_2 \cup S$ that contains elements from S , and any such circuit intersects a series pair of $M \setminus e$ in a single element. This shows that $(M \setminus e/S)|(L_1 \cup L_2) = M|(L_1 \cup L_2)$, so $co(M \setminus e)$ is a smaller counterexample. From this contradiction we conclude that $cl(L_1 \cup L_2) = E(M)$.

If $cl(L_1) \cup cl(L_2) = E(M)$, then $(cl(L_1), cl(L_2) - cl(L_1))$ is a 2-separation of M . Therefore there is an element e in neither $cl(L_1)$ nor $cl(L_2)$. Since $\{x, c_1, \dots, c_6\}$ is a basis of M , it follows that $\{c_1, \dots, c_6\}$ is a basis of M/x , and $\{y_1, c_1, c_2, c_3\}$ and $\{y_2, c_4, c_5, c_6\}$ are circuits of M/x . Let C be the fundamental circuit of e relative to $\{c_1, \dots, c_6\}$ in M/x . Then $C - e$ is not contained in $\{c_1, c_2, c_3\}$ or in $\{c_4, c_5, c_6\}$, by our choice of e . If $C' \subseteq C - e$ and $|C'| \geq 4$, then C' is a circuit in $M/((C \cup x) - C')$, and Lemma 4.1 implies M has an $AG(3, 2)$ -minor. Therefore $|C - e| \leq 3$. By relabelling as necessary, we can assume that either

- (a) $C = \{e, c_1, c_4, c_5\}$, or
- (b) $C = \{e, c_1, c_4\}$.

In case (a) we let C' be $C \Delta \{y_1, c_1, c_2, c_3\} = \{e, y_1, c_2, c_3, c_4, c_5\}$. Otherwise we let C' be $C \Delta \{y_1, c_1, c_2, c_3\} \Delta \{y_2, c_4, c_5, c_6\} = \{e, y_1, y_2, c_2, c_3, c_5, c_6\}$. In either case, C' is a disjoint union of circuits and $\{e, y_1\} \subseteq C'$. Furthermore, C' contains exactly two elements from each of $\{c_1, c_2, c_3\}$, and $\{c_4, c_5, c_6\}$. We prove that C' is a circuit of M/x .

Let $C'' \subseteq C'$ be a circuit of M/x that contains y_1 . Note that $C'' \neq \{y_1, c_1, c_2, c_3\}$, so C'' contains either e or y_2 . In case (a), $y_2 \notin C'$ and $C'' = C'$, for otherwise $C' - \{e, y_1\} = \{c_2, c_3, c_4, c_5\}$ is dependent. Next consider case (b). If $y_2 \notin C''$, then $e \in C''$, and the circuit in C' containing y_2 must be the unique circuit in $\{y_2, c_1, \dots, c_6\}$, namely $\{y_2, c_4, c_5, c_6\}$. This is impossible since $c_4 \notin C'$. Therefore $y_2 \in C''$. If $e \notin C''$, then the circuit in C' that contains e must be C , which is impossible by the choice of C' . Therefore $e, y_2 \in C''$, so $C'' = C'$ since $C' - \{e, y_1, y_2\}$ is independent in M/x . Now C' is a circuit in M/x , so $\{c_1, \dots, c_6\}$ contains a 4-element circuit in either $M/\{x, e, y_1\}$ or $M/\{x, e, y_1, y_2\}$. The result follows by Lemma 4.1. \square

APPENDIX A. LOW RANK EXAMPLES

At rank 6, the graphic matroid is the unique maximum sized matroid, but at rank 5 there are 3 others; these three can all be described as follows: first take a basis in $GF(2)^4$ and add the third point on every line determined by the six pairs of basis elements; clearly this is a representation of the matroid $M(K_5)$. Then add one further point, still in rank 4, yielding the matroid $M(K_5)^+$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Now embed $GF(2)^4$ as the hyperplane $x_1 = 0$ in $GF(2)^5$ and let $e = (1, 0, 0, 0, 0)^T$. Then construct three new matroids as follows: the first one is obtained by adding the points on the lines between e and three non-collinear points in the Fano sub plane of $M(K_5)^+$. The second is obtained by adding the points on the lines between e and any three original basis elements whose span is *not* the Fano sub plane of $M(K_5)^+$.

The final one does not have $M(K_5)^+$ in the induced hyperplane, but one of the points is “lifted” into the fifth dimension, but rather there are *five* affine points, namely e which is joined by long lines to three points in $M(K_5)^+$ and one additional point. The binary matrix shows it as well as a verbal explanation: the 4×11 portion of each matrix is the original $M(K_5)^+$.

$$\left[\begin{array}{cccccccccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccccccccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccccccccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

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