ON THE RELATIVE IMPORTANCE OF EXCLUDED MINORS

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Abstract. If $\mathcal{E}$ is a set of matroids, then $\mathrm{Ex}(\mathcal{E})$ denotes the set of matroids that have no minor isomorphic to a member of $\mathcal{E}$. If $\mathcal{E}' \subseteq \mathcal{E}$, we say that $\mathcal{E}'$ is *superfluous* if $\mathrm{Ex}(\mathcal{E} - \mathcal{E}') - \mathrm{Ex}(\mathcal{E})$ contains only finitely many 3-connected matroids. We determine the superfluous subsets of six well-known collections of excluded minors.

Dedicated, with affection, to "*Mathematician, gone 60, left fox with leg trouble.* (5, 7)"

1. Introduction

For a set $\mathcal{E}$ of matroids, let $\mathrm{Ex}(\mathcal{E})$ be the set of matroids such that $M \in \mathrm{Ex}(\mathcal{E})$ if and only if M has no minor isomorphic to a member of $\mathcal{E}$. Thus, if $\mathcal{P} = \{U_{2,4}, F_7, F_7^*, M(K_{3,3}), M(K_5), M^*(K_{3,3}), M^*(K_5)\}$, then $\mathrm{Ex}(\mathcal{P})$ is the set of cycle matroids of planar graphs. Hall's classical theorem on the graphs without a $K_{3,3}$-minor [5] can be interpreted as saying that

$$\mathrm{Ex}(\mathcal{P} - \{M(K_5)\}) - \mathrm{Ex}(\mathcal{P})$$

contains only a single 3-connected matroid, namely $M(K_5)$ itself. This motivates the following definition: if $\mathcal{E}$ is a set of matroids, then $\mathcal{E}' \subseteq \mathcal{E}$ is a *superfluous* subset of $\mathcal{E}$ if $\mathrm{Ex}(\mathcal{E} - \mathcal{E}') - \mathrm{Ex}(\mathcal{E})$ contains only finitely many 3-connected matroids. Thus $\{M(K_5)\}$ is a superfluous subset of $\mathcal{P}$. Obviously every subset of a superfluous subset is itself superfluous. In this article we determine the superfluous subsets of six well-known collections of excluded minors.

We will concentrate on the excluded minors for classes of matroids representable over partial fields. Partial fields were introduced by Semple and Whittle [15], prompted by Whittle's investigation of classes of ternary matroids [20, 21]. A *partial field* is a pair (R, G), where R is a commutative ring with identity, and G is a subgroup of the multiplicative group of R, such that $-1 \in G$. Note that every field, $\mathbb{F}$, can be seen as a partial field,

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$(\mathbb{F}, \mathbb{F} - \{0\})$. For more information on partial fields, and matroid representations over them, we refer to [14]. The reader should know that M is representable over a partial field if and only if M^* is. All undefined matroids appearing in the paper can be found in the appendix of Oxley [10]. We assume that the reader is familiar with the terminology and notation from that source. We use the terms *line* and *plane* to refer to rank-2 and rank-3 flats of the ground set.

To date, the class of matroids representable over a partial field has been characterized via excluded minors in only six cases. Those cases are: the fields GF(2), GF(3), and GF(4), the regular partial field, and two of the partial fields discovered by Whittle, namely the sixth-roots-of-unity partial field, and the near-regular partial field. We will determine the superfluous subsets of all these collections of excluded minors.

First of all, Tutte [19] showed that the only excluded minor for the class of GF(2)-representable matroids is $U_{2,4}$. It is clear that the only superfluous subset in this case is the empty set. For a more interesting example, we examine the *regular* partial field, $\mathbb{U}_0 := (\mathbb{Z}, \{1, -1\})$. Tutte also proved that the set of excluded minors for $\mathbb{U}_0$-representable matroids is $\{U_{2,4}, F_7, F_7^*\}$. It is a well-known application of Seymour's Splitter Theorem [18] that F_7 is a splitter for the class $\mathrm{Ex}(\{U_{2,4}, F_7^*\})$. The next theorem follows easily from this fact and the fact that infinitely many binary matroids are not regular.

Theorem 1.1. *The only non-empty superfluous subsets of* $\{U_{2,4}, F_7, F_7^*\}$ *are* $\{F_7\}$ *and* $\{F_7^*\}$*. The only* 3*-connected matroid in* $\mathrm{Ex}(\{U_{2,4}, F_7^*\}) - \mathrm{Ex}(\{U_{2,4}, F_7, F_7^*\})$ *is* F_7*.*

By duality, the only 3-connected matroid in $\mathrm{Ex}(\{U_{2,4}, F_7\}) - \mathrm{Ex}(\{U_{2,4}, F_7, F_7^*\})$ is F_7^*. From here on we will omit such dual statements.

Next we consider the excluded-minor characterization of GF(3)-representable matroids, due to Bixby [1] and Seymour [17].

Theorem 1.2. *The set of excluded minors for* GF(3)*-representable matroids is* $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$*.*

In this paper we will prove the following:

Theorem 1.3. *The only non-empty superfluous subsets of* $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$ *are* $\{F_7\}$ *and* $\{F_7^*\}$*. The only* 3*-connected matroid in* $\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7^*\}) - \mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7, F_7^*\})$ *is* F_7*.*

At this point we should observe that a 3-connected matroid of rank and corank at least three has a $U_{2,5}$-minor if and only if it has a $U_{3,5}$-minor (see [10, Proposition 12.2.15]), so $U_{2,5}$ is not superfluous only because $\mathrm{Ex}(\{U_{3,5}, F_7, F_7^*\}) - \mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7, F_7^*\})$ contains arbitrarily long lines. This raises the question if $\mathrm{Ex}(\mathcal{E} - X) - \mathrm{Ex}(\mathcal{E})$ is highly structured for other choices of $\mathcal{E}$ and $X \subseteq \mathcal{E}$. For instance, it is possible that there is only a finite number of internally 4-connected members.

This is certainly not always the case: if all members of $\mathcal{E} - \{F_7, F_7^*\}$ are non-binary, then $\mathrm{Ex}(\mathcal{E} - \{F_7, F_7^*\}) - \mathrm{Ex}(\mathcal{E})$ contains all binary matroids. In the remaining cases in this paper we make no attempt to characterize the full nature of $\mathrm{Ex}(\mathcal{E} - X) - \mathrm{Ex}(\mathcal{E})$. We focus purely on the finite/infinite dichotomy captured by the definition of "superfluous".

The set of excluded minors for GF(4)-representable matroids was determined by Geelen, Gerards, and Kapoor [3].

Theorem 1.4. *The set of excluded minors for the class of* GF(4)*-representable matroids is* $\{U_{2,6}, U_{4,6}, F_7^-, (F_7^-)^*, P_6, P_8, P_8^=\}$.

Let $\mathcal{O}$ be the set of excluded minors in Theorem 1.4. Geelen, Oxley, Vertigan, and Whittle showed the following:

Theorem 1.5 ([4, Theorem 1.1]). *Let M be a* 3*-connected matroid. Then one of the following holds:*

(i) M is GF(4)*-representable;*
(ii) M has a minor isomorphic to one of $\mathcal{O} - \{P_8, P_8^=\}$;
(iii) M is isomorphic to $P_8^=$;
(iv) M is isomorphic to a minor of $S(5, 6, 12)$.

This implies that $\{P_8, P_8^=\}$ is a superfluous subset of $\mathcal{O}$. We complement this theorem by showing that it is best possible:

Theorem 1.6. *The only superfluous subsets of $\mathcal{O}$ are the subsets of $\{P_8, P_8^=\}$. The only* 3*-connected matroids in* $\mathrm{Ex}(\mathcal{O} - \{P_8, P_8^=\}) - \mathrm{Ex}(\mathcal{O})$ *are isomorphic to $P_8^=$, or are minors of $S(5, 6, 12)$.*

Let $\mathbb{S} := (\mathbb{C}, \{z \in \mathbb{C} \mid z^6 = 1\})$ be the *sixth-roots-of-unity* partial field, so that a matroid is $\mathbb{S}$-representable if and only if it is both GF(3)- and GF(4)-representable. By combining Theorems 1.2 and 1.4, Geelen, Gerards, and Kapoor derived the following result [3, Corollary 1.4].

Theorem 1.7. *The set of excluded minors for the class of $\mathbb{S}$-representable matroids is* $\{U_{2,5}, U_{3,5}, F_7, F_7^*, F_7^-, (F_7^-)^*, P_8\}$.

Let $\mathcal{S}$ be the set of excluded minors in Theorem 1.7. In this paper we prove the following:

Theorem 1.8. *The only superfluous subsets of $\mathcal{S}$ are the subsets of $\{F_7, P_8\}$ and $\{F_7^*, P_8\}$. The only* 3*-connected matroids in* $\mathrm{Ex}(\mathcal{S} - \{F_7, P_8\}) - \mathrm{Ex}(\mathcal{S})$ *are isomorphic to F_7, or are minors of $S(5, 6, 12)$.*

Let $\mathbb{U}_1 := (\mathbb{Q}(\alpha), \{\pm\alpha^i(1-\alpha)^j \mid i, j \in \mathbb{Z}\})$ be the *near-regular* partial field. A matroid is $\mathbb{U}_1$-representable if and only if it is representable over GF(3), GF(4), and GF(5). The next theorem is proved in [6].

Theorem 1.9. *The set of excluded minors for the class of $\mathbb{U}_1$-representable matroids is*

$$\{U_{2,5}, U_{3,5}, F_7, F_7^*, F_7^-, (F_7^-)^*, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*, \Delta_3(\mathrm{AG}(2,3)\backslash e), P_8\}.$$

The matroid $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ in this theorem is obtained from $\mathrm{AG}(2,3)\backslash e$ by performing a Δ-Y exchange on $\mathrm{AG}(2,3)\backslash e$. It is represented over GF(3) by $[I_4\ A]$, where A is the following matrix.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] \end{array}. \tag{1}$$

Let $\mathcal{N}$ be the set featured in Theorem 1.9. In this paper we prove the following:

Theorem 1.10. *The only superfluous subsets of $\mathcal{N}$ are the subsets of $\{F_7, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*\}$ and $\{F_7^*, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*\}$. The only 3-connected matroids in $\mathrm{Ex}(\mathcal{N} - \{F_7, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*\}) - \mathrm{Ex}(\mathcal{N})$ are isomorphic to F_7, $\mathrm{AG}(2,3)\backslash e$, $(\mathrm{AG}(2,3)\backslash e)^*$, $\mathrm{AG}(2,3)$, or $(\mathrm{AG}(2,3))^*$.*

The paper is built up as follows. In Section 2 we use Seymour's Splitter Theorem to prove that certain subsets are superfluous. To prove that a subset $\{M\}$ is not superfluous, we need to generate an infinite number of 3-connected matroids in $\mathrm{Ex}(\mathcal{E} - \{M\}) - \mathrm{Ex}(\mathcal{E})$. We do so by the simple expedient of growing arbitrarily long fans. Section 3 proves the technical lemmas that allow us to do so. In Section 4 we introduce several matroids to which our method of growing fans will be applied, and in Section 5 we will round up the results. Note that the proofs in Sections 2 and 4 are finite case-checks that could be replaced by computer checks. However, at the moment of writing no sufficiently reliable software for this existed.

2. Applying the splitter theorem

The following result is very well-known [10, Proposition 12.2.3].

Proposition 2.1. *The matroid F_7 is a splitter for the class $\mathrm{Ex}(\{U_{2,4}, F_7^*\})$.*

Our next result, which seems not to be in the literature, proves a generalization of Proposition 2.1.

Theorem 2.2. *The matroid F_7 is a splitter for the class $\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7^*\})$.*

Proof. By Seymour's Splitter Theorem we only have to check that F_7 has no 3-connected single-element extensions and coextensions in $\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7^*\})$. If M is a 3-connected matroid such that $M\backslash e \cong F_7$, then either e is on exactly one line of F_7, or e is on no line of F_7. In either case M/e has a $U_{2,5}$-minor.

We may now assume that M is a 3-connected matroid such that $M/e \cong F_7$ and M belongs to $\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7^*\})$. Let $\mathcal{M}$ be the class of matroids that are either binary or ternary. Now $\mathcal{M}$ is a minor-closed class, and its excluded minors are determined in [8]. Certainly M is not binary, since that would lead to a contradiction to Proposition 2.1. Moreover, M is not ternary, as it has an F_7-minor. Therefore M is not contained in $\mathcal{M}$. Hence [16, Theorem 4.1] implies that M contains a 3-connected excluded minor for $\mathcal{M}$. There are only four such excluded minors, and as M does not have $U_{2,5}$ or $U_{3,5}$ as a minor, M must have as a minor one of the matroids obtained from the affine geometry $\mathrm{AG}(3,2)$ or from T_{12} by relaxing a circuit-hyperplane. As M has only 8 elements, M must be isomorphic to the unique relaxation of $\mathrm{AG}(3,2)$. But this matroid has an F_7^*-minor ([10, Page 646]). This contradiction completes the proof. $\square$

We can make short work of the case in which we do not exclude P_8. Geelen et al. [4, Theorem 1.5] proved the following result:

Theorem 2.3. *If M is a 3-connected matroid in $\mathrm{Ex}(\{U_{2,6}, U_{4,6}, P_6, F_7^-, (F_7^-)^*\})$, and M has a P_8-minor, then M is a minor of $S(5,6,12)$.*

Since each of $U_{2,6}$, $U_{4,6}$, and P_6 has a minor in $\{U_{2,5}, U_{3,5}\}$, we immediately have

Corollary 2.4. *If M is a 3-connected matroid in $\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7^-, (F_7^-)^*\})$, and M has a P_8-minor, then M is a minor of $S(5,6,12)$.*

Next, we determine what happens if we do not exclude $\mathrm{AG}(2,3)\backslash e$. Our starting point is the automorphism group of $\mathrm{AG}(2,3)\backslash e$. Note that it is transitive on elements of the ground set ([10, Page 653]). For each element p in $\mathrm{AG}(2,3)\backslash e$, there is a unique element p' such that p and p' are not on a 3-point line of $\mathrm{AG}(2,3)\backslash e$. Any automorphism will map $\{p, p'\}$ to another such pair, so specifying the image of p also specifies the image of p'. Consider automorphisms of the diagram in Figure 1 that point-wise fix 1 and 8. It is easy to confirm that the permutations below (presented in cyclic notation),

$$(1)(2,4)(3,7)(5,6)(8) \tag{2}$$

and

$$(1)(2,3,5)(4,6,7)(8) \tag{3}$$

are two such automorphisms. The next result follows easily from this discussion.

Lemma 2.5. *Let p and p' be points in $\mathrm{AG}(2,3)\backslash e$ such that there is no 3-point line containing p and p'. The subgroup of the automorphism group of $\mathrm{AG}(2,3)\backslash e$ that point-wise fixes p and p' is transitive on $E(\mathrm{AG}(2,3)\backslash e) - \{p, p'\}$.*

We wish to find automorphisms mapping a basis B to a basis B'. This cannot be done for arbitrary bases B and B', but the following lemma gives sufficient conditions for the automorphism to exist.

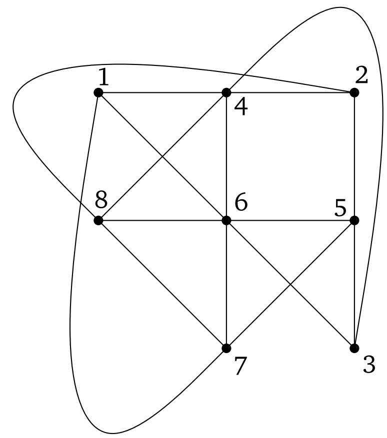

FIGURE 1. The matroid $\mathrm{AG}(2,3)\backslash e$.

Lemma 2.6. *Let B and B' be bases of $\mathrm{AG}(2,3)\backslash e$ such that every pair $p, q \in B$, and every pair $k, l \in B'$ spans a 3-point line. There is an automorphism of $\mathrm{AG}(2,3)\backslash e$ mapping B to B'.*

Proof. If x is any element of $\mathrm{AG}(2,3)\backslash e$, then let x' be the point that is in no 3-point line with x. Let $B = \{p, q, r\}$. The hypotheses of the lemma imply that $|\{p, q, r, p', q', r'\}| = 6$. Let e_{pq} be the unique point such that $\{p, q, e_{pq}\}$ is a circuit. Define e_{pr} and e_{qr} symmetrically. Then $|\{p, q, r, e_{pq}, e_{pr}, e_{qr}\}| = 6$. As $\mathrm{AG}(2,3)\backslash e$ has only 8 points, we can relabel as necessary, and assume e_{qr} is in $\{p', q', r'\}$. Since e_{qr} is in a non-trivial line with q and r, it follows that $e_{qr} = p'$, so that $\{p', q, r\}$ is a circuit. Let $B' = \{k, l, m\}$. By relabeling and using the same arguments, we can assume that $\{k', l, m\}$ is a 3-point line of $\mathrm{AG}(2,3)\backslash e$.

Consider the automorphism that maps k to p. It must map k' to p'. By composing this automorphism with an automorphism that fixes p and p', and referring to Lemma 2.5, we can assume that l is mapped to q. But an automorphism maps lines to lines, so then m must be mapped to r, and the result follows. □

In the proof of the next lemma we will show several times that a matroid $M = M[I\ A]$ is isomorphic to one of $\Delta_3(\mathrm{AG}(2,3)\backslash e)$, P_8, F_7^-, or $(F_7^-)^*$. Unless the isomorphism is obvious (i.e. one merely needs to permute rows and columns), we will specify which isomorphism we use. For this we use the representation of $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ with elements labeled as in Equation (1). Moreover, we will label the elements of P_8, F_7^-, $(F_7^-)^*$ so that $P_8 = [I_4\ A_8]$, $F_7^- = [I_3\ A_7]$, and $(F_7^-)^* = [-A_7^T\ I_4]$, where A_7 and A_8 are the following matrices over GF(3).

$$A_8 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_7 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Lemma 2.7. *Let M be a 3-connected $\mathbb{S}$-representable matroid such that $M/f \cong \mathrm{AG}(2,3)\backslash e$ for some $f \in E(M)$. Then M has $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ as minor.*

Proof. Suppose that M is a counterexample. Let $M' := M\backslash f$.

Claim 2.7.1. *There exists a set $X \subseteq E(M) - f$ such that $|X| = 5$ and $r(X) = 3$.*

Proof. Suppose M' has no 5-point planes. First we show that M' has no 3-point lines. Observe that each line of M' is a line of $\mathrm{AG}(2,3)\backslash e$, so M' has no 4-point lines. Suppose $\{x, y, z\}$ is a line of M'. If x is on another 3-point line, then the union of those lines would be a 5-point plane, a contradiction. It follows that $M'/x\backslash y$ is simple. Furthermore, z is in no 3-point line in $M'/x\backslash y$, or else the union of this line with $\{x, y\}$ is a 5-point plane in M'. Therefore $M'/x\backslash y/z$ is simple, has rank 2, and has 5 points. Therefore M' has a $U_{2,5}$-minor, which is impossible since it is $\mathbb{S}$-representable. Hence M' has no 3-point lines.

Let e be an arbitrary point in $E(M')$. Then M'/e is a simple rank-3 matroid with 7 points. Since M' has no 5-point planes, M'/e has no 4-point lines. Hence M'/e cannot be the union of two lines, so it is 3-connected. Then M'/e is isomorphic to one of the matroids F_7, F_7^-, P_7, or O_7 (see [3, Page 292]). Since M'/e is $\mathbb{S}$-representable, it is not isomorphic to F_7 or F_7^-. Furthermore, O_7 has a 4-point line restriction, so M'/e must be isomorphic to P_7. By the uniqueness of representation over GF(3), we can assume that the following GF(3)-matrix A' is such that $M' = [I_4\ A']$.

$$A' := \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ e \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \end{bmatrix} \end{array}.$$

As M' has no 3-point lines, all of α, β, and γ are non-zero. By scaling the row labeled e, we assume that $\alpha = 1$. Also, $\gamma \neq \delta$ as $\{1, 6, 7\}$ is not a triangle.

If $\beta = 1$, then $\gamma \neq 1$, or else $M'\backslash 7 \cong (F_7^-)^*$. Therefore $\gamma = -1$. If $\delta = 0$, then A' represents P_8, which is impossible as M is GF(4)-representable. Therefore $\delta = 1$. By the discussion above, $M'/1 \cong P_7$. But in $M'/1$, the sets $\{2, 4, e\}$, $\{3, 5, e\}$, and $\{6, 7, e\}$ are triangles containing e, whereas

$\{3, 5, e\}, \{4, 5, 6\}$, and $\{2, 5, 7\}$ are triangles containing 5. This is a contradiction, since P_7 has only one element that is on three lines. Therefore $\beta = -1$. It follows that $\delta \neq 0$, or else $\{4, 5, 7\}$ is a triangle of M'.

Assume that $\gamma = -1$, from which it follows that $\delta = 1$. Then we find that $M' \cong P_8$, with isomorphism

$$1 \to 1 \quad 2 \to 2 \quad 3 \to 5 \quad 4 \to 7 \quad 5 \to 8 \quad 6 \to 3 \quad 7 \to 6 \quad e \to 4.$$

Therefore we must have $\gamma = 1$, and hence $\delta = -1$. But then again $M' \cong P_8$, with isomorphism

$$1 \to 1 \quad 2 \to 5 \quad 3 \to 3 \quad 4 \to 8 \quad 5 \to 6 \quad 6 \to 2 \quad 7 \to 7 \quad e \to 4.$$

From this final contradiction we conclude that the claim holds. □

Let X be a set of 5 points of a plane of M', and $Y := E(M') - X$. Note that $f \notin \mathrm{cl}_M(X)$, as M/f has no rank-2 flat with 5 elements.

Since M/f is isomorphic to $\mathrm{AG}(2, 3)\backslash e$, we can distinguish three cases. Either Y is a 3-point line of M/f; or Y is a basis of M/f, and every pair of elements of Y spans a 3-point line in M/f; or Y is a basis of M/f, and there is exactly one pair of elements in Y that does not span a 3-point line of M/f. We can use Lemmas 2.5 and 2.6, and the fact that the automorphism group of $\mathrm{AG}(2, 3)\backslash e$ is transitive on 3-point lines ([10, Page 653]), and thereby assume that either $Y = \{4, 6, 7\}$ or $Y = \{4, 6, 8\}$ or $Y = \{4, 5, 6\}$, where the elements of $\mathrm{AG}(2, 3)\backslash e$ are labeled as in Figure 1.

Case I. Suppose $Y = \{4, 6, 7\}$, so that $X = \{1, 2, 3, 5, 8\}$. Since f is not a coloop and not in a series pair, there are two elements in Y that are not spanned by X in M'. Let σ be the automorphism in Equation (3), so that Y is an orbit of σ. There is some $i \in \{0, 1, 2\}$ such that σ^i takes the two elements in $Y - \mathrm{cl}_{M'}(X)$ to $\{4, 6\}$. Now σ^i induces a relabeling of the elements of M' that set-wise fixes X. After applying this relabeling, M/f is still equal to $\mathrm{AG}(2, 3)\backslash e$, as labeled in Figure 1. Moreover, X is a 5-point plane of M' that does not contain 4 or 6. By the uniqueness of representations over GF(3) we can assume that $M = M[I\ A]$ for some GF(3)-matrix of the form

$$A := \begin{array}{c} \\ f \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & \alpha & \beta & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{array}\right] \end{array}$$

with $\alpha \neq 0$. If $\alpha = 1$ then $M\backslash\{5, 7\} \cong (F_7^-)^*$, with isomorphism

$$1 \to 5 \quad 2 \to 7 \quad 3 \to 6 \quad 4 \to 4 \quad 6 \to 2 \quad 8 \to 3 \quad f \to 1.$$

Hence $\alpha = -1$. But now $M\backslash 7 \cong \Delta_3(\mathrm{AG}(2, 3)\backslash e)$. This completes the analysis in Case I.

From now on, we assume that Y is not a triangle of M/f. We will also assume that if X spans an element $y \in Y$, then there is no triangle T of M/f

that contains $Y - y$. To justify this assumption, note that if $y \in \mathrm{cl}_{M'}(X)$, then $(Y-y)\cup f$ must be a triad of M, so that $r_M(X\cup y) = 3$. Furthermore, Y is not a triangle in M/f, so T contains exactly one element of X. Therefore, if T exists, we can replace X with $(X - T) \cup y$, and replace Y with T, and reduce to Case I.

Case II. Suppose $Y = \{4, 6, 8\}$. Since any pair of elements from $\{4, 6, 8\}$ is in a triangle of M/f, we can assume that X spans no element of Y, by the argument in the previous paragraph. Hence we have $M = M[I\ A]$ for some GF(3)-matrix of the form

$$A := \begin{array}{c} \\ f \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & \alpha & 0 & \beta \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{array}\right] \end{array},$$

where α and β are non-zero.

If $(\alpha, \beta) = (1, 1)$, then $M\backslash 5 \cong \Delta_3(\mathrm{AG}(2, 3)\backslash e)$, with isomorphism

$$1 \to 1 \quad 2 \to 2 \quad 3 \to 4 \quad 4 \to 3 \quad 6 \to 8 \quad 7 \to 7 \quad 8 \to 6 \quad f \to 5.$$

If $(\alpha, \beta) = (1, -1)$, then $M\backslash 5 \cong P_8$, with isomorphism

$$1 \to 2 \quad 2 \to 3 \quad 3 \to 4 \quad 4 \to 6 \quad 6 \to 1 \quad 7 \to 5 \quad 8 \to 8 \quad f \to 7,$$

contradicting GF(4)-representability of M.

If $(\alpha, \beta) = (-1, 1)$, then $M/1\backslash 5 \cong F_7^-$, with isomorphism

$$2 \to 2 \quad 3 \to 3 \quad 4 \to 1 \quad 6 \to 7 \quad 7 \to 6 \quad 8 \to 5 \quad f \to 4.$$

If $(\alpha, \beta) = (-1, -1)$, then $M\backslash 5 \cong \Delta_3(\mathrm{AG}(2, 3)\backslash e)$, with isomorphism

$$1 \to 2 \quad 2 \to 7 \quad 3 \to 5 \quad 4 \to 4 \quad 6 \to 3 \quad 7 \to 6 \quad 8 \to 8 \quad f \to 1.$$

Thus M has a $\Delta_3(\mathrm{AG}(2, 3)\backslash e)$-minor.

Case III. Suppose $Y = \{4, 5, 6\}$. Since $\{4, 6, 7\}$ and $\{5, 6, 8\}$ are triangles of M/f, we assume that neither 4 nor 5 is in the span of X, by the argument immediately preceding Case II. Hence $M = M[I\ A]$ for some GF(3)-matrix of the form

$$A := \begin{array}{c} \\ f \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccccc} 1 & \alpha & \beta & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{array}\right] \end{array},$$

where $\alpha \neq 0$. If $\alpha = 1$ then $M\backslash\{6, 8\} \cong (F_7^-)^*$, with isomorphism

$$1 \to 5 \quad 2 \to 6 \quad 3 \to 7 \quad 4 \to 1 \quad 5 \to 4 \quad 7 \to 3 \quad f \to 2.$$

Therefore $\alpha = -1$. But now $M\backslash 6 \cong \Delta_3(\mathrm{AG}(2, 3)\backslash e)$, with isomorphism

$$1 \to 8 \quad 2 \to 3 \quad 3 \to 2 \quad 4 \to 7 \quad 5 \to 1 \quad 7 \to 4 \quad 8 \to 6 \quad f \to 5.$$

The result follows. □

We must now study coextensions of $\mathrm{AG}(2,3)$. Luckily our previous analysis can be used for this.

Lemma 2.8. *Let M be a 3-connected $\mathbb{S}$-representable matroid such that $M/f \cong \mathrm{AG}(2,3)$ for some $f \in E(M)$. Then M has an element $g \neq f$ such that $M\backslash g$ is 3-connected.*

Proof. Let M be as stated, and suppose the result is false, so for each element $g \neq f$, $M\backslash g$ is not 3-connected. Since $M\backslash g/f$ is 3-connected, g must be in a triad with f. Two distinct triads T_1 and T_2, both containing f, intersect only in f, or else $M/f \cong \mathrm{AG}(2,3)$ has a triad. From this we find that $M\backslash f$ can be partitioned into series pairs. However, this matroid has an odd number of elements, a contradiction. □

Corollary 2.9. *Let M be a 3-connected $\mathbb{S}$-representable matroid such that $M/f \cong \mathrm{AG}(2,3)$ for some $f \in E(M)$. Then M has $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ as minor.*

Proof. Let g be an element as in Lemma 2.8. Then $M\backslash g$ is a matroid satisfying all the conditions of Lemma 2.7, and the result follows. □

Now we combine the previous results and the Splitter Theorem to prove the following theorem.

Theorem 2.10. *Let M be a 3-connected matroid in*

$$\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7, F_7^*, F_7^-, (F_7^-)^*, \Delta_3(\mathrm{AG}(2,3)\backslash e), P_8\}).$$

Then either M is near-regular, or one of M and M^* is isomorphic to a member of $\{\mathrm{AG}(2,3)\backslash e, \mathrm{AG}(2,3)\}$.

Proof. By the excluded-minor characterization of $\mathbb{S}$-representable matroids (Theorem 1.7), it follows that M is $\mathbb{S}$-representable. We assume that M is not $\mathbb{U}_1$-representable. Then Theorem 1.9 implies that M has a minor isomorphic to $\mathrm{AG}(2,3)\backslash e$ or its dual. By duality, we assume that M has an $\mathrm{AG}(2,3)\backslash e$-minor. If $M \cong \mathrm{AG}(2,3)\backslash e$, we are done, so we assume otherwise. By Seymour's Splitter Theorem, M has a 3-connected minor M', such that M' is a single-element extension or coextension of $\mathrm{AG}(2,3)\backslash e$. Lemma 2.7 implies that M' is a single-element extension of $\mathrm{AG}(2,3)\backslash e$. Thus M' is simple and $r(M') = 3$. Moreover $|E(M')| = 9$, so [12, Theorem 2.1] implies that $M' \cong \mathrm{AG}(2,3)$. If $M = M'$, we are done, so we assume that M has a 3-connected minor M'', such that M'' is a single-element extension or coextension of $\mathrm{AG}(2,3)$. But $r(M'') > 3$, or else we have contradicted [12, Theorem 2.1]. Therefore $M''/f \cong \mathrm{AG}(2,3)$, for some element f. Corollary 2.9 implies that M'' has a $\Delta_3(\mathrm{AG}(2,3)\backslash e)$-minor, a contradiction. □

3. Creating bigger fans

In this section we prove two results that allow us to replace a fan by a bigger fan while keeping a certain minor N, without losing 3-connectivity, and without introducing an undesired minor N' (subject to the conditions that N' is 3-connected and has no 4-element fans). We will use Brylawski's generalized parallel connection [2] for this. We refer the reader to Oxley [10, Section 11.4] for definitions and elementary properties, including the following:

Lemma 3.1. *Let M and N be matroids having a common restriction with ground set T, such that T is a modular flat of N. Let $M' := P_T(N, M)$.*

(i) A subset $F \subseteq E(M')$ is a flat of M' if and only if $F \cap E(N)$ is a flat of N and $F \cap E(M)$ is a flat of M;
(ii) $M'|E(N) = N$ and $M'|E(M) = M$;
(iii) If $e \in E(N) - T$ then $M'\backslash e = P_T(N\backslash e, M)$;
(iv) If $e \in E(N) - \mathrm{cl}_N(T)$ then $M'/e = P_T(N/e, M)$;
(v) If $e \in E(M) - T$ then $M'\backslash e = P_T(N, M\backslash e)$;
(vi) If $e \in E(M) - \mathrm{cl}_M(T)$ then $M'/e = P_T(N, M/e)$.

Let M be a matroid on the ground set E. A subset of E is *fully closed* if it is closed in M and M^*. If $X \subseteq E$, then $\mathrm{fcl}(X)$ is the intersection of all fully closed sets that contain X. We can obtain $\mathrm{fcl}(X)$ by applying the closure operator to X, applying the coclosure operator to the result, and so on, until we cease to add any new elements. We omit the elementary proof of the following lemma.

Lemma 3.2. *Let M be a simple, cosimple, connected matroid, and let (A, B) be a 2-separation of M. Then $(\mathrm{fcl}_M(A), B - \mathrm{fcl}_M(A))$ is a 2-separation.*

Definition 3.3. Let M be a matroid, and $F = (x_1, x_2, \dots, x_k)$ an ordered subset of $E(M)$, with $k \geq 3$. We say F is a *fan* of M if, for all $i \in \{1, \dots, k-2\}$, $T_i := \{x_i, x_{i+1}, x_{i+2}\}$ is either a triangle or a triad, and if T_i is a triad, then T_{i+1} is a triangle; if T_i is a triangle then T_{i+1} is a triad.

Assume that $F = (x_1, \dots, x_k)$ is a fan. Then F is a fan of M^*. We say that F is a *maximal* fan if there is no fan $(y_1, \dots, y_l)$ such that $l > k$ and $\{x_1, \dots, x_k\} \subseteq \{y_1, \dots, y_l\}$. We say x_i is a *rim element* if $1 < i < k$ and x_i is contained in exactly one triangle that is contained in $\{x_1, \dots, x_k\}$, or if $i \in \{1, k\}$ and x_i is contained in no such triangle. We say x_i is a *spoke element* if it is not a rim element. The following is an easy consequence of Lemma 3.2.

Lemma 3.4. *Let M be a simple, cosimple, connected matroid, let $F = (x_1, \dots, x_k)$ be a fan of M, and let (A, B) be a 2-separation of M. Then M has a 2-separation (A', B') with $\{x_1, \dots, x_k\} \subseteq A'$.*

In what follows, the elements of the wheel $M(\mathcal{W}_n)$ and whirl $\mathcal{W}^n$ are labeled $\{s_1, r_1, s_2, \dots, s_n, r_n\}$ where, for all indices i (interpreted modulo n), $\{s_i, r_i, s_{i+1}\}$ is a triangle and $\{r_i, s_{i+1}, r_{i+1}\}$ is a triad. Hence, $\{s_1, \dots, s_n\}$ is the set of spokes and $\{r_1, \dots, r_n\}$ is the set of rim elements.

Theorem 3.5. *Let M be a 3-connected matroid, and let $F = (x_1, \dots, x_k)$ be a fan of M with $T := \{x_1, x_2, x_3\}$ a triangle. Let $n \geq 3$ be an integer,* *and relabel the elements s_1, r_n, s_n of $M(\mathcal{W}_n)$ by x_1, x_2, x_3, in that order. Let $M' := P_T(M(\mathcal{W}_n), M)$, and $M'' := M' \backslash x_2$. Then M'' has the following properties:*

(i) $(x_1, r_1, s_2, r_2, \ldots, s_{n-1}, r_{n-1}, x_3, \ldots, x_k)$ *is a fan of* M''*;*
(ii) M *is isomorphic to a minor of* M''*, with the isomorphism fixing all elements but* x_2*; and*
(iii) M'' *is* 3*-connected.*

Proof. Let M, F, T, n, M', and M'' be as stated, and define $N := M(\mathcal{W}_n)$. Since T is a modular flat of N, we know $M' = P_T(N, M)$ is defined. It follows from Lemma 3.1 that $(s_1, r_1, \ldots, s_{n-1}, r_{n-1}, s_n)$ is a fan of M' and of M''. If $k = 3$, then (i) holds. If $k \geq 4$, then we only need to show that $\{r_{n-1}, s_n, x_4\}$ is a triad of M''. Consider $H := E(M') - \{r_{n-1}, s_n, r_n, x_4\}$. Since $H \cap E(N)$ and $H \cap E(M)$ are hyperplanes of their respective matroids, H is a flat of M'. Since $\mathrm{cl}_{M'}(H \cup s_n) = E(M')$, it follows that $\{r_{n-1}, s_n, r_n, x_4\}$ is a cocircuit of M'. But then $\{r_{n-1}, s_n, x_4\}$ is a cocircuit of M'', as desired.

Statement (ii) is a straightforward consequence of Lemma 3.1. Statement (iii) follows immediately from [13, Corollary 2.8]. □

We will denote the matroid M'', as described in the statement of Theorem 3.5, by $\boxtimes_T^n(M)$. Theorem 3.5 shows that we can make a fan arbitrarily long while keeping 3-connectivity. Our next task is to show that we can do so without introducing certain minors. The following lemma, whose elementary proof we omit, will be useful:

Lemma 3.6. *Let N be a* 3*-connected matroid without* 4*-element fans. Let M be a* 3*-connected matroid having N as minor, and let F be a* 4*-element fan of M. Then $|F \cap E(N)| \leq 3$.*

Recall that if T is a coindependent triangle of the matroid M, then $\Delta_T(M)$ is the matroid obtained from M by a Δ-Y exchange (see [10, Section 11.5]).

Theorem 3.7. *Let N be a* 3*-connected matroid with no* 4*-element fan. Let M be a* 3*-connected matroid with at least* 5 *elements that does not have an N-minor. Let $F = (x_1, \ldots, x_k)$ be a fan of M, where $T := \{x_1, x_2, x_3\}$ is a triangle, and let $n \geq 3$ be an integer. If $\boxtimes_T^n(M)$ has an N-minor, then so does $\Delta_T(M)$.*

Proof. We will assume that $n \geq 3$ has been chosen so that it is as small as possible, subject to the condition that $\boxtimes_T^n(M)$ has an N-minor. Let N' be a minor of $\boxtimes_T^n(M)$ that is isomorphic to N.

First assume that $n \geq 4$. Since $\{r_1, s_2, r_2, s_3\}$ is a 4-element fan of $\boxtimes_T^n(M)$, it follows from Lemma 3.6 that this set is not contained in $E(N)$. We claim that $\boxtimes_T^n(M)/r_1 \backslash s_2$ has an N-minor. Assume this is not the case. If $\boxtimes_T^n(M)/r_1$ has an N-minor, then, as $\{s_1, s_2\}$ is a parallel pair, $\boxtimes_T^n(M)/r_1 \backslash s_2$ has an N-minor. Therefore $\boxtimes_T^n(M)/r_1$ does not have an N-minor. Similarly, $\{r_1, r_2\}$ is a series pair in $\boxtimes_T^n(M) \backslash s_2$, so we assume that $\boxtimes_T^n(M) \backslash s_2$ has no N-minor. As $\{s_2, s_3\}$ is a parallel pair in $\boxtimes_T^n(M)/r_2$, this means that $\boxtimes_T^n(M)/r_2$

has no N-minor. Moreover, $\{r_2, r_3\}$ is a series pair in $\boxtimes^n_T(M)\backslash s_3$, so this matroid does not have an N-minor. As $\{s_2, r_2\}$ is a series pair in $\boxtimes^n_T(M)\backslash r_1$, and we concluded that $\boxtimes^n_T(M)/r_2$ has no N-minor, neither does $\boxtimes^n_T(M)\backslash r_1$. Since $\{r_1, s_1\}$ is a parallel pair in $\boxtimes^n_T(M)/s_2$, and deleting r_1 destroys all N-minors, $\boxtimes^n_T(M)/s_2$ has no N-minor. Deleting r_2 creates the series pair $\{r_1, s_2\}$, and contracting r_1 destroys all N-minors, so $\boxtimes^n_T(M)\backslash r_2$ does not have an N-minor. Lastly, contracting s_3 creates the parallel pair $\{s_2, r_2\}$, so $\boxtimes^n_T(M)/s_3$ does not have an N-minor, or else $\boxtimes^n_T(M)\backslash s_2$ does. From this discussion, we conclude that $\{r_1, s_2, r_2, s_3\} \subseteq E(N')$, contradicting our earlier conclusion. Therefore $\boxtimes^n_T(M)/r_1\backslash s_2$ has an N-minor.

Since contracting r_1 and deleting s_2 from $M(\mathcal{W}_n)$ produces a copy of $M(\mathcal{W}_{n-1})$, it follows easily from Lemma 3.1 that $\boxtimes^n_T(M)/r_1\backslash s_2$ is isomorphic to $\boxtimes^{n-1}_T(M)$. Thus our assumption on the minimality of n is contradicted. Now we must assume that $n = 3$.

If $\{r_1, s_2, r_2\} \not\subseteq E(N')$, then it is readily seen that M has an N-minor, contrary to hypothesis. It follows that $\{r_1, s_2, r_2\} \subseteq E(N')$.

Since N' has no 4-element fans, $s_1 \notin E(N')$. Then we must have that N' is a minor of $\boxtimes^n_T(M)\backslash s_1$. Likewise, N' is a minor of $\boxtimes^n_T(M)\backslash s_3$. So N' is a minor of $P_T(M(\mathcal{W}_3), M)\backslash T$. Since $|E(M)| \geq 5$, any triangle of M is coindependent ([10, Lemma 8.7.5]). Therefore $P_T(M(\mathcal{W}_3), M)\backslash T$ is isomorphic to $\Delta_T(M)$, and we are done. □

4. Infinite families

In this section we describe a collection of matroids to which we can apply our operation of growing fans. Recall that $\mathcal{O}$, $\mathcal{S}$, and $\mathcal{N}$, respectively, denote the sets of excluded minors for GF(4)-representable, sixth-roots-of-unity, and near-regular matroids, as listed in Theorems 1.4, 1.7, and 1.9.

Let M_8 be the rank-3 matroid shown in Figure 2. Then M_8 is represented over GF(3) by $[I_3\ A]$, where A is the following matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \end{array}\right] \end{array}$$

Lemma 4.1. *Let T be the triangle $\{3, 6, 8\}$ of M_8. If $n \geq 3$ is an integer, then $\boxtimes^n_T(M_8)$ is 3-connected, and has an F_7^--minor but no minor in $(\mathcal{O} \cup \mathcal{S} \cup \mathcal{N}) - \{F_7^-\}$.*

Proof. Clearly M_8 is 3-connected, and $M_8\backslash 8$ is isomorphic to F_7^-. By Theorem 3.5, then, $\boxtimes^n_T(M_8)$ is 3-connected and has an F_7^--minor for any $n \geq 3$.

Now assume that $\boxtimes^n_T(M_8)$ has a minor in $(\mathcal{O} \cup \mathcal{S} \cup \mathcal{N}) - \{F_7^-\}$. Therefore either M_8 or $\Delta_T(M_8)$ has such a minor, by Theorem 3.7. By observing that M_8 and $\Delta_T(M_8)$ are both ternary, considering ranks, and counting triangles, we can rule out minors isomorphic to $U_{2,6}$, $U_{4,6}$, P_6, $P_8^{=}$, $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, $(\mathrm{AG}(2,3)\backslash e)^*$, P_8, $\mathrm{AG}(2,3)\backslash e$, and $\Delta_3(\mathrm{AG}(2,3)\backslash e)$.

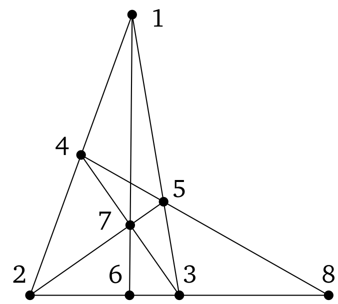

FIGURE 2. Geometric representation of M_8.

The only matroid left to check is $(F_7^-)^*$. Obviously M_8 does not have an $(F_7^-)^*$-minor. Assume that $\Delta_T(M_8)$ does. As $(F_7^-)^*$ has no triangles, $\Delta_T(M_8)\backslash 2$ must be isomorphic to $(F_7^-)^*$. Now $\{3,6,8\}$ is a triad of this matroid, and performing a Y-Δ exchange on this triad should produce a copy of F_7^-. Instead it produces a copy of $M_8\backslash 2$, which contains disjoint triangles, and is therefore not isomorphic to F_7^-. □

Let M_9 be the matroid represented by $[I_4\ A]$ over GF(3), where A is the following matrix.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Then M_9 is represented by the geometric diagram in Figure 3.

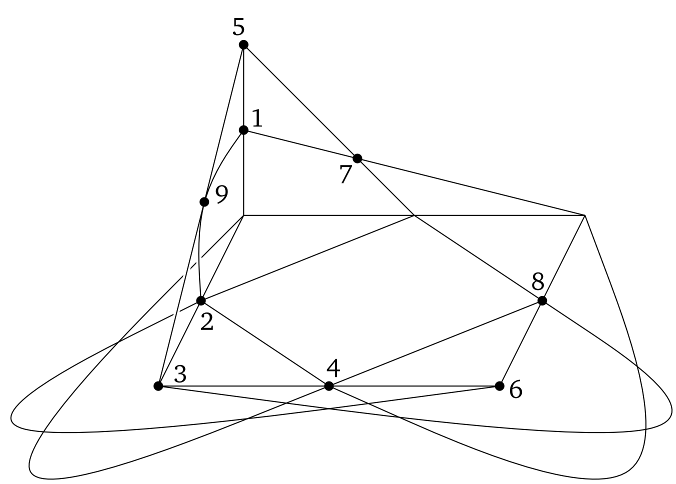

FIGURE 3. Geometric representation of M_9.

Lemma 4.2. *Let T be the triangle $\{3,5,9\}$ of M_9. If $n \geq 3$ is an integer, then $\boxtimes^n_T(M_9)$ is 3-connected, and has an $\Delta_3(\mathrm{AG}(2,3)\backslash e)$-minor, but no minor in $\mathcal{N} - \{\Delta_3(\mathrm{AG}(2,3)\backslash e)\}$.*

Proof. Note that M_9 is 3-connected and ternary, and $M_9\backslash 9 \cong \Delta_3(\mathrm{AG}(2,3)\backslash e)$, so by Theorems 3.5 and 3.7 it suffices to check that neither M_9 nor $\Delta_T(M_9)$ has a minor isomorphic to one of F_7^-, $(F_7^-)^*$, P_8, $\mathrm{AG}(2,3)\backslash e$, or $(\mathrm{AG}(2,3)\backslash e)^*$.

In $M_9/7$, the sets $\{3,5,8,9\}$ and $\{1,2,4,9\}$ are 4-point lines. Therefore any 7-element restriction of $M_9/7$ has either a 4-point line or two disjoint triangles. It follows that $M_9/7$ has no minor in $\mathcal{N}$. Similarly $M_9/8$ has no minor in $\mathcal{N}$.

The triangles of M_9 are $\{1,2,9\}$, $\{3,5,9\}$, and $\{3,4,6\}$. It follows easily that every 8-element restriction of M_9 contains at least one triangle, so M_9 does not have P_8 as minor. The rank of M_9 is too low to have $(\mathrm{AG}(2,3)\backslash e)^*$ as minor. Suppose M_9 has $\mathrm{AG}(2,3)\backslash e$ as minor. We need to contract one element. But this cannot be on a 3-point line, and elements 7 and 8 were ruled out above.

Suppose M_9 has a $(F_7^-)^*$-minor. To obtain this minor we must delete two elements so that no triangles remain. Deleting 9 gives us $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ again, so we must delete 3 and one of $\{1,2\}$. But $M_9\backslash\{1,3\}$ has disjoint triads $\{2,4,6\}$ and $\{5,7,9\}$, whereas $M_9\backslash\{2,3\}$ has disjoint triads $\{1,7,8\}$ and $\{4,5,9\}$. Hence neither is isomorphic to $(F_7^-)^*$.

Therefore we assume that M_9 has an F_7^--minor. We must contract a single element from M_9, and then delete a single element to obtain a copy of F_7^-. If we contract either 3 or 9, then we produce two disjoint parallel pairs, which cannot be rectified with a single deletion. If we contract one of 1, 2, 4, or 6 then we create a single parallel pair, so up to isomorphism we must delete, respectively, 2, 1, 6, or 4 to obtain a copy of F_7^-. But in these minors, the triangle $\{3,5,9\}$ is disjoint from, respectively, the triangles $\{6,7,8\}$, $\{4,6,8\}$, $\{1,2,7\}$, and $\{1,7,8\}$. If we contract 5, then up to isomorphism we must delete 3 to obtain a copy of F_7^-, but in this minor $\{1,4,8\}$ and $\{2,6,7\}$ are disjoint triangles. Thus M_9 does not have a minor in $\mathcal{N} - \{\Delta_3(\mathrm{AG}(2,3)\backslash e)\}$.

Assume that $\Delta_T(M_9)$ has a minor N' that is isomorphic to a ternary member of $\mathcal{N} - \{\Delta_3(\mathrm{AG}(2,3)\backslash e)\}$. If $T \nsubseteq E(N')$, then an element $x \in T$ is contracted to obtain N'. But $\Delta_T(M_9)/x \cong M_9\backslash x$, by [11, Lemma 2.13], and we are back in the previous case. Hence T is a triad of N', and therefore N' is isomorphic to $(F_7^-)^*$ or $(\mathrm{AG}(2,3)\backslash e)^*$. It follows easily from [11, Corollary 2.17] and Seymour's Splitter Theorem, that $\nabla_T(N')$ is a minor of $\nabla_T(\Delta_T(M_9)) = M_9$. If $N' \cong (F_7^-)^*$, then $\nabla_T(N) \cong F_7^-$, and this leads to a contradiction. Therefore $N' \cong (\mathrm{AG}(2,3)\backslash e)^*$. The definition of Y-Δ exchange implies that $\nabla_T(N') \cong (\Delta_3(\mathrm{AG}(2,3)\backslash e))^*$. But $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ is a self-dual matroid, so M_9 has a minor isomorphic to $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ that contains $\{3,5,9\}$ in its ground set. To obtain this minor, we must delete a single element, but in each case the result has two triangles, namely

$\{3,5,9\}$ and at least one of $\{1,2,9\}$ and $\{3,4,6\}$. This is a contradiction as $\Delta_3(\mathrm{AG}(2,3)\backslash e)$ has only one triangle. □

For a third infinite class, consider the following matrix, A, over GF(8). Here α is an element that satisfies $\alpha^3+\alpha+1=0$. Let M_7 be $[I_3\ A]$. A geometric representation of M_7 can be found in Figure 4.

$$\begin{array}{c} \\ 1\\ 2\\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 1\\ 1 & 0 & 1 & \alpha\\ 0 & 1 & \alpha & \alpha^2 \end{array}\right] \end{array}$$

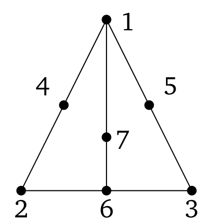

FIGURE 4. Geometric representation of M_7.

Lemma 4.3. *Let T be the triangle $\{1,2,4\}$ of M_7. If $n \geq 3$ is an integer, then $\boxtimes_T^n(M_7)$ is 3-connected, and has a P_6-minor, but no minor in $\mathcal{O}-\{P_6\}$.*

The proof is again a straightforward check and we skip it.

5. Proofs of the main results

Proof of Theorem 1.1. If $M \in \mathrm{Ex}(\{U_{2,4}, F_7^*\}) - \mathrm{Ex}(\{U_{2,4}, F_7, F_7^*\})$ is 3-connected, then M has an F_7-minor, and Proposition 2.1 implies that M is isomorphic to F_7. Therefore $\{F_7\}$ is certainly superfluous. Dually, $\{F_7^*\}$ is superfluous. Since $\mathrm{Ex}(\{F_7, F_7^*\})-\mathrm{Ex}(\{U_{2,4}, F_7, F_7^*\})$ contains all non-binary rank-2 uniform matroids, $\{U_{2,4}\}$ is contained in no superfluous subset. Similarly, $\mathrm{Ex}(\{U_{2,4}\}) - \mathrm{Ex}(\{U_{2,4}, F_7, F_7^*\})$ contains all binary projective geometries. Therefore $\{F_7, F_7^*\}$ is contained in no superfluous subset. The result follows. □

Proof of Theorem 1.3. Theorem 2.2 implies that the only 3-connected matroid in $\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7^*\}) - \mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7, F_7^*\})$ is F_7 itself. Thus $\{F_7\}$ and, by duality, $\{F_7^*\}$ are superfluous subsets. On the other hand, $\mathrm{Ex}(\{U_{3,5}, F_7, F_7^*\}) - \mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7, F_7^*\})$ contains all the non-ternary rank-2 uniform matroids, so $\{U_{2,5}\}$ and, by duality, $\{U_{3,5}\}$ is not contained in any superfluous subset. Finally, $\mathrm{Ex}(\{U_{2,5}, U_{3,5}\})-\mathrm{Ex}(\{U_{2,5}, U_{3,5}, F_7, F_7^*\})$ contains all binary projective geometries, so $\{F_7, F_7^*\}$ is not superfluous. □

Proof of Theorem 1.6. Theorem 1.5 implies that if M is a 3-connected matroid in $\mathrm{Ex}(\mathcal{O}-\{P_8, P_8^{=}\})-\mathrm{Ex}(\mathcal{O})$, then M is isomorphic to $P_8^{=}$ or a minor of $S(5,6,12)$. Thus $\{P_8, P_8^{=}\}$ is superfluous. As $\mathrm{Ex}(\mathcal{O}-\{U_{2,6}\})-\mathrm{Ex}(\mathcal{O})$ contains all rank-2 uniform matroids with at least 6 elements, $\{U_{2,6}\}$, and by duality $\{U_{4,6}\}$, is not contained in any superfluous subset. By Lemma 4.1, the set $\mathrm{Ex}(\mathcal{O}-\{F_7^-\})-\mathrm{Ex}(\mathcal{O})$ contains all matroids of the form $\boxtimes_T^n(M_8)$, so $\{F_7^-\}$, and by duality $\{(F_7^-)^*\}$, is not contained in any superfluous subset. Finally, Lemma 4.3 shows that $\mathrm{Ex}(\mathcal{O}-\{P_6\})-\mathrm{Ex}(\mathcal{O})$ contains an infinite number of 3-connected matroids, so $\{P_6\}$ is not contained in any superfluous subset. □

Proof of Theorem 1.8. Let M be a 3-connected matroid in $\mathrm{Ex}(\mathcal{S}-\{F_7, P_8\})-\mathrm{Ex}(\mathcal{S})$. If M has an F_7-minor, then Theorem 2.2 implies that $M \cong F_7$. Hence we assume that M does not have an F_7-minor, so that M has a P_8-minor. Corollary 2.4 says that M is a minor of $S(5,6,12)$. Therefore $\{F_7, P_8\}$, and by duality $\{F_7^*, P_8\}$, is superfluous. However, $\mathrm{Ex}(\mathcal{S}-\{U_{2,5}\})-\mathrm{Ex}(\mathcal{S})$ contains infinitely many uniform matroids, and $\mathrm{Ex}(\mathcal{S}-\{F_7^-\})-\mathrm{Ex}(\mathcal{S})$ contains all matroids of the form $\boxtimes_T^n(M_8)$. Duality implies that none of $\{U_{2,5}\}$, $\{U_{3,5}\}$, $\{F_7^-\}$, $\{(F_7^-)^*\}$ is contained in a superfluous subset. Finally, $\mathrm{Ex}(\mathcal{S}-\{F_7, F_7^*\})-\mathrm{Ex}(\mathcal{S})$ contains all binary projective geometries, so $\{F_7, F_7^*\}$ is contained in no superfluous subset. □

Proof of Theorem 1.10. Let M be a 3-connected matroid in

$$\mathrm{Ex}(\mathcal{N}-\{F_7, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*\})-\mathrm{Ex}(\mathcal{N}).$$

If M has an F_7-minor, then Theorem 2.2 implies that $M \cong F_7$. Otherwise Theorem 2.10 says that M is isomorphic to $\mathrm{AG}(2,3)\backslash e$, $\mathrm{AG}(2,3)$, or the dual of one of these matroids. Therefore $\{F_7, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*\}$, and by duality $\{F_7^*, \mathrm{AG}(2,3)\backslash e, (\mathrm{AG}(2,3)\backslash e)^*\}$, is superfluous. As $\mathrm{Ex}(\mathcal{N}-\{U_{2,5}\})-\mathrm{Ex}(\mathcal{N})$ contains infinitely many uniform matroids, and $\mathrm{Ex}(\mathcal{N}-\{F_7^-\})-\mathrm{Ex}(\mathcal{N})$ contains all matroids of the form $\boxtimes_T^n(M_8)$, none of $\{U_{2,5}\}$, $\{U_{3,5}\}$, $\{F_7^-\}$, $\{(F_7^-)^*\}$ is contained in a superfluous subset. Moreover, $\mathrm{Ex}(\mathcal{N}-\{\Delta_3(\mathrm{AG}(2,3)\backslash e)\})-\mathrm{Ex}(\mathcal{N})$ contains all matroids of the form $\boxtimes_T^n(M_9)$, by Lemma 4.2. Therefore $\{\Delta_3(\mathrm{AG}(2,3)\backslash e)\}$ is contained in no superfluous subset. Again, we observe that $\mathrm{Ex}(\mathcal{N}-\{F_7, F_7^*\})-\mathrm{Ex}(\mathcal{N})$ contains infinitely many binary matroids, so the proof is complete. □

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