WELL-QUASI-ORDERING IN LATTICE PATH MATROIDS

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ABSTRACT. Lattice path matroids form a subclass of transversal matroids and were introduced by Bonin, de Mier and Noy [3]. Transversal matroids are not well-quasi-ordered, even when the branch-width is restricted. Though lattice path matroids are also not well-quasi-ordered, we prove that lattice path matroids of bounded branch-width are well-quasi-ordered.

1. INTRODUCTION

Well-quasi-ordering is at the heart of major projects undertaken in discrete mathematics in the recent years. A *quasi-ordering* is a relation that is reflexive and transitive. A *well-quasi-ordering* is a quasi-ordering, $\leq$, with the property that if $a_0, a_1, \ldots$ is an infinite sequence, then there exists i and j such that $a_i \leq a_j$.

The expansive Robertson-Seymour graph-minors project was a major accomplishment in discrete mathematics. It proved that graphs are well-quasi-ordered under the minor relation.

Recently, Geelen, Gerards and Whittle announced a proof (Theorem 6, [10]) that the class of F-representable matroids is well-quasi-ordered under the minor relation, where F is any finite field. This is connected to their proof of Rota's conjecture. One of the crucial steps that brought them closer to the proof of the former conjecture was their 2002 proof [6] that a class of F-representable matroids with bounded branch-width is well-quasi-ordered. We will accomplish the same goal for the class of lattice path matroids.

The class of lattice path matroids is an attractive class discovered by Bonin, de Mier and Noy [3] with nice structural properties. It is a subclass of transversal matroids, but surprisingly is closed under duality and minors, unlike the class of transversal matroids. Interestingly enough, lattice path matroids are not well-quasi-ordered (for examples, refer to Section 2.1). However, we prove that they are well-quasi-ordered when certain restrictions are placed on the class.

Theorem 1.1. *Lattice path matroids of bounded branch-width are well-quasi-ordered.*

The proof of the theorem uses the elegant minimal bad sequence argument that Nash-Williams employs to prove that finite trees are well-quasi-ordered [8].

On the other hand, transversal matroids do not behave so well under those same restrictions. To observe this, consider the well-known polygon matroid anti-chain (Example 14.1.2, [9]) that starts with the matroids represented in Figure 1.

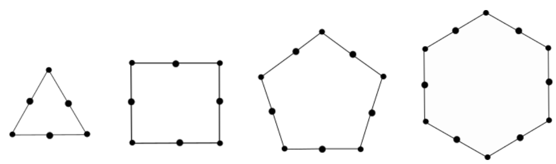

Figure 1. The first four matroids in the polygon matroid anti-chain

This is a class of rank-3 matroids, and hence has bounded branch-width (at most 3). Since branch-width of the dual of a matroid is the same as that of the original matroid, the dual class of these matroids also has bounded branch-width. Since strict gammoids are exactly the duals of transversal matroids (Corollary 2.4.5, [9]), it is enough to show that this infinite class indeed consists of strict gammoids, and this is an easy exercise. Thus we have found an infinite class of transversal matroids of bounded branch-width that form an anti-chain.

2. Lattice Path Matroids

The class of lattice path matroids can be best understood in terms of lattice paths.

All lattice paths considered here start at $(0, 0)$ and use the steps $E = (1, 0)$ (or moving right) and $N = (0, 1)$ (or moving up), which are called *East* and *North* respectively. The paths are written as words or strings in the alphabet $\{E, N\}$.

In the definitions following, if X is a lattice path with $m + r$ steps, then $\text{pre}_i(X)$ denotes the first i steps of X and $\text{suf}_i(X)$ denotes steps $i+1$ to $m+r$. Hence, we can consider the path P as $P = \text{pre}_i(P)\,\text{suf}_i(P)$, where juxtaposition indicates concatenation of strings. Also if X is any lattice path, then $r(X)$ = number of North steps in X, and $m(X)$ = number of East steps in X.

Let $P = p_1p_2 \dots p_{m+r}$ and $Q = q_1q_2 \dots q_{m+r}$ be two lattice paths from $(0, 0)$ to (m, r), where P never goes above Q. In other words, for every i, the number of North steps in $\mathrm{pre}_i(P)$ is never more than that in $\mathrm{pre}_i(Q)$ (and the number of East steps in $\mathrm{pre}_i(Q)$ is never more than that in $\mathrm{pre}_i(P)$). Let $p_{u_1}, p_{u_2}, \ldots, p_{u_r}$ be the set of North steps of P, with $u_1 < u_2 < \cdots < u_r$. Let $q_{l_1}, q_{l_2}, \ldots, q_{l_r}$ be the set of North steps of Q, with $l_1 < l_2 < \cdots < l_r$. Let N_i be the interval $[l_i, u_i]$ of integers. Let $M[P, Q]$ be the transversal matroid that has ground set $[m + r] = \{1, \ldots, m + r\}$ and presentation $(N_i : i \in [r])$. A *lattice path matroid* is a transversal matroid that is isomorphic to $M[P, Q]$ for some such pair of lattice paths P and Q.

Figure 2 gives an illustration of a lattice path presentation. Here $P = EENENEEENENEENN$ and $Q = NENNENEENEENEEE$. Also $N_1 = [1, 3]$, $N_2 = [3, 5]$, $N_3 = [4, 9]$, $N_4 = [6, 11]$, $N_5 = [9, 14]$ and $N_6 = [12, 15]$.

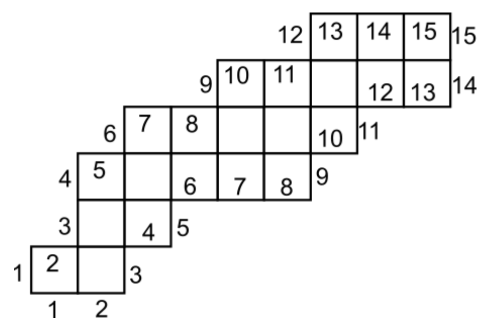

FIGURE 2. A lattice path matroid $M(P, Q)$, where the paths P and Q start at $(0, 0)$ and end at $(9, 6)$

We say that $[P, Q]$ is a *lattice path presentation* that corresponds to the matroid $M[P, Q]$. The size of a presentation is nothing but the size of the ground set of the corresponding matroid. We use r and m to denote the rank and co-rank of $M[P, Q]$ respectively. We blur the distinction between the path presentation $[P, Q]$ and the matroid $M[P, Q]$ when doing so will not create a confusion.

When thought of as arising from the particular presentation of bounding paths P and Q, the elements are in their natural order. However this order is not evident in the matroid structure.

Let X be a subset of the ground set $[m + r]$ of the lattice path matroid $M[P, Q]$. The lattice path $P(X)$ is the word $s_1 s_2 \ldots s_{m+r}$ in the alphabet $\{E, N\}$, where

$$s_i = \begin{cases} N, & \text{if} \quad i \in X \\ E, & \text{otherwise.} \end{cases}$$

This leads to the following characterisation of bases of lattice path matroids:

A subset B of $[m+r]$ with $|B| = r$ is a basis of $M[P,Q]$ if and only if $P(B)$ stays in the region bounded by P and Q (Theorem 3.3, [3]). That is, $\mathrm{pre}_i(P(X))$ has no more North steps than $\mathrm{pre}_i(Q)$ for all i, and no more East steps than $\mathrm{pre}_i(P)$ for all i. Note that the paths P and Q also stay in the desired region and hence correspond to bases.

Minors: Let $[P,Q]$ be a lattice path presentation. Single element deletions and contractions can be described in terms of bounding paths of $M = M[P,Q]$ as follows: An isthmus is an element x for which some N_i is $\{x\}$. So, to delete or contract x, delete the corresponding North step from both the bounding paths. Correspondingly, to delete or contract a loop, which is an element that is in no set N_i, delete that East step from both the bounding paths.

If x is neither a loop nor an isthmus, the upper bounding path of $M\backslash x$ is formed by deleting from Q the first East step that is at or after x and the lower bounding path is obtained by deleting from P the last East step that is at or before x. Dually, the upper bounding path of M/x is formed by deleting from Q the last North step that is at or before x and the lower bounding path is obtained by deleting from P the first North step that is at or after x.

Lattice path matroids are closed under minors, duals and direct sums (Theorem 3.1, [1]) but are not closed under the operations of truncation, elongation and free extension.

We define a lattice path presentation to be a minor of another if it can be obtained by the operations of deletions and contractions, as described above, from the other.

Nested matroids form a subclass of lattice path matroids that has appeared under different names in varying contexts. A nested matroid is obtained from the empty matroid by iterating the operations of adding co-loops and taking free extensions. Bonin and de Mier [1] defined them in terms of lattice path matroids as a matroid of the form $M[P,Q]$, where $P = E^m N^r$ and named them generalised Catalan matroids. They later proved that nested matroids are well-quasi-ordered (Theorem 5.4, [2]).

2.1 Well-Quasi-Ordering.

An infinite sequence $a_1, a_2, a_3, \dots$ is *bad* if there does not exist an $i < j$ such that $a_i \leq a_j$. Otherwise, the sequence is good. A quasi-order is well-quasi-ordered if and only if it does not have a bad sequence.

An infinite sequence $a_1, a_2, \dots$ is called an anti-chain when there does not exist i, j such that $i \neq j$ and $a_i \leq a_j$. Also, a sequence $a_1, a_2, \dots$

is infinitely strictly decreasing if $a_i > a_{i+1}$ for $i \geq 1$. Equivalently, a quasi-order is a well-quasi-order is and only if there exists no infinite anti-chain or infinite decreasing sequence (See, for example, [4], Prop 12.1.1). Hence, a graph class or matroid class is well-quasi-ordered if and only if it does not have an infinite antichain, as graph and matroid classes do not contain infinite decreasing sequences.

As we mentioned in the introduction, lattice path matroids are not well-quasi-ordered. There is a subclass of lattice path matroids named notch matroids by Bonin and de Mier. Their paper includes an excluded minor characterisation for notch matroids (Theorem 8.8, [1]). Among the excluded minors are three infinite families of lattice path matroids, which are listed below:

- for $n \geq 4$, the rank-n matroid $F_n := [E^{n-3}N^2EN^{n-2}, N^{n-2}EN^2E^{n-3}]$,
- for $n \geq 2$, the rank-n matroid $G_n := [E^nNE^2N^{n-1}, N^{n-1}E^2NE^n]$, and
- for $n \geq 3$, the rank-n matroid $H_n := [E^{n-2}NE^2N^{n-1}, N^{n-2}EN^2E^{n-1}]$.

Thus we conclude that these infinite families each form an anti-chain in the class of lattice path matroids and hence the class is not well-quasi-ordered.

3. Square-width

Let $[P, Q]$ be a pair of lattice paths that correspond to the lattice path matroid $M[P, Q]$.

We say that $[P, Q]$ has a $k \times k$ *square at* $i \in [m + r]$ when $\text{pre}_i(P)$ has exactly k more East steps than $\text{pre}_i(Q)$, and $\text{pre}_i(Q)$ has exactly k more North steps than $\text{pre}_i(P)$. This corresponds to a $k \times k$ square in the region of the integer lattice bounded by P and Q. A $k \times k$ square is *proper* if $i \in [k + 1, m + r - k - 1]$. We say that $[P, Q]$ has a $k \times k$ *square at the top* if the last k steps of P are North and the last k steps of Q are East. Similarly, $[P, Q]$ has a $k \times k$ *square at the bottom* if the first k steps of P are East and those of Q are North.

A lattice path presentation is said to have *square-width* k when the largest square it contains is a $k \times k$ square. The square-width of $[P, Q]$ is closely associated with branch-width of the matroid $M[P, Q]$, as we see below.

Lemma 3.1. *Let $[P, Q]$ be a path presentation with a $k \times k$ square. Then $M[P, Q]$ has a $U_{k,2k}$-minor.*

Proof. Let $M[P,Q]$ be a minimal counter-example to our hypothesis, with P and Q being lattice paths from $(0,0)$ to (m,r). Let the corners of the $k\times k$ square be at (i,j), $(i+k,j)$, $(i,j+k)$ and $(i+k,j+k)$. If $i>0$, then the first element is not part of the $k\times k$ square. This implies that $M[P,Q]\backslash 1$ contains a $k\times k$ square, which in turn implies that $M[P,Q]\backslash 1$ contains a $U_{k,2k}$-minor. But then so would $M[P,Q]$, which contradicts our assumption. Thus $i=0$. Similarly, $j=0$ as otherwise, $M[P,Q]/1$ would contain a $k\times k$ square.

Now, if $2k<m+r$, the path presentation of either $M[P,Q]\backslash(m+r)$ or $M[P,Q]/(m+r)$ contains a $k\times k$ square, which again leads to a contradiction. Hence, $2k=m+r$. Also, since $k\leq\min\{m,r\}$, it follows that $m=r=k$. Thus P and Q bound a $k\times k$ square, and so $M[P,Q]$ is in fact isomorphic to $U_{k,2k}$ and the proof is complete. □

Corollary 3.2. *Let $M=M[P,Q]$ be a lattice path matroid and assume that $bw(M)\leq k$. Then the square-width is less than $\lceil 3k/2\rceil$.*

Proof. Assume that the square width j is at least $\lceil 3k/2\rceil$. Then by Lemma 3.1, $M[P,Q]$ has $U_{j,2j}$ minor. But the branch-width of $U_{j,2j}$ is $\lceil 2j/3\rceil+1$ (Exercise 14.2.5, [9]), and hence the branch-width of $M[P,Q]$ is at least $k+1$. But this is a contradiction to our assumption that the branch-width of $M[P,Q]$ is at most k. □

It is not difficult, but unnecessary for Theorem 1.1, to prove that a class of lattice-path matroids has bounded branch-width if and only if it has bounded square-width. In fact, if $[P,Q]$ has square-width at most k, then $M[P,Q]$ has branch-width at most $k+1$.

Definition 3.1. *Let $[P,Q]$ be a lattice path presentation on $[m+r]$ with a proper $k\times k$ square at i. Then we define two new lattice path presentations from $[P,Q]$ as follows:*

$$B_i(P)=\mathrm{pre}_i(P)N^k$$
$$B_i(Q)=\mathrm{pre}_i(Q)E^k$$

and

$$T_i(P)=E^k\,\mathrm{suf}_i(P)\ \textit{and}$$
$$T_i(Q)=N^k\,\mathrm{suf}_i(Q).$$

Note that $M([B_i(P),B_i(Q)])$ is a lattice path matroid on the ground set $[i+k]$ but we relabel $M([T_i(P),T_i(Q)])$ to be a lattice path matroid on the ground set $[i-k,m+r]$, so as to retain the same elements as in the original matroid. This will prove beneficial in the gluing operation that follows.

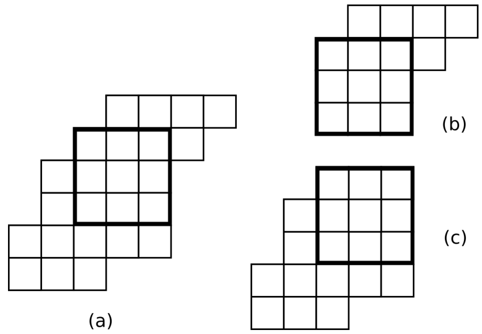

FIGURE 3. (a) A proper 3×3 square in $[P, Q]$ (b) $[T_i(P), T_i(Q)]$ (c) $[B_i(P), B_i(Q)]$

In Figure 3, $\mathrm{pre}_7(P) = EEENEEN$, $\mathrm{pre}_7(Q) = NNENNEN$, $\mathrm{suf}_7(P) = NNENEN$ and $\mathrm{suf}_7(Q) = ENEEEE$. Thus $B_7(P) = \mathrm{pre}_7(P)N^3$, $B_7(Q) = \mathrm{pre}_7(Q)E^3$, $T_7(P) = E^3\,\mathrm{suf}_7(P)$ and $T_7(Q) = N^3\,\mathrm{suf}_7(Q)$.

An intuitive property of these new path presentations is proved below:

Lemma 3.3. *Let $[B_i(P), B_i(Q)]$ and $[T_i(P), T_i(Q)]$ be as in Definition 3.1. Then $M([B_i(P), B_i(Q)]), M([T_i(P), T_i(Q)])$ are minors of $M([P, Q])$.*

Proof. We know that $P = \mathrm{pre}_i(P)\,\mathrm{suf}_i(P)$ and $Q = \mathrm{pre}_i(Q)\,\mathrm{suf}_i(Q)$, where $1 < i < m + r$. Since there is a $k \times k$ square at i, $\mathrm{pre}_i(P)$ has k more East steps than $\mathrm{pre}_i(Q)$. Hence, $\mathrm{suf}_i(Q)$ has k more East steps than $\mathrm{suf}_i(P)$. Similarly, $\mathrm{suf}_i(P)$ has k more North steps than $\mathrm{suf}_i(Q)$. Recall that deletion requires the removal of East steps from both paths, and contraction requires removal of North steps from both paths.

Let J be the subset of $\{i+1, \ldots, m+r\}$ such that P contains E at exactly those steps. Let $[P', Q']$ be $[P, Q] \backslash J$. Then, $\mathrm{suf}_i(P^{'})$ contains no East steps but $\mathrm{suf}_i(Q^{'})$ has exactly k. Now let I be the set of positions after i where $Q^{'}$ contains N. Define $[P^{''}, Q^{''}]$ to be $[P^{'}, Q^{'}] \backslash I$. So $\mathrm{suf}_i(Q^{''})$ contains no North steps and $\mathrm{suf}_i(P^{''})$ contains exactly k. Hence, $[P^{''}, Q^{''}] = [\mathrm{pre}_i(P)N^k, \mathrm{pre}_i(Q)E^k]$. Then $[P^{''}, Q^{''}]$ is nothing but $[B_i(P), B_i(Q)]$.

The same deductions as above imply that $[T_i(P), T_i(Q)]$ is a minor of $[P, Q]$. $\square$

Definition 3.2. *If we have a pair of lattice path presentations $[P_B, Q_B]$ and $[P_T, Q_T]$ such that the following conditions are satisfied:*

(i) *The last k steps in P_B are North steps, that is, $P_B = P^{'}_B N^k$*

(ii) *The last k steps in Q_B are East steps, that is,* $Q_B = Q_B^{'}E^k$
(iii) *The first k steps in P_T are East steps, that is,* $P_T = E^k P_T^{'}$
(iv) *The first k steps in Q_T are North steps, that is,* $Q_T = N^k Q_T^{'}$.

Then we define $GL([P_B, Q_B], [P_T, Q_T])$ *to be* $[P, Q]$*, where* $P = P_B^{'}P_T^{'}$*, and* $Q = Q_B^{'}Q_T^{'}$*. When they are defined as above, we say that* $[P_B, Q_B]$ *has a* $k \times k$ *square at the top and* $[P_T, Q_T]$ *has a* $k \times k$ *square at the bottom and that* $GL([P_B, Q_B], [P_T, Q_T])$ *is obtained by gluing* $[P_B, Q_B]$ *and* $[P_T, Q_T]$*.*

In other words, if we start with a lattice path presentation $[P, Q]$ and 'pull them apart' at a $k \times k$ square to give rise to two new lattice paths $[B_i(P), B_i(Q)]$ and $[T_i(P), T_i(Q)]$, then $GL([B_i(P), B_i(Q)], [T_i(P), T_i(Q)])$ will lead us back to the lattice path $[P, Q]$ that we originally had. This fact is illustrated in the following lemma:

Lemma 3.4. *Let* $[P, Q]$ *be a path presentation with a* $k \times k$ *square at* i*, where* $P = \mathrm{pre}_i(P)\,\mathrm{suf}_i(P)$ *and* $Q = \mathrm{pre}_i(Q)\,\mathrm{suf}_i(Q)$*. Construct the two lattice paths* $[B_i(P), B_i(Q)]$ *and* $[T_i(P), T_i(Q)]$*. Then* $GL([B_i(P), B_i(Q)], [T_i(P), T_i(Q)]) = [P, Q]$*.*

Proof. By Definition 3.1, paths $B_i(P), B_i(Q), T_i(P)$ and $T_i(Q)$ satisfy conditions (i)–(iv) in Definition 3.2. Thus $P_B^{'} = \mathrm{pre}_i(P)$, $Q_B^{'} = \mathrm{suf}_i(P)$, $P_T^{'} = \mathrm{suf}_i(P)$ and $Q_T^{'} = \mathrm{suf}_i(Q)$. Thus $P_B^{'}P_T^{'} = \mathrm{pre}_i(P)\,\mathrm{suf}_i(P) = P$ and $Q_B^{'}Q_T^{'} = \mathrm{pre}_i(Q)\,\mathrm{suf}_i(Q) = Q$. This completes the proof. $\square$

We are now well-equipped to prove the central lemma which proves pivotal in proving the main theorem.

Lemma 3.5. *Let* $[P, Q]$ *be a path presentation with a proper* $k \times k$ *square at* i*. Let* $[B_i(P), B_i(Q)]$ *and* $[T_i(P), T_i(Q)]$ *be as in Definition 3.1. Let* $[P_B, Q_B]$ *be a minor of* $[B_i(P), B_i(Q)]$ *with a* $k \times k$ *square at the top and* $[P_T, Q_T]$ *be a minor of* $[T_i(P), T_i(Q)]$*, with a* $k \times k$ *square at the bottom. Then* $GL([P_B, Q_B], [P_T, Q_T])$ *is a minor of* $[P, Q]$*.*

Proof. Let b_1 be the size of $[B_i(P), B_i(Q)]$ and b_2 be the size of $[P_B, Q_B]$. Similarly, let t_1 be the size of $[T_i(P), T_i(Q)]$ and t_2 be the size of $[P_T, Q_T]$. Now, let $n = (b_1 - b_2) + (t_1 - t_2)$. We then proceed by induction on n. Note that $[B_i(P), B_i(Q)]$ has a $k \times k$ square at the top and $[T_i(P), T_i(Q)]$, has a $k \times k$ square at the bottom, by definition.

When $n = 0$, the lemma holds trivially. Assume that it holds true for $n = k$. We prove the result for $n = k + 1$, where $k \geq 0$.

When $n = k + 1$, $(b_1 - b_2) + (t_1 - t_2) = k + 1$. Assume that $b_1 - b_2 > 0$. (The case that $t_1 - t_2 > 0$ is similar.) Thus

$[P_B, Q_B] = [B_i(P), B_i(Q)]/I\backslash J$, where I, J are disjoint sets of elements in $[B_i(P), B_i(Q)]$, with $I \cup J \neq \emptyset$ as $b_1 - b_2 = |I \cup J|$.

Now choose $e \in I \cup J$. If $e \in I$, then let L be $[B_i(P), B_i(Q)]/(I-e)\backslash J$. Otherwise, let L be $[B_i(P), B_i(Q)]/I\backslash(J-e)$. Since $[P_B, Q_B]$ has a $k \times k$ square at the top, so does L. Thus $L = [P(L), Q(L)] = [L_P N^k, L_Q E^k]$ for some lattice paths L_P and L_Q.

Note the difference in sizes between L and $[B_i(P), B_i(Q)]$ is $b_1 - b_2 - 1$. By induction, $GL(L, [P_T, Q_T])$ is a minor of $[P, Q]$. It suffices to show that $GL([P_B, Q_B], [P_T, Q_T])$ is a minor of $GL(L, [P_T, Q_T])$.

We will show that $GL([P_B, Q_B], [P_T, Q_T])$ is $GL(L, [P_T, Q_T])$ with e either deleted or contracted. Note that the first k steps of P_T are East steps and the first k steps of Q_T are North steps. Hence $[P_T, Q_T] = [E^k T_P, N^k T_Q]$ for some lattice paths T_P and T_Q.

We first consider the case where e is a step in the sub-strings L_P and L_Q. Assume that $e \in I$. We consider the different cases where e is a North step and an East step in the sub-strings L_P and L_Q.

Case (i): e is an East step in both L_P and L_Q

We can decompose the paths as $L_P = P_0 E P_1$ and $L_Q = Q_0 E Q_1$, where the East step represents the position of e position. Then, it is obvious that we have to remove the East steps from both paths. Thus,

$$L = [L_P N^k, L_Q E^k] = [P_0 E P_1 N^k, Q_0 E Q_1 E^k].$$

Therefore,

$$L\backslash e = [P_0 P_1 N^k, Q_0 Q_1 E^k] = [P_B, Q_B].$$

Then,

$$GL(L\backslash e, [P_T, Q_T]) = [P_0 P_1 T_P, Q_0 Q_1 T_Q] = GL([P_B, Q_B], [P_T, Q_T]).$$

On the other hand,

$$GL(L, [P_T, Q_T]) = [P_0 E P_1 T_P, Q_0 E Q_1 T_Q] \text{ and thus}$$

$$GL(L, [P_T, Q_T])\backslash e = [P_0 P_1 T_P, Q_0 Q_1 T_Q] = GL([P_B, Q_B], [P_T, Q_T]).$$

Case (ii): e is an East step in L_P, but a North step in L_Q

Then L in this case can be represented as :

$$[L_P N^k, L_Q E^k] = [P_0 E P_1 N^k, Q_0 N Q_1 E^k].$$

When deleting e, we remove the East step from L_P. Since, $L_Q = Q_0 N Q_1$, Q_1 will contain an East step for otherwise, suppose that Q_1 does not contain any East step. Then we will have to remove an East step from the last k East steps following L_Q which will result in destroying the $k \times k$ square in L. Then we rewrite L_Q as $L_Q = Q_0' E Q_1'$, where E is the first East step in Q_1. Hence,

$$L = [L_P N^k, L_Q E^k] = [P_0 E P_1 N^k, Q_0' E Q_1' E^k].$$

Therefore,

$$L\backslash e = [P_0P_1N^k, Q_0'Q_1'E^k] = [P_B, Q_B].$$

Then,

$$GL(L\backslash e, [P_T, Q_T]) = [P_0P_1T_P, Q_0'Q_1'T_Q] = GL([P_B, Q_B], [P_T, Q_T]).$$

Similar to the case above,

$$GL(L, [P_T, Q_T]) = [P_0EP_1T_P, Q_0'EQ_1'T_Q] \text{ and thus}$$

$$GL(L, [P_T, Q_T])\backslash e = [P_0P_1T_P, Q_0'Q_1'T_Q] = GL([P_B, Q_B], [P_T, Q_T]).$$

Case (iii): e is a North step in both P and Q

As before, we can represent L as $[L_PN^k, L_QE^k] = [P_0NP_1N^k, Q_0NQ_1E^k]$. Suppose L_P has only North steps in P_0. Then, Q_0 will also contain only North steps. This implies that e is a coloop, and deleting a coloop involves removal of the corresponding North step from both paths. (This holds true for contraction too.)

On the other hand, say P_0 has an East step. Then, $L_P = P_0'EP_1'$, where E represents the first East step before position e. And by the previous argument, Q_1 will contain an East step. Thus,

$$L = [L_PN^k, L_QE^k] = [P_0'EP_1'N^k, Q_0'EQ_1'E^k] \text{and}$$

$$L\backslash e = [P_0'P_1'N^k, Q_0'Q_1'E^k] = [P_B, Q_B].$$

Hence,

$$GL(L\backslash e, [P_T, Q_T]) = [P_0'P_1'T_P, Q_0'Q_1'T_Q] = GL([P_B, Q_B], [P_T, Q_T]).$$

And,

$$\begin{aligned} GL(L, [P_T, Q_T])\backslash e &= [P_0'EP_1'T_P, Q_0'EQ_1'T_Q]\backslash e \\ &= [P_0'P_1'T_P, Q_0'Q_1'T_Q] \\ &= GL([P_B, Q_B], [P_T, Q_T]). \end{aligned}$$

Case (iv): e is a North step in P, but an East step in Q

The same conclusion follows by the arguments made in previous cases.

From the above cases we observe that when e is an element deleted in L_P and L_Q, the result holds true. So we are finished with the case when $e \in I$. When e is to be contracted ($e \in J$), almost identical arguments hold. Hence, we conclude that the result holds true when e is a step in the sub-strings L_P and L_Q.

Now, if e is not an element in L_P and L_Q, but is an element from the last k North/East steps that create the square in L, then we claim that this is the same as deleting or contracting the last element in the paths L_P and L_Q.

Case (a) : e is deleted

We know that deletion of an element involves removal of an East step from both paths. The last k steps in $Q(L)$ are East steps (as L has a $k \times k$ square at the top). But so does Q_B, and thus we require that $Q(L)$ has only East steps as the last $k+1$ steps, that is, the last element in L_Q is an East step, otherwise deleting e from L destroys the $k \times k$ square. Hence deletion of any of the last k East steps from $Q(L)$ is the same as deleting the last step in L_Q.

Now, $P(L)$ has North steps only as the last k steps. Then the last step in L_P can either be a North step or an East step. If it is an East step, then since deletion of an element in a lower bounding path is equivalent to deleting the East step at or before the corresponding element, deletion of any of the last k North steps will result in the removal of the last step in L_P, which is an East step. Thus deletion of e is the deletion of the last step in L_P if the last step is East.

But, if the last step in L_P is a North step, then deletion is the removal of the first East step that comes before the last $k+1$ North steps. But once again, this is the same as deletion of the last step in L_P. Thus deleting any element e that is in L but not in $[L_P, L_Q]$ and then using the gluing operation to join the different path presentations together is the same as gluing them together and then deleting the last element of L_P and L_Q.

In conclusion,

$$L = [P_0EN^{k'}, Q_0EE^k], \text{where } k' \geq k.$$

Then,

$$L \backslash e = [P_0N^{k'}, Q_0E^k] = [P_B, Q_B].$$

Thus,

$$\begin{aligned} GL(L\backslash e, [P_T, Q_T]) &= [P_0N^{k''}T_P, Q_0T_Q] \\ &= GL([P_B, Q_B], [P_T, Q_T]), \text{where } k'' = k' - k. \end{aligned}$$

Also,

$$GL(L, [P_T, Q_T]) = [P_0EN^{k'}T_P, Q_0EE^kT_Q] \text{ and thus}$$
$$GL(L, [P_T, Q_T])\backslash e = [P_0N^{k''}T_P, Q_0T_Q] = GL([P_B, Q_B], [P_T, Q_T]).$$

Case(b) : e was contracted

This case yields to a similar argument as case (i).

□

4. Proof of the main theorem

With the aid of the above lemma, we are now ready to prove that the class of lattice path matroids with bounded square-width is well-quasi-ordered. We use the *minimal bad sequence* argument in the proof of the same. We require a lemma about bad sequences to explain the minimal bad sequence argument. This result is well-known, but we include a proof for completeness.

Lemma 4.1. *Let $a_1, a_2, \ldots,$ be an infinite sequence with no bad subsequences. Then there exists an infinite sequence $i_1 < i_2 < i_3 < \ldots$, such that $a_{i_s} \leq a_{i_{s+1}}$ for all s.*

Proof. We begin by constructing a directed graph as follows: if $a_i < a_j$, where $i < j$, and there does not exist a k such that $i < k < j$ and $a_i \leq a_k \leq a_j$, then we have a directed edge from vertex a_i to vertex a_j.

Now, G has to be a directed graph with finitely many connected components. Else, suppose that G has infinitely many connected components $G_1, G_2, \ldots$. Then selecting a vertex from each component provides us with a bad sequence.

Clearly, at least one among the finitely many components of G must have infinite number of vertices. Hence, by König's Lemma [7](see also [5]), this infinite graph either contains a vertex of infinite degree or an infinite simple path. If there exists a vertex of infinite degree, then its neighbours form a bad sequence. Thus G cannot contain a vertex of infinite degree. Hence it contains an infinite path, which completes our proof.

□

We now prove the main theorem:

Theorem 4.2. *Let $\mathcal{L}_k$ be the class of lattice path matroids with square-width at most k. Then, $\mathcal{L}_k$ is well-quasi-ordered.*

Proof. We prove this by induction on k. When $k = 0$, the class consists of matroids that arise from path presentations where $P = Q$. Thus they can be represented by a combination of horizontal lines that go right or vertical lines that go up. Then the corresponding matroids consist of only loops and co-loops. Hence these matroids are nested matroids which form a well-quasi-ordered class as stated before.

We assume that the result is true for k, i.e., $\mathcal{L}_k$ is well-quasi-ordered. We now prove that $\mathcal{L}_{k+1}$ is well-quasi-ordered.

Suppose the contrary that $\mathcal{L}_{k+1}$ is not well-quasi-ordered. Then there exists a bad sequence in $\mathcal{L}_{k+1}$. Clearly, every bad sequence has only finitely many path presentations that belong to $\mathcal{L}_k$. Thus we can safely remove this finite subsequence from the bad sequence without altering the property of being bad. Also, as uniform matroids are well-quasi-oredred, we can consider only bad sequences that contain no such matroids. Hence, from now on we only consider bad sequences that are made up entirely of path presentations of square width $k+1$, that are not uniform.

We now construct a minimal bad sequence as follows: Assume we have chosen $L_1, \dots, L_{i-1}$ to be the initial segment of our minimal bad sequence. Then, L_i is the smallest L such that there exists a bad sequence that starts with $L_1, \dots, L_{i-1}, L$. We denote this sequence by $L_1, L_2, L_3, \dots$, where $L_i = [P_i, Q_i]$. It can be seen easily that $L_1, L_2, L_3, \dots$ is a bad sequence in itself. If not, there exists $i < j$ such that $L_i \leq L_j$. By virtue of construction of the minimal bad sequence, L_j is the smallest presentation in the j^{th} position among all bad sequences that start with $L_1, \dots, L_{j-1}$. Thus L_i is a minor of L_j, which contradicts the fact that $L_1, L_2, \dots L_j$ appears at the beginning of a bad sequence. Each L_i has a proper $k+1$ square because it is not a uniform matroid, as uniform matroids correspond to rectangualr presentations. Say L_i has a proper $k+1$ square at $j(i)$.

We apply the pulling apart operation as defined in Definition 3.1 to the sequence $L_1, L_2, L_3, \dots$ to obtain two new sequences $B_1, B_2, B_3, \dots$, where $B_i = [B_{j(i)}(P_i), B_{j(i)}(Q_i)]$ and $T_1, T_2, T_3, \dots$, where $T_i = [T_{j(i)}(P_i), T_{j(i)}(Q_i)]$. Since the sequence $L_1, L_2, L_3, \dots$ is the minimal bad sequence, $B_1, B_2, B_3, \dots$ cannot contain a bad subsequence. This can be seen as follows : let there exist a bad subsequence of $B_1, B_2, B_3, \dots$, say $B_{i_1}, B_{i_2}, B_{i_3}, \dots$, then $L_1, \dots, L_{i_1-1}$, $B_{i_1}, B_{i_2}, \dots$ is a bad sequence. If it were not a bad sequence, then there would exist L_k and B_{i_j} such that $L_k \leq B_{i_j}$. But by Lemma 3.3, $L_k \leq B_{i_j} \leq L_{i_j}$. Now, B_{i_1} is smaller than L_{i_1} which contradicts the fact that $L_1, L_2, L_3, \dots$ is the minimal bad sequence.

Thus in $B_1, B_2, B_3, \dots$, there exists no bad subsequence. Hence by Lemma 4.1, there exists a subsequence $i_1 < i_2 < i_3 < \dots$ such that $B_{i_1} \leq B_{i_2} \leq B_{i_3} \leq \dots$. By the definition of bad sequences, for some $s < t$, $T_{i_s} \leq T_{i_t}$, or else $T_{i_1}, T_{i_2}, T_{i_3}, \dots$ would be a bad sequence, which leads to a contradiction as before. Thus $B_{i_s} \leq B_{i_t}$ and $T_{i_s} \leq T_{i_t}$. By Lemma 3.5, $GL(B_{i_s}, T_{i_s}) = L_{i_s}$ is a minor of $GL(B_{i_t}, T_{i_t}) = L_{i_t}$. This is a contradiction to our assumption that $L_1, L_2, L_3, \ldots$ is a bad sequence. Thus L_{k+1} is well-quasi-ordered.

□

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