

# THE EXCLUDED MINORS FOR NEAR-REGULAR MATROIDS

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ABSTRACT. In unpublished work, Geelen proved that a matroid is near-regular if and only if it has no minor isomorphic to  $U_{2,5}$ ,  $U_{3,5}$ ,  $F_7$ ,  $F_7^*$ ,  $F_7^-$ ,  $(F_7^-)^*$ ,  $AG(2,3)\setminus e$ ,  $(AG(2,3)\setminus e)^*$ ,  $\Delta_T(AG(2,3)\setminus e)$ , or  $P_8$ . We provide a proof of this characterization.

## 1. INTRODUCTION

Suppose that  $\mathcal{F}$  is a set of fields, and that  $\mathcal{M}(\mathcal{F})$  is the class of matroids that are representable over every field in  $\mathcal{F}$ . It is well-known that the family of binary matroids contains exactly two classes that arise in this way: the binary matroids themselves, and the regular matroids. A striking result due to Whittle [Whi97] shows that the family of ternary matroids contains exactly six such classes of matroids: the classes of ternary matroids, regular matroids, near-regular matroids, dyadic matroids, sixth-roots-of-unity matroids, and those matroids obtained from dyadic and sixth-roots-of-unity matroids using direct sums and 2-sums.

It is natural to ask for excluded-minor characterizations of the families mentioned above. The excluded minors for binary, ternary, and regular matroids have been known for some time [Bix79, Sey79, Tut58]. Geelen, Gerards, and Kapoor [GGK00] characterized the excluded minors for  $\text{GF}(4)$ -representable matroids.

**Theorem 1.1.** *The excluded minors for representability over  $\text{GF}(4)$  are  $U_{2,6}$ ,  $U_{4,6}$ ,  $P_6$ ,  $F_7^-$ ,  $(F_7^-)^*$ ,  $P_8$ , and  $P_8''$ .*

(Here  $P_6$  is the rank-3 matroid with six elements, and a triangle as its only non-spanning circuit. Other matroids mentioned in the article are defined in Section 7.1.) Since the class of sixth-roots-of-unity matroids is exactly  $\mathcal{M}(\{\text{GF}(3), \text{GF}(4)\})$ , Theorem 1.1 leads to an excluded minor characterization of the sixth-roots-of-unity matroids [GGK00, Corollary 1.4].

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In this article we consider the class of near-regular matroids, which is exactly  $\mathcal{M}(\{\text{GF}(3), \text{GF}(4), \text{GF}(5)\})$ . By adapting the proof of Theorem 1.1, Geelen was able to characterize the excluded minors for near-regularity. However, this result remained unpublished until now. We present a proof of Geelen's theorem.

**Theorem 1.2** (Geelen). *The excluded minors for the class of near-regular matroids are  $U_{2,5}$ ,  $U_{3,5}$ ,  $F_7$ ,  $F_7^*$ ,  $F_7^-$ ,  $(F_7^-)^*$ ,  $\text{AG}(2,3)\setminus e$ ,  $(\text{AG}(2,3)\setminus e)^*$ ,  $\Delta_T(\text{AG}(2,3)\setminus e)$ , and  $P_8$ .*

We now give an informal outline of the proof. The classes of regular, near-regular, sixth-roots-of-unity, and dyadic matroids can all be characterized as the matroids representable over a particular partial field. Partial fields were introduced by Semple and Whittle [SW96]. They are much like fields, except that addition is not always defined. If the subdeterminants of a matrix over a partial field are all defined, then there is a corresponding matroid, whose ground set consists of the rows and columns of the matrix. Two matrices representing the same matroid are equivalent if they are equal up to pivots, scaling, and applications of partial field automorphisms. Kahn [Kah88] showed that a stable matroid is uniquely representable over  $\text{GF}(4)$ , up to equivalence, and this fact plays a crucial role in the proof of Theorem 1.1. (A stable matroid is one that cannot be expressed as a direct sum or a 2-sum of two nonbinary matroids.)

In order to proceed with our proof, we must establish a similar uniqueness of representations for near-regular matroids. For this purpose we use Whittle's tool of stabilizers [Whi99]. In Section 3 we prove an analogue of Kahn's theorem by showing that a stable near-regular matroid is uniquely representable over the near-regular partial field.

We reduce the proof of Theorem 1.2 to a finite case check by proving that any excluded minor for near-regularity has at most eight elements. We suppose that  $M$  is a counterexample to this proposition. Theorem 3.1 in [GGK00] shows that there are elements  $u$  and  $v$ , such that  $M\setminus u$ ,  $M\setminus v$ , and  $M\setminus\{u, v\}$  are all stable, and  $M\setminus\{u, v\}$  is connected and nonbinary. At this point, Geelen et al. construct the unique  $\text{GF}(4)$ -representable matroid  $N$  such that  $M\setminus u = N\setminus u$  and  $M\setminus v = N\setminus v$ . Our proof is slightly different, in that our matroid  $N$  need not be near-regular. However,  $N$  is representable over the field  $\mathbb{Q}(\alpha)$ , as is every near-regular matroid. Whittle's characterization reveals that the counterexample  $M$  cannot be  $\mathbb{Q}(\alpha)$ -representable, so  $M$  and  $N$  are genuinely different.

The core of the proof is contained in Section 6. This part of the proof follows the proof of Theorem 1.1 very closely, only deviating when that proof calls upon the structure of  $\text{GF}(4)$ . We are advantaged here by the fact that our counterexample must be ternary. In the proof of Theorem 1.1, there is no a priori reason why the counterexample need be representable over any field. Our fundamental tool is the uniqueness of the matroid  $N$ . Suppose that  $M'$  is some small proper (and hence near-regular) minor of

$M$ , such that  $M' \setminus u$ ,  $M' \setminus v$ , and  $M' \setminus \{u, v\}$  are all stable, and  $M' \setminus \{u, v\}$  is connected and nonbinary. By using the same technique as before, we can construct a  $\mathbb{Q}(\alpha)$ -representable matroid  $N'$  such that  $M' \setminus u = N' \setminus u$  and  $M' \setminus v = N' \setminus v$ . The uniqueness of  $N$  guarantees that  $N'$  is the minor of  $N$  that corresponds to  $M'$ , and that  $M' = N'$ . If we can find some certificate that  $M'$  and  $N'$  are not equal, then we have arrived at a contradiction. This contradiction forces us to conclude that  $M'$  is not near-regular, and that therefore  $M' = M$ . Because we have a bound on the size of  $M'$ , this induces a bound on the size of  $M$ .

In order to invoke the uniqueness of  $N$ , certain connectivity conditions have to be satisfied. To obtain these conditions we use blocking sequences, which we review in Section 5.

Once we have completed the work of Section 6, finishing the proof is relatively straightforward. In Section 7 we first introduce the matroids listed in Theorem 1.2, and we show that they are in fact excluded minors for the class of near-regular matroids. Then it remains only to perform the finite case-check. All undefined matroid terms are as in Oxley [Oxl92].

## 2. PRELIMINARIES

**2.1. Partial fields.** The classes of regular, near-regular, dyadic, and sixth-roots-of-unity matroids have a common characteristic: for every such class, there is a field  $\mathbb{F}$ , and a subgroup  $\mathbf{G}$  of  $\mathbb{F}^*$ , such that a matroid belongs to the class if and only if it can be represented by a matrix  $A$  over  $\mathbb{F}$ , where all the nonzero subdeterminants of  $A$  belong to  $\mathbf{G}$ . Partial fields provide a unified framework for studying this phenomenon. They were introduced by Semple and Whittle [SW96], and studied further by Pendavingh and Van Zwam [PZa, PZb].

Semple and Whittle developed partial fields axiomatically. We treat them somewhat differently: Vertigan showed that every partial field can be thought of as a ring along with a subgroup of units (see [PZb, Theorem 2.16]), and we use this description as our definition.

**Definition 2.1.** A *partial field* is a pair  $(\mathbb{O}, \mathbf{G})$ , where  $\mathbb{O}$  is a commutative ring with identity, and  $\mathbf{G}$  is a subgroup of the group of units  $\mathbb{O}^*$  of  $\mathbb{O}$ , such that  $-1 \in \mathbf{G}$ .

Note that every field  $\mathbb{F}$  is also a partial field  $(\mathbb{F}, \mathbb{F}^*)$ . Suppose that  $\mathbb{P} = (\mathbb{O}, \mathbf{G})$  is a partial field. We also use  $\mathbb{P}$  to denote the set  $\mathbf{G} \cup 0$ , so we say that  $p \in \mathbb{O}$  is an element of  $\mathbb{P}$  (and we write  $p \in \mathbb{P}$ ), if  $p \in \mathbf{G}$  or  $p = 0$ . Thus,  $p + q$  may not be an element of  $\mathbb{P}$ , even though both  $p$  and  $q$  are contained in  $\mathbb{P}$ . If  $p + q$  is in  $\mathbb{P}$ , then we say that  $p + q$  is *defined*. We use  $\mathbb{P}^*$  to denote the set of nonzero elements of  $\mathbb{P}$ ; thus  $\mathbb{P}^* = \mathbf{G}$ .

**Definition 2.2.** Suppose that  $\mathbb{P}$  is a partial field. We say that  $p \in \mathbb{P}$  is a *fundamental element* if  $1 - p \in \mathbb{P}$ .

Note that  $p + q$  is defined if and only if  $-q/p$  is a fundamental element, since  $p + q = p(1 - (-q/p))$ .

**Definition 2.3.** Suppose that  $\mathbb{P}_1$  and  $\mathbb{P}_2$  are partial fields. A function  $\psi : \mathbb{P}_1 \rightarrow \mathbb{P}_2$  is a *partial-field homomorphism* if

- (i)  $\psi(1) = 1$ ;
- (ii) for all  $p, q \in \mathbb{P}_1$ ,  $\psi(pq) = \psi(p)\psi(q)$ ; and
- (iii) for all  $p, q \in \mathbb{P}_1$  such that  $p + q$  is defined,  $\psi(p) + \psi(q) = \psi(p + q)$ .

In particular, if  $\mathbb{P}_1 = (\mathbb{O}_1, \mathbf{G}_1)$ ,  $\mathbb{P}_2 = (\mathbb{O}_2, \mathbf{G}_2)$ , and  $\psi : \mathbb{O}_1 \rightarrow \mathbb{O}_2$  is a ring homomorphism such that  $\psi(\mathbf{G}_1) \subseteq \mathbf{G}_2$ , then the restriction of  $\psi$  to  $\mathbb{P}_1$  is a partial-field homomorphism. It is easy to verify that if  $\psi$  is a partial-field homomorphism then  $\psi(0) = 0$  and  $\psi(-1) = -1$ .

A partial field *isomorphism* from  $\mathbb{P}_1$  to  $\mathbb{P}_2$  is a bijective homomorphism  $\psi$  with the additional property that  $\psi(p) + \psi(q)$  is defined if and only if  $p + q$  is defined. We use  $\mathbb{P}_1 \cong \mathbb{P}_2$  to denote the fact that  $\mathbb{P}_1$  and  $\mathbb{P}_2$  are isomorphic. An *automorphism* of a partial field  $\mathbb{P}$  is an isomorphism from  $\mathbb{P}$  to itself.

**2.2. Representation matrices.** Suppose that  $A$  is a matrix with entries from a partial field  $\mathbb{P}$ , and that the rows and columns of  $A$  are labeled by the (ordered) sets  $X$  and  $Y$  respectively, where  $X \cap Y = \emptyset$ . If the determinant of every square submatrix of  $A$  is contained in  $\mathbb{P}$ , then we say that  $A$  is an  $X \times Y$   $\mathbb{P}$ -matrix. If  $A$  is a  $\mathbb{P}$ -matrix, then the *rank* of  $A$ , written  $\text{rank}(A)$ , is the largest value  $k$  such that  $A$  contains a nonzero  $k \times k$  subdeterminant.

Since we will frequently work with submatrices, it is useful to introduce some notation. If  $X' \subseteq X$  and  $Y' \subseteq Y$ , then  $A[X', Y']$  is the submatrix of  $A$  induced by  $X'$  and  $Y'$ . In particular, we define  $A_{xy}$  to be  $A[\{x\}, \{y\}]$ . If  $Z \subseteq X \cup Y$ , then  $A[Z] := A[Z \cap X, Z \cap Y]$ , and  $A - Z := A[X \setminus Z, Y \setminus Z]$ . If  $A$  is a matrix over the partial field  $\mathbb{P}$ , and  $\psi$  is a function on  $\mathbb{P}$ , then  $\psi(A)$  is obtained by operating on each entry in  $A$  with  $\psi$ . The following theorem follows from [SW96, Theorem 3.6] (see also [PZa, Theorem 2.8]).

**Lemma 2.4.** *Let  $\mathbb{P}$  be a partial field, and let  $A$  be an  $X \times Y$   $\mathbb{P}$ -matrix. Let*

$$\mathcal{B} := \{X\} \cup \{X \Delta Z \mid |X \cap Z| = |Y \cap Z|, \det(A[Z]) \neq 0\}.$$

*Then  $\mathcal{B}$  is the set of bases of a matroid on  $X \cup Y$ .*

Let  $M$  be the matroid of Lemma 2.4. We say that  $M$  is *representable* over  $\mathbb{P}$ , or is  $\mathbb{P}$ -*representable*, and we say that  $M$  is *represented* by  $A$ . We remark that this terminology is not standard: the usual convention is that a matroid represented by a matrix  $A$  has the set of columns of  $A$  as its ground set. Throughout this article, when we say that  $M$  is represented by  $A$ , we mean that  $M$  is the matroid of Lemma 2.4, so the ground set of  $M$  is the set of rows and columns of  $A$ , and the set of rows of  $A$  is a basis of  $M$ . If  $M$  is represented by  $A$  (in our sense), then it is represented (in the standard sense) by the matrix obtained from  $A$  by appending an  $|X| \times |X|$  identity matrix. For this reason, we write  $M = M[I|A]$  if  $A$  is a  $\mathbb{P}$ -matrix, and  $M$  is the matroid in Lemma 2.4. If  $A$  is an  $X \times Y$   $\mathbb{P}$ -matrix, and  $M$  is the matroid

represented by  $A$ , then  $M^*$  is represented by  $A^T$ , the transpose of  $A$ , where the rows and columns of  $A^T$  are labeled with  $Y$  and  $X$  respectively.

**Proposition 2.5.** [SW96, Proposition 4.2]. *Let  $\mathbb{P}$  be a partial field. The class of  $\mathbb{P}$ -representable matroids is closed under duality, taking minors, direct sums, and 2-sums.*

The next result follows from [SW96, Proposition 5.1] or [PZb, Proposition 2.10].

**Proposition 2.6.** *Let  $\mathbb{P}_1, \mathbb{P}_2$  be partial fields and let  $\psi : \mathbb{P}_1 \rightarrow \mathbb{P}_2$  be a homomorphism. Let  $A$  be a  $\mathbb{P}_1$ -matrix. Then*

- (i)  $\psi(A)$  is a  $\mathbb{P}_2$ -matrix;
- (ii) If  $A$  is square then  $\det(A) = 0$  if and only if  $\det(\psi(A)) = 0$ ; and
- (iii)  $M[I|A] = M[I|\psi(A)]$ .

**Definition 2.7.** Let  $A$  be an  $X \times Y$   $\mathbb{P}$ -matrix, and let  $x \in X, y \in Y$  be such that  $A_{xy} \neq 0$ . Then we define  $A^{xy}$  to be the  $(X \triangle \{x, y\}) \times (Y \triangle \{x, y\})$  matrix given by

$$(A^{xy})_{uv} = \begin{cases} A_{xy}^{-1} & \text{if } uv = yx \\ A_{xy}^{-1} A_{xv} & \text{if } u = y, v \neq x \\ -A_{xy}^{-1} A_{uy} & \text{if } v = x, u \neq y \\ A_{uv} - A_{xy}^{-1} A_{uy} A_{xv} & \text{otherwise.} \end{cases}$$

We say that  $A^{xy}$  is obtained from  $A$  by *pivoting* over  $xy$ . Note that after pivoting,  $x$  labels a column, and  $y$  labels a row. Suppose that  $\mathbb{P}$  is a partial field and that  $A$  is an  $X \times Y$   $\mathbb{P}$ -matrix. *Scaling* means multiplying the rows or columns of  $A$  by nonzero members of  $\mathbb{P}$ . The next result is Proposition 3.3 in [SW96], or Proposition 2.5 in [PZb].

**Proposition 2.8.** *If  $A$  is a  $\mathbb{P}$ -matrix, and  $A'$  is obtained from  $A$  by scaling and pivoting, then  $A'$  is a  $\mathbb{P}$ -matrix.*

**Definition 2.9.** Let  $\mathbb{P}$  be a partial field, and let  $A, A'$  be  $\mathbb{P}$ -matrices. Then  $A$  and  $A'$  are *scaling-equivalent* if  $A'$  can be obtained from  $A$  by scaling. If  $A'$  can be obtained from  $A$  by scaling, pivoting, permuting columns and rows (permuting labels at the same time), and applying automorphisms of  $\mathbb{P}$ , then we say that  $A$  and  $A'$  are *equivalent*.

The next result follows easily from [SW96, Proposition 3.5] and Proposition 2.6.

**Proposition 2.10.** *Suppose that  $A$  and  $A'$  are equivalent  $\mathbb{P}$ -matrices. Then  $M[I|A] = M[I|A']$ .*

**Definition 2.11.** Let  $M$  be a matroid and suppose that  $\mathbb{P}$  is a partial field. We say that  $M$  is *uniquely representable over  $\mathbb{P}$*  if, whenever  $A, A'$  are  $\mathbb{P}$ -matrices such that  $M = M[I|A] = M[I|A']$ , then  $A$  and  $A'$  are equivalent.

**2.3. Bipartite graphs and twirls.** Let  $M$  be a rank- $r$  matroid with ground set  $E$ , and let  $\mathcal{B}$  be its set of bases. Suppose that  $B \in \mathcal{B}$ . Let  $G_B(M) = (V, E)$  be the bipartite graph with vertices  $V := B \cup (E \setminus B)$  and edges  $E := \{(x, y) \in B \times (E \setminus B) \mid B \Delta \{x, y\} \in \mathcal{B}\}$ .

Let  $A$  be an  $X \times Y$  matrix. We associate with  $A$  a bipartite graph  $G(A) = (V, E)$ , where  $V := X \cup Y$  and  $E := \{(x, y) \in X \times Y \mid A_{xy} \neq 0\}$ . Thus each edge,  $e$ , of  $G(A)$  corresponds to a nonzero entry,  $A_e$ , of  $A$ . We note here that if  $A_{xy} \neq 0$ , and  $y'$  and  $x'$  are neighbors of  $x$  and  $y$  respectively such that  $y'$  and  $x'$  are not adjacent in  $G(A)$ , then  $y'$  and  $x'$  are adjacent in  $G(A^{xy})$ .

**Lemma 2.12.** *Let  $\mathbb{P}$  be a partial field,  $A$  an  $X \times Y$   $\mathbb{P}$ -matrix, and let  $M = M[I|A]$ .*

- (i)  $G_X(M) = G(A)$ .
- (ii) *Let  $T$  be a forest of  $G(A)$  with edges  $e_1, \dots, e_k$ . Suppose that  $p_1, \dots, p_k$  are elements of  $\mathbb{P}^*$ . There exists a  $\mathbb{P}$ -matrix  $A'$  such that  $A'$  is scaling-equivalent to  $A$ , and  $A'_{e_i} = p_i$  for  $1 \leq i \leq k$ .*

*Proof.* Suppose that  $x \in X$  and  $y \in Y$ . Then  $xy$  is an edge of  $G(A)$  if and only if the determinant of  $A[\{x\}, \{y\}]$  is nonzero, which is true if and only if  $X \Delta \{x, y\}$  is a basis of  $M$ . This is equivalent to  $xy$  being an edge of  $G_X(M)$ .

We prove the second statement by induction on  $k$ . The result is trivially true if  $T$  contains no edges. By relabeling as required, we can assume that in the forest  $T$ , the edge  $e_k$  is incident with a degree-one vertex  $v$ . By induction, there is a matrix  $A''$  obtained from  $A$  by scaling, with the property that  $A''_{e_i} = p_i$  for  $1 \leq i \leq k - 1$ . Certainly  $A''_{e_k}$  is nonzero, let us say that it is equal to the element  $\beta \in \mathbb{P}^*$ . Now we multiply the row or column labeled by  $v$  with  $p_k \beta^{-1}$  to produce  $A'$ .  $\square$

Let  $A$  be a matrix and suppose that  $T$  is a forest of  $G(A)$ . We say that  $A$  is  $T$ -normalized if  $A_{xy} = 1$  for all  $xy \in T$ . By Lemma 2.12 there is always a  $T$ -normalized matrix  $A'$  that is scaling-equivalent to  $A$ .

We make repeated use of the following (easy) fact.

**Proposition 2.13.** *Let  $G$  be a graph, and suppose that  $S$  is a set of edges that contains a maximal forest of  $G$ . If  $e$  is an edge not contained in  $S$ , then there is an induced cycle of  $G$  that contains  $e$ , and such that the edges of this cycle are contained in  $S \cup e$ .*

**Definition 2.14.** Let  $A$  be a square  $\mathbb{P}$ -matrix. Then  $A$  is a *twirl* if  $G(A)$  is a cycle and  $\det(A) \neq 0$ .

Recall that the rank- $r$  *whirl* is denoted by  $\mathcal{W}^r$ . A whirl is representable over a field  $\mathbb{F}$  if and only if  $|\mathbb{F}| \geq 3$ . Note that if  $A$  is a twirl then  $M[I|A]$  is a whirl.

**Proposition 2.15.** [GGK00, Proposition 4.5]. *Let  $A$  be an  $X \times Y$  matrix that is a twirl, and let  $x, y$  be such that  $A_{xy} \neq 0$ .*

- (i) If  $|X \cup Y| = 4$  then  $A^{xy}$  is a twirl.
- (ii) If  $|X \cup Y| > 4$  then  $A^{xy} - \{x, y\}$  is a twirl.

**2.4. Near-regular matroids.** Recall that  $\mathbb{Q}(\alpha)$  is the field obtained from the rational numbers by extending with the transcendental element  $\alpha$ . Let  $\mathbb{Z}[\alpha, 1/\alpha, 1/(1-\alpha)]$  be the subring of  $\mathbb{Q}(\alpha)$  induced by  $\alpha$ ,  $1/\alpha$ ,  $1/(1-\alpha)$ , and the integers.

**Definition 2.16.** The *near-regular* partial field is

$$\mathbb{U}_1 := \left( \mathbb{Z} \left[ \alpha, \frac{1}{\alpha}, \frac{1}{1-\alpha} \right], \langle -1, \alpha, 1-\alpha \rangle \right).$$

Here  $\langle -1, \alpha, 1-\alpha \rangle$  denotes the subgroup of units generated by  $-1$ ,  $\alpha$ , and  $1-\alpha$ . Thus  $\mathbb{U}_1$  consists of zero, and elements of the form  $\pm\alpha^i(1-\alpha)^j$ , where  $i$  and  $j$  are integers. We note that  $\mathbb{U}_1$  is a special case of a class of partial fields studied by Semple [Sem97].

A  $\mathbb{U}_1$ -matrix is said to be *near-unimodular*. A matroid is *near-regular* if it is representable over  $\mathbb{U}_1$ . Whittle's characterization shows, amongst other things, that a matroid is near-regular if and only if it is representable over every field with cardinality at least three.

**Theorem 2.17.** [Whi97, Theorem 1.4]. *Let  $M$  be a matroid. The following are equivalent:*

- (i)  $M$  is representable over  $\text{GF}(3)$ ,  $\text{GF}(4)$ , and  $\text{GF}(5)$ ;
- (ii)  $M$  is representable over  $\text{GF}(3)$  and  $\text{GF}(8)$ ;
- (iii)  $M$  is near-regular; and
- (iv)  $M$  is representable over all fields except, possibly,  $\text{GF}(2)$ .

Next we collect some basic facts about the near-regular partial field. The first result follows from Lemmas 2.23 and 4.3 in [PZb].

**Proposition 2.18.** *The fundamental elements of  $\mathbb{U}_1$  are*

$$\left\{ 0, 1, \alpha, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha-1}{\alpha}, \frac{1}{\alpha} \right\}.$$

**Proposition 2.19.** *Let  $\alpha_i$  and  $\alpha_j$  be fundamental elements of  $\mathbb{U}_1$  that are equal to neither 1 nor 0. There is an automorphism of  $\mathbb{U}_1$  that takes  $\alpha_i$  to  $\alpha_j$ .*

*Proof.* Obviously an automorphism of  $\mathbb{U}_1$  permutes the fundamental elements. Consider a function  $\psi : \mathbb{Q}(\alpha) \rightarrow \mathbb{Q}(\alpha)$  which acts as the identity on 0 and 1, takes  $\alpha$  to another fundamental element of  $\mathbb{U}_1$ , and which respects addition and multiplication. The following table shows how  $\psi$  acts upon the

element  $\alpha^i(1 - \alpha)^j$  of  $\mathbb{U}_1$ .

$\psi(\alpha)$	$\psi(\alpha^i(1 - \alpha)^j)$
$\alpha$	$\alpha^i(1 - \alpha)^j$
$1 - \alpha$	$\alpha^j(1 - \alpha)^i$
$1/(1 - \alpha)$	$(-1)^j \alpha^j (1 - \alpha)^{-(i+j)}$
$\alpha/(\alpha - 1)$	$(-1)^i \alpha^i (1 - \alpha)^{-(i+j)}$
$(\alpha - 1)/\alpha$	$(-1)^i \alpha^{-(i+j)} (1 - \alpha)^i$
$1/\alpha$	$(-1)^j \alpha^{-(i+j)} (1 - \alpha)^j$

Now it is clear that the restriction of  $\psi$  to  $\mathbb{U}_1$  is indeed an automorphism. Since the inverse of an automorphism is another automorphism, and so is the composition of two automorphisms, the result follows easily.  $\square$

Recall that a matrix over the rationals is *totally unimodular* if every sub-determinant belongs to  $\{0, 1, -1\}$ . A matroid is *regular* if and only if it can be represented by a totally unimodular matrix. It is well-known that regular matroids are representable over all fields ([Oxl92, Theorem 6.6.3]).

**Proposition 2.20.** *Suppose that  $A$  is a near-unimodular matrix that is not equivalent to a totally unimodular matrix. If  $\psi$  is an automorphism of  $\mathbb{U}_1$  such that  $\psi(A) = A$ , then  $\psi$  is the trivial automorphism.*

*Proof.* Suppose that the rows and columns of  $A$  are labeled with  $X$  and  $Y$ . We assume that  $\psi$  is not the identity function on  $\mathbb{U}_1$ , so that  $\psi(\alpha) \neq \alpha$ . Let  $T$  be a maximal forest of  $G(A)$ . By examining the proof of Lemma 2.12, we see that while  $T$ -normalizing  $A$ , we only ever multiply a row or column by the inverse of a nonzero entry of  $A$ . If  $\beta$  is a nonzero entry of  $A$ , then  $\psi(\beta) = \beta$ , and therefore  $\psi(\beta^{-1}) = \beta^{-1}$ . It follows easily that normalizing  $A$  does not affect the assumption that  $\psi(A) = A$ . Moreover, normalizing  $A$  does not produce a totally unimodular matrix, as  $A$  is not equivalent to such a matrix. Henceforth we assume that  $A$  is  $T$ -normalized.

Let  $S$  be the set of nonzero entries of  $A$  that are equal to 1 or  $-1$ . There is an edge  $e$  in  $G(A)$  not contained in  $S$ . As  $S$  contains the edge-set of  $T$ , Proposition 2.13 asserts that there is a set  $C \subseteq X \cup Y$  such that  $G(A[C])$  is an induced cycle of  $G(A)$  containing  $e$ , and the edges of  $G(A[C])$  are contained in  $S \cup e$ .

Suppose that  $A_e = (-1)^k \alpha^i (1 - \alpha)^j$  for integers  $i, j$ , and  $k$ . Then

$$(1) \quad \psi(\alpha^i(1 - \alpha)^j) = \alpha^i(1 - \alpha)^j.$$

By examining the table in the proof of Proposition 2.19, we see that if  $\psi(\alpha)$  is equal to  $1/(1 - \alpha)$  or  $(\alpha - 1)/\alpha$ , then the only solution to Equation (1) is  $i = j = 0$ . This is a contradiction as  $e \notin S$ . Therefore we suppose that  $\psi(\alpha) = 1 - \alpha$ . Then  $\psi(\alpha^i(1 - \alpha)^j) = \alpha^j(1 - \alpha)^i$ , so  $i = j$ .

Since every nonzero entry in  $A[C]$ , other than  $A_e$ , is in  $\{1, -1\}$ , and  $G(A[C])$  is a cycle, it follows that the determinant of  $A[C]$  is, up to multiplication by  $-1$ , equal to  $A_e \pm 1$ . As this determinant belongs to  $\mathbb{U}_1$ , it



follows that either  $A_e$  or  $-A_e$  is a fundamental element. But no fundamental element, other than 1, is of the form  $\pm\alpha^i(1-\alpha)^i$ , and we have a contradiction.

Similarly, if  $\psi(\alpha)$  is equal to  $\alpha/(\alpha-1)$  or  $1/\alpha$ , then  $i$  and  $j$  must satisfy either  $2j = -i$ , or  $2i = -j$ . In either case we arrive at a similar contradiction.  $\square$

The next result is an adaptation of Lemma 4.3 in [GGK00].

**Lemma 2.21.** *Let  $A$  be a near-unimodular  $X \times Y$  matrix. Then there is some  $C \subseteq X \cup Y$  such that  $A[C]$  is a twirl if and only if  $M[I|A]$  is nonbinary.*

*Proof.* If  $A$  contains a twirl, then  $M[I|A]$  contains a whirl-minor, and is therefore nonbinary. For the converse, let  $T$  be a maximal forest of  $G(A)$ , and assume that  $A$  is  $T$ -normalized. Let  $S$  be the set of nonzero entries in  $A$  that are equal to 1 or  $-1$ . As  $M[I|A]$  is nonbinary, it is certainly not regular, and therefore  $A$  is not totally unimodular. Hence there is an edge  $e$  of  $G(A)$  such that  $e \notin S$ . Proposition 2.13 provides a subset  $C \subseteq X \cup Y$  such that  $G(A[C])$  is a cycle containing  $e$ , and the edges of  $G(A[C])$  are contained in  $S \cup e$ . Then  $A[C]$  is a twirl.  $\square$

The following analogue of Lemma 4.4 in [GGK00] is proved in a similar way to Lemma 2.21.

**Lemma 2.22.** *Let  $A$  be an  $X \times Y$   $\mathbb{U}_1$ -matrix, and suppose that  $A[C]$  is a twirl for some  $C \subseteq X \cup Y$ . Let  $v_0, \dots, v_p$  be the vertices of  $A[C]$  in cyclic order. Suppose that  $x \in (X \cup Y) \setminus C$  and the neighbors of  $x$  in  $C$  are  $v_{i_1}, \dots, v_{i_k}$ , where  $k \geq 2$ . Then there exists a twirl of the form  $A[\{x, v_{i_j}, \dots, v_{i_{j+1}}\}]$  (where  $1 \leq j \leq k-1$ ) or  $A[\{x, v_0, \dots, v_{i_1}, v_{i_k}, \dots, v_p\}]$ .*

**2.5. Stabilizers.** The notion of a stabilizer, introduced by Whittle [Whi99], is an indispensable tool for controlling inequivalent representations.

**Definition 2.23.** Let  $\mathbb{P}$  be a partial field, and let  $M$  and  $N$  be 3-connected  $\mathbb{P}$ -representable matroids such that  $N$  is a minor of  $M$ . Suppose that the ground set of  $N$  is  $X' \cup Y'$ , where  $X'$  is a basis of  $N$ . We say that  $N$  is a  $\mathbb{P}$ -stabilizer for  $M$  if, whenever  $A_1$  and  $A_2$  are  $X \times Y$   $\mathbb{P}$ -matrices (where  $X' \subseteq X$  and  $Y' \subseteq Y$ ) such that

- (i)  $M = M[I|A_1] = M[I|A_2]$ ;
- (ii)  $A_1[X', Y']$  is scaling-equivalent to  $A_2[X', Y']$ ; and
- (iii)  $N = M[I|A_1[X', Y']] = M[I|A_2[X', Y']]$ ,

then  $A_1$  is scaling-equivalent to  $A_2$ .

We say that  $N$  is a  $\mathbb{P}$ -stabilizer for a class of matroids if  $N$  is a  $\mathbb{P}$ -stabilizer for every 3-connected member of the class.

Whittle proved that verifying that a matroid is a stabilizer can be accomplished with a finite case-check. (See also [PZa, Theorem 3.10].)

**Theorem 2.24** (Stabilizer Theorem, Whittle [Whi99]). *Let  $\mathbb{P}$  be a partial field, and let  $N$  be a 3-connected  $\mathbb{P}$ -representable matroid. Let  $M$  be a 3-connected  $\mathbb{P}$ -representable matroid having an  $N$ -minor. Then exactly one of the following is true:*

- (i)  $N$  stabilizes  $M$ .
- (ii)  $M$  has a 3-connected minor  $M'$  such that
  - (a)  $N$  does not stabilize  $M'$ ;
  - (b)  $N$  is isomorphic to  $M'/x$ ,  $M'\setminus y$ , or  $M'/x\setminus y$ , for some  $x, y \in E(M')$ ; and
  - (c) If  $N$  is isomorphic to  $M'/x\setminus y$  then at least one of  $M'/x$ ,  $M'\setminus y$  is 3-connected.

Since  $U_{2,4}$  has no 3-connected, near-regular one-element extensions or coextensions, the following result follows easily:

**Corollary 2.25.**  $U_{2,4}$  is a  $\mathbb{U}_1$ -stabilizer for the class of near-regular matroids.

**2.6. The  $\Delta$ - $Y$  operation.** Suppose that  $M$  is a matroid and that  $T$  is a coindependent triangle of  $M$ . Let  $N$  be an isomorphic copy of  $M(K_4)$  such that  $E(N) \cap E(M) = T$  and  $T$  is a triangle of  $N$ . Then the *generalized parallel connection* of  $M$  and  $N$ , denoted  $P_T(N, M)$ , is defined. This is the matroid on the ground set  $E(M) \cup E(N)$  whose flats are exactly the sets  $F$  such that  $F \cap E(N)$  and  $F \cap E(M)$  are flats of  $N$  and  $M$  respectively. Suppose that  $T = \{a, b, c\}$ . If  $x \in T$ , then there is a unique element,  $x'$ , of  $N$ , that is in no triangle with  $x$ . We swap the labels on  $x$  and  $x'$  in  $P_T(M, N)$ , for each  $x \in T$ . Thus  $P_T(M, N) \setminus T$  and  $M$  have the same ground set. We say that  $P_T(M, N) \setminus T$  is produced by a  $\Delta$ - $Y$  operation on  $M$ , and we denote the resulting matroid with  $\Delta_T(M)$ . The  $\Delta$ - $Y$  operation has been studied by Akkari and Oxley [AO93] and generalized by Oxley, Semple, and Vertigan [OSV00].

Suppose that  $T$  is an independent triad of the matroid  $M$ . Then  $\Delta_T(M^*)$  is defined, and  $(\Delta_T(M^*))^*$  is said to be produced from  $M$  by a  $Y$ - $\Delta$  operation, and is denoted by  $\nabla_T(M)$ . The next results follow by combining Lemmas 2.6 and 2.11, and Theorem 1.1 in [OSV00].

**Lemma 2.26.** *Suppose that  $T$  is a coindependent triangle of  $M$ . Then*

$$r(\Delta_T(M)) = r(M) + 1.$$

*Moreover,  $T$  is an independent triad in  $\Delta_T(M)$ , and  $\nabla_T(\Delta_T(M)) = M$ .*

**Lemma 2.27.** *Suppose that  $\mathbb{P}$  is a partial field and that  $M$  is an excluded minor for the class of  $\mathbb{P}$ -representable matroids. If  $T$  is a coindependent triangle of  $M$  then  $\Delta_T(M)$  is also an excluded minor for the class of  $\mathbb{P}$ -representable matroids.*

## 3. UNIQUE REPRESENTATIONS

In this section we prove an analogue of Kahn's theorem by showing that stable near-regular matroids are uniquely representable over  $\mathbb{U}_1$ . Brylawski and Lucas [BL76] prove that binary matroids are uniquely representable over any field. The proof of the following result sketches the straightforward adaptation of their argument to partial fields.

**Proposition 3.1.** *Suppose that  $\mathbb{P}$  is a partial field, and that the  $X \times Y$   $\mathbb{P}$ -matrices  $A_1$  and  $A_2$  both represent the binary matroid  $M$ . Let  $T$  be a maximal forest of  $G(A_1) = G(A_2)$ . Suppose that both  $A_1$  and  $A_2$  are  $T$ -normalized. Then  $A_1 = A_2$ . Hence  $M$  is uniquely representable over  $\mathbb{P}$ .*

*Proof.* We claim that  $A_1 = A_2$  and that every nonzero entry of  $A_1$  and  $A_2$  belongs to  $\{1, -1\}$ . Let  $S$  be the set of edges of  $G(A_1) = G(A_2)$  such that  $xy \in S$  if and only if  $(A_1)_{xy} \in \{1, -1\}$  and  $(A_2)_{xy} = (A_1)_{xy}$ . If our claim is false, then there is an edge  $e$  of  $G(A_1)$  not in  $S$ . Since  $S$  contains the edge-set of  $T$ , Proposition 2.13 implies that there is a set  $C \subseteq X \cup Y$  such that  $G(A_1[C])$  is a cycle containing  $e$ , the edges of which are contained in  $S \cup e$ .

Let  $A$  be the  $X \times Y$   $\text{GF}(2)$ -matrix obtained from  $A_1$  by replacing every nonzero entry with 1. As  $M$  is binary,  $A$  represents  $M$  over  $\text{GF}(2)$ . Since  $G(A[C])$  is a cycle, it is easy to see that  $A[C]$  has zero determinant over  $\text{GF}(2)$ . Therefore the determinant of  $A_1[C]$  is also zero. Let  $\beta = (A_1)_e$ . Every nonzero entry of  $A_1[C]$ , other than  $(A_1)_e$ , belongs to  $\{1, -1\}$ . Now it is easy to see that the determinant of  $A_1[C]$  is, up to multiplication by  $-1$ , equal to  $\beta \pm 1$ . Thus  $\beta \in \{1, -1\}$ . However, the same argument shows that  $(A_2)_e$  is equal to  $\beta$ , and we have a contradiction to the fact that  $e \notin S$ .  $\square$

The direct sum or 2-sum of two uniquely representable matroids need not be uniquely representable (for example, the 2-sum of two copies of  $U_{2,4}$  is not uniquely representable over  $\text{GF}(4)$ ). But we do have the following partial result.

**Proposition 3.2.** *Let  $\mathbb{P}$  be a partial field, and suppose that the matroid  $M_1$  is uniquely representable over  $\mathbb{P}$ . Let  $M_2$  be a  $\mathbb{P}$ -representable matroid, and suppose that, whenever  $A_1$  and  $A_2$  are two  $T$ -normalized  $X \times Y$   $\mathbb{P}$ -representations of  $M_2$ , then  $A_1 = A_2$ . (Here  $T$  is a maximal forest of  $G(A_1) = G(A_2)$ .) Then  $M_1 \oplus_1 M_2$  and  $M_1 \oplus_2 M_2$  are uniquely  $\mathbb{P}$ -representable.*

*Proof.* We present the proof that  $M_1 \oplus_2 M_2$  is uniquely representable. The proof for  $M_1 \oplus_1 M_2$  is similar (and easier).

Let  $A_1$  and  $A_2$  be two  $\mathbb{P}$ -representations of  $M_1 \oplus_2 M_2$ . Let  $X$  be a basis of  $M_1 \oplus_2 M_2$ , and let  $Y = E(M_1 \oplus_2 M_2) - X$ . By pivoting, we can assume that  $A_1$  and  $A_2$  are  $X \times Y$  matrices. Thus  $(A_1)_{xy}$  is nonzero if and only if  $(A_2)_{xy}$  is nonzero. For  $i = 1, 2$ , let  $X_i$  and  $Y_i$  be equal to  $X \cap E(M_i)$  and  $Y \cap E(M_i)$  respectively. It is straightforward to prove (see Lemma 5.2) that, by relabeling as necessary, we can assume that  $A_i[X_2, Y_1]$  is the zero

matrix, and  $A_i[X_1, Y_2]$  has rank one. Therefore the nonzero columns of  $A_i[X_1, Y_2]$  are equal, up to scaling; the same comment applies to the rows.

Let  $y \in Y_2$  be such that  $A_i[X_1, \{y\}]$  is nonzero for  $i = 1, 2$ . (Note that such a  $y$  exists, for otherwise we can reduce to the direct-sum case.) By considering the result of contracting  $X_2$ , it is easy to see that  $A_1[X_1, Y_1 \cup y]$  and  $A_2[X_1, Y_1 \cup y]$  are representations of  $M_1$ . By unique representability, we can apply scalings and automorphisms of  $\mathbb{P}$  to  $A_2$ , and assume that  $A_2[X_1, Y_1 \cup y] = A_1[X_1, Y_1 \cup y]$ . Now, since  $A_2[X_1, \{y\}] = A_1[X_1, \{y\}]$ , and the nonzero columns of  $A_i[X_1, Y_2]$  are parallel to  $A_i[X_1, \{y\}]$ , for  $i = 1, 2$ , we can scale columns of  $A_2$  so that  $A_2[X_1, Y] = A_1[X_1, Y]$ .

Let  $x \in X_1$  be such that  $A_i[\{x\}, Y_2]$  is nonzero for  $i = 1, 2$ . By considering the result of contracting  $X_1 - x$ , we see that  $A_i[X_2 \cup x, Y_2]$  represents  $M_2$ .

**Claim 3.2.1.** *Let  $T$  be a forest of  $G(A_1) = G(A_2)$ , and assume that  $T$  contains all the edges incident with  $x$ . By performing row and column scalings, we can  $T$ -normalize both  $A_1$  and  $A_2$ , without affecting the assumption  $A_2[X_1, Y] = A_1[X_1, Y]$ .*

*Proof.* The proof of the claim is inductive on the number of edges in  $T$ . If  $T$  contains only those edges incident with  $x$ , then we can  $T$ -normalize by multiplying column  $y$  by  $1/(A_1)_{xy} = 1/(A_2)_{xy}$  in both  $A_1$  and  $A_2$ , for every neighbor  $y$  of  $x$ . This proves the base case of the argument.

Suppose that  $T$  contains edges that are not incident with  $x$ . Let  $u$  be a degree-one vertex in  $T$  that is not adjacent to  $x$ , and let  $v$  be the vertex of  $T$  adjacent to  $u$ . By the inductive hypothesis, we can assume that  $A_1$  and  $A_2$  are both  $(T - uv)$ -normalized, and the assumption  $A_2[X_1, Y] = A_1[X_1, Y]$  still holds. If  $u \in X_2$  then we can scale row  $u$  in  $A_i$  by  $1/(A_i)_{uv}$ , for  $i = 1, 2$ . The resulting matrices are  $T$ -normalized, and agree on the submatrices induced by  $X_1$  and  $Y$ . If  $u \in X_1$  then we can multiply row  $u$  in both  $A_1$  and  $A_2$  by  $1/(A_1)_{uv} = 1/(A_2)_{uv}$ , and we see that the claim holds for  $T$ . A similar argument holds if  $u \in Y_1$ . Thus we suppose that  $u \in Y_2$ . Since  $u$  is not adjacent to  $x$ , it follows that  $(A_i)_{xu} = 0$  for  $i = 1, 2$ . Therefore  $A_i[X_1, \{u\}]$  is the zero column, since the nonzero rows of  $A_i[X_1, Y_2]$  are parallel. It follows that we can multiply column  $u$  by  $1/(A_i)_{vu}$  for  $i = 1, 2$  without changing  $A_i[X_1, Y]$ . This completes the proof of the claim.  $\square$

Now we let  $T'$  be a maximal forest of the subgraph of  $G(A_1) = G(A_2)$  induced by  $X_2 \cup Y_2 \cup x$ . Assume that  $T'$  contains all the edges incident with  $x$ . We extend  $T'$  to a maximal forest  $T$ , of  $G(A_1) = G(A_2)$ , where  $T$  also contains all edges incident with  $x$ . By Claim 3.2.1, we can  $T$ -normalize  $A_1$  and  $A_2$  without affecting the assumption that  $A_2[X_1, Y] = A_1[X_1, Y]$ .

Since  $A_1[X_2 \cup x, Y_2]$  and  $A_2[X_2 \cup x, Y_2]$  are  $T'$ -normalized, the hypotheses imply that  $A_2[X_2 \cup x, Y_2] = A_1[X_2 \cup x, Y_2]$ . Now we see that, by pivoting, scaling rows and columns, and possibly applying an automorphism, we have converted  $A_1$  and  $A_2$  into identical matrices. The result follows.  $\square$

**Definition 3.3.** Let  $M$  be a matroid. Then  $M$  is *stable* if it can not be expressed as the direct sum or 2-sum of two nonbinary matroids.

**Lemma 3.4.** *Let  $M$  be a stable near-regular matroid. Then  $M$  is uniquely representable over  $\mathbb{U}_1$ .*

*Proof.* Let  $M$  be a stable near-regular matroid, and suppose that the lemma holds for all smaller matroids. We start by assuming that  $M$  is 3-connected. If  $M$  is binary, then the result follows immediately from Proposition 3.1. Therefore we suppose that  $M$  is nonbinary, and therefore has a  $U_{2,4}$ -minor. Let  $A_1$  and  $A_2$  be  $X \times Y$   $\mathbb{U}_1$ -matrices that represent  $M$ . By pivoting, we can assume that there are 2-element subsets  $X' \subseteq X$  and  $Y' \subseteq Y$ , such that  $A_i[X', Y']$  represents  $U_{2,4}$  for  $i = 1, 2$ . By scaling, we can assume that

$$A_i[X', Y'] = \begin{bmatrix} 1 & 1 \\ p_i & 1 \end{bmatrix}$$

for some  $p_i \in \mathbb{U}_1$ . Since  $\det(A_i[X', Y']) = 1 - p_i$  is defined,  $p_1$  and  $p_2$  are fundamental elements. By Proposition 2.19, we can apply an automorphism of  $\mathbb{U}_1$  to  $A_2$ , and assume that  $A_2[X', Y'] = A_1[X', Y']$ . Now the lemma follows immediately from Corollary 2.25.

Hence we assume that  $M$  is not 3-connected, and can therefore be expressed as a direct sum or a 2-sum of  $M_1$  and  $M_2$ . Since  $M$  is stable, we can assume that  $M_2$  is binary. It is easy to see that  $M_1$  must be stable. Therefore  $M_1$  is uniquely representable over  $\mathbb{U}_1$  by the inductive hypothesis. The result now follows from Propositions 3.1 and 3.2.  $\square$

#### 4. THE SETUP

In this section we collect the results that underlie our proof strategy. An excluded minor  $M$  for near-regularity with more than eight elements has a ‘‘companion’’ matroid  $N$  that is representable over  $\mathbb{Q}(\alpha)$ . Our main objective here is to develop the tools for constructing  $N$ .

Note that if an excluded minor for near-regularity is not ternary, then it is an excluded minor for the class of ternary matroids. Now the following lemmas follow immediately from Reid’s characterization of ternary matroids [Bix79, Sey79], and Proposition 2.5.

**Lemma 4.1.** *Let  $M$  be an excluded minor for the class of near-regular matroids, and assume  $M$  is not isomorphic to  $U_{2,5}$ ,  $U_{3,5}$ ,  $F_7$ , or  $F_7^*$ . Then  $M$  is ternary.*

**Lemma 4.2.** *Let  $M$  be an excluded minor for the class of near-regular matroids. Then  $M$  is 3-connected.*

**Lemma 4.3.** *Let  $M$  be an excluded minor for the class of near-regular matroids. Then  $M^*$  is an excluded minor for the class of near-regular matroids.*

**Definition 4.4.** Suppose that  $M$  is a matroid, and that  $u, v \in E(M)$ . We will say that  $u, v$  is a *deletion pair* if

- (i)  $\{u, v\}$  is coindependent;
- (ii) Each of  $M \setminus u, M \setminus v, M \setminus \{u, v\}$  is stable; and
- (iii)  $M \setminus \{u, v\}$  is connected and nonbinary.

Our definition here is slightly different from that used in [GGK00]. The next result follows from [GGK00, Theorem 3.1].

**Lemma 4.5.** *Let  $M$  be a 3-connected nonbinary matroid such that  $r(M) \geq 4$  or  $r^*(M) \geq 4$ . Then, for some  $M' \in \{M, M^*\}$ , there is a pair of elements  $u, v$  such that  $M' \setminus \{u, v\}$  is connected, and each of  $M' \setminus u, M' \setminus v, M' \setminus \{u, v\}$  is a 0-, 1-, or 2-element coextension of a 3-connected nonbinary matroid. Hence  $u, v$  is a deletion pair for  $M'$ .*

Lemmas 4.6 and 4.9 are analogues of Lemmas 2.2 and 2.3 in [GGK00]. Suppose that  $A$  is a matrix (not necessarily a  $\mathbb{U}_1$ -matrix) over the field  $\mathbb{Q}(\alpha)$ , and that all the entries of  $A$  belong to  $\mathbb{U}_1$ . If  $\psi$  is a homomorphism from  $\mathbb{U}_1$  to some other partial field, then we use  $\psi(A)$  to denote the matrix obtained by applying  $\psi$  to all the entries of  $A$ .

**Lemma 4.6.** *Suppose  $M$  is a matroid, and that  $u, v$  is a deletion pair of  $M$  such that  $M \setminus u$  and  $M \setminus v$  are near-regular. Let  $X$  be a basis of  $M \setminus \{u, v\}$ , and define  $Y := E(M) \setminus X$ . Then there exists an  $X \times Y$  matrix  $A$  over  $\mathbb{Q}(\alpha)$  such that*

- (i)  $M[I|A - u] = M \setminus u$ ;
- (ii)  $M[I|A - v] = M \setminus v$ ; and
- (iii)  $A - u$  and  $A - v$  are near-unimodular.

Moreover,  $A$  is unique up to row and column scaling and applying automorphisms of  $\mathbb{U}_1$ .

*Proof.* Let  $A_1$  be a near-unimodular  $X \times (Y \setminus u)$  matrix representing  $M \setminus u$ . Likewise, let  $A_2$  be a near-unimodular  $X \times (Y \setminus v)$  matrix representing  $M \setminus v$ . If  $u$  is a loop, then it is straightforward to confirm that the matrix obtained from  $A_1$  by adding a zero column satisfies the statements of the lemma. Therefore we assume that  $u$  (and  $v$ , by symmetry) is not a loop. Now  $A_1 - v$  and  $A_2 - u$  are near-unimodular matrices representing  $M \setminus \{u, v\}$ . Since  $M \setminus \{u, v\}$  is stable by the definition of a deletion pair, it follows from Lemma 3.4 that by scaling, and applying automorphisms of  $\mathbb{U}_1$  to  $A_2$ , we can assume that  $A_2 - u = A_1 - v$ . Propositions 2.6 and 2.8 imply that  $A_2$  remains near-unimodular after these operations. Let  $A$  be the matrix obtained from  $A_1$  by adding the column  $A_2[X, \{u\}]$ . Since  $A - u = A_1$  and  $A - v = A_2$  the conditions of the lemma clearly hold.

To prove that  $A$  is unique, we first assume that  $A$  is  $T$ -normalized, for some spanning tree  $T$  of  $G(A)$  that has  $u$  and  $v$  as degree-one vertices. (Such a tree exists because  $M \setminus \{u, v\}$ , and hence  $G(A - \{u, v\})$ , is connected; neither  $u$  nor  $v$  is a loop; and because  $u$  and  $v$  are not adjacent.) Let  $A'$  be some other  $X \times Y$  matrix over  $\mathbb{Q}(\alpha)$  that satisfies the conditions of the lemma. Since  $A - u$  and  $A' - u$  both represent  $M \setminus u$  over  $\mathbb{U}_1$ , and  $M \setminus u$  is

stable, we can, by scaling and applying automorphisms of  $\mathbb{U}_1$  to  $A'$ , assume that  $A' - u = A - u$ . Similarly, as  $A' - v$  and  $A - v$  both represent the stable matroid  $M \setminus v$ , there are nonsingular diagonal matrices  $D_1$  and  $D_2$ , and an automorphism  $\psi$  of  $\mathbb{U}_1$ , such that  $D_1 \psi(A' - v) D_2 = A - v$ .

Let  $xy$  be an edge in  $T - \{u, v\}$ . Then

$$(2) \quad 1 = (A - v)_{xy} = (D_1)_{xx} \psi((A' - v)_{xy}) (D_2)_{yy} \\ = (D_1)_{xx} \psi(1) (D_2)_{yy} = (D_1)_{xx} (D_2)_{yy}.$$

Let  $\gamma = (D_1)_{xx}$ , so that  $(D_2)_{yy} = 1/\gamma$ . Let  $w$  be some vertex in  $T - \{u, v\}$ . It is easy to prove, using Equation (2), and induction on the length of the path in  $T - \{u, v\}$  joining  $w$  to  $x$ , that if  $w \in X$  then  $(D_1)_{ww} = \gamma$ , and if  $w \in Y$  then  $(D_2)_{ww} = 1/\gamma$ . Thus  $A - v = D_1 \psi(A' - v) D_2$  is obtained from  $A' - v$  by applying  $\psi$ , and possibly scaling the column  $u$  by a nonzero constant. Thus  $\psi(A' - \{u, v\}) = A - \{u, v\} = A' - \{u, v\}$ . Since  $A' - \{u, v\}$  represents the nonbinary matroid  $M \setminus \{u, v\}$ , it follows that  $A' - \{u, v\}$  is near-unimodular but not totally unimodular. Proposition 2.20 implies that  $\psi$  is the trivial automorphism. Thus  $A - v$  can be obtained from  $A' - v$  by possibly scaling the column  $u$ . Now, as  $A'[X, v] = A[X, v]$ , it follows that  $A'$  and  $A$  are equal, up to scaling and automorphisms of  $\mathbb{U}_1$ .  $\square$

We will need a few more properties of the matrix appearing in Lemma 4.6. First of all, we need to be able to modify the choice of the basis  $X$ . The straightforward proof of the next result is omitted.

**Lemma 4.7.** *Suppose that  $M$  is a matroid, and that  $u, v$  is a deletion pair of  $M$  such that  $M \setminus u$  and  $M \setminus v$  are near-regular. Let  $X$  be a basis of  $M \setminus \{u, v\}$ , and let  $Y = E(M) \setminus X$ . Let  $A$  be the  $X \times Y$   $\mathbb{Q}(\alpha)$ -matrix such that  $M[I|A - u] = M \setminus u$ ,  $M[I|A - v] = M \setminus v$ , and  $A - u$  and  $A - v$  are near-unimodular. Suppose that  $x \in X$ ,  $y \in Y \setminus \{u, v\}$  and that  $A_{xy} \neq 0$ . Then  $M[I|A^{xy} - u] = M \setminus u$ ,  $M[I|A^{xy} - v] = M \setminus v$ , and  $A^{xy} - u$  and  $A^{xy} - v$  are near-unimodular.*

Consider the function from  $\mathbb{U}_1$  to  $\text{GF}(3)$  which takes 0 to 0, 1 to 1, and  $\alpha$  to  $-1$ . It is not difficult to confirm that this induces a partial-field homomorphism from  $\mathbb{U}_1$  to  $\text{GF}(3)$ . Indeed, if  $\phi : \mathbb{U}_1 \rightarrow \text{GF}(3)$  is a partial-field homomorphism, then  $\phi(0) = 0$  and  $\phi(1) = 1$ , by elementary properties of homomorphisms, and  $\phi(\alpha)$  cannot be equal to 0, as  $\phi(\alpha)$  must have a multiplicative inverse. Nor, for the same reason, can  $\phi(1 - \alpha)$  be equal to 0. Thus  $\phi(\alpha) = -1$ , so there is a unique partial-field homomorphism from  $\mathbb{U}_1$  to  $\text{GF}(3)$ .

**Lemma 4.8.** *Suppose  $M$  is a ternary matroid, and that  $u, v$  is a deletion pair of  $M$  such that  $M \setminus u$  and  $M \setminus v$  are near-regular. Let  $X$  be a basis of  $M \setminus \{u, v\}$ , and let  $Y = E(M) \setminus X$ . Let  $A$  be the  $X \times Y$   $\mathbb{Q}(\alpha)$ -matrix such that  $M[I|A - u] = M \setminus u$ ,  $M[I|A - v] = M \setminus v$ , and  $A - u$  and  $A - v$  are near-unimodular. Let  $\phi$  be the homomorphism from  $\mathbb{U}_1$  to  $\text{GF}(3)$ . Then  $M = M[I|\phi(A)]$ .*

*Proof.* We assume that  $\phi(A)$  is  $T$ -normalized for some maximal forest  $T$  of  $G(\phi(A))$ , where  $u$  and  $v$  are degree-one vertices of  $T$ . Let  $A'$  be an  $X \times Y$   $\text{GF}(3)$ -matrix that represents  $M$ . Then both  $A' - u$  and  $\phi(A) - u$  represent  $M \setminus u$  over  $\text{GF}(3)$  (by Proposition 2.6). Since representations are unique over  $\text{GF}(3)$  ([BL76]), and  $\text{GF}(3)$  has no non-trivial automorphisms, by scaling we can assume that  $A' - u = \phi(A) - u$ . Now there are nonsingular diagonal matrices  $D_1$  and  $D_2$  such that  $D_1(A' - v)D_2 = \phi(A) - v$ . Just as in the proof of Lemma 4.6, we can prove that  $A' - v = \phi(A) - v$ , up to scaling of the column  $u$ . The result follows.  $\square$

**Lemma 4.9.** *Let  $M$  be a matroid, and let  $u, v$  be a deletion pair for  $M$  such that  $M \setminus u$  and  $M \setminus v$  are near-regular. Let  $X$  be a basis of  $M \setminus \{u, v\}$ , and let  $Y = E(M) \setminus X$ . Let  $A$  be the  $X \times Y$   $\mathbb{Q}(\alpha)$ -matrix such that  $M[I|A - u] = M \setminus u$ ,  $M[I|A - v] = M \setminus v$ , and  $A - u$  and  $A - v$  are near-unimodular. Now assume that  $X' \subseteq X$  and  $Y' \subseteq Y - \{u, v\}$  are such that*

- (i)  $u, v$  is a deletion pair for  $M/X' \setminus Y'$ ; and
- (ii)  $M/X' \setminus Y' \neq M[I|A - (X' \cup Y')]$ .

*Then  $M/X' \setminus Y'$  is not near-regular.*

*Proof.* Suppose that  $M/X' \setminus Y'$  is near-regular, and that  $A'$  is an  $(X \setminus X') \times (Y \setminus Y')$   $\mathbb{U}_1$ -matrix that represents  $M/X' \setminus Y'$ . Deleting  $u$  or  $v$  from  $A'$  produces a near-unimodular matrix that represents  $M \setminus u$  or  $M \setminus v$  respectively. But the same statements apply to  $A - (X' \cup Y')$ . The uniqueness guaranteed by Lemma 4.6 means that  $M[I|A'] = M[I|A - (X' \cup Y')]$ , so we have a contradiction to the hypotheses of the lemma.  $\square$

**Lemma 4.10.** *Let  $M$  be an excluded minor for the class of near-regular matroids such that  $M$  is representable over  $\text{GF}(3)$  and  $\text{GF}(4)$ . Then  $M$  is not representable over  $\mathbb{Q}(\alpha)$ .*

*Proof.* Let  $\mathcal{M}$  be the set of matroids representable over  $\text{GF}(3)$ ,  $\text{GF}(4)$ , and  $\mathbb{Q}(\alpha)$ . We claim that this is precisely the class of near-regular matroids. Theorem 1.5 of [Whi97] shows that  $\mathcal{M}$  is exactly the set of matroids representable over both  $\text{GF}(3)$  and  $\text{GF}(q)$ , for some  $q \in \{2, 3, 4, 5, 7, 8\}$ . It cannot be the case that  $q = 2$ , for then  $\mathcal{M}$  would be the set of regular matroids. Since  $\mathcal{M}$  contains  $U_{2,4}$  this is impossible.

Consider the matroid  $\text{AG}(2, 3)$ . It is representable over the field  $\mathbb{F}$  if and only if  $\mathbb{F}$  contains a solution to  $x^2 - x + 1 = 0$  ([Oxl92, p. 515]). Since  $\mathbb{Q}(\alpha)$  contains no such solution, it follows that  $\text{AG}(2, 3)$  is not  $\mathbb{Q}(\alpha)$ -representable, and therefore does not belong to  $\mathcal{M}$ . However,  $\text{AG}(2, 3)$  is representable over  $\text{GF}(3)$ ,  $\text{GF}(4)$ , and  $\text{GF}(7)$  (since  $x = 3$  is a solution to  $x^2 - x + 1 = 0$ ). Thus  $q$  cannot be equal to 3, 4, or 7. We conclude that  $q$  is equal to either 5 or 8. In either case Theorem 2.17 implies that  $\mathcal{M}$  is the class of near-regular matroids, as desired. The result follows immediately.  $\square$



## 5. CONNECTIVITY

Much of this paper consists of recovering connectivity in situations where it seems to have been lost. Our tool for this is the blocking sequence. Suppose that  $M$  is a matroid on the ground set  $E$ . We introduce a similar notation to that used for induced submatrices. Suppose  $E = B \cup Y$  where  $B \cap Y = \emptyset$  and  $B$  is a basis of  $M$ . Let  $Z$  and  $Z'$  be subsets of  $E$ . Then  $M_B[Z] := M/(B \setminus Z)(Y \setminus Z)$ , and  $M_B - Z := M_B[E \setminus Z]$ . Moreover,  $M_B[Z] - Z' = M_B[Z \setminus Z']$ .

**Definition 5.1.** Let  $M$  be a matroid,  $B$  a basis of  $M$ , and suppose that  $X$  and  $Y$  are subsets of  $E(M)$ . Then

$$\lambda_B(X, Y) := r_{M/(B \setminus Y)}(X \setminus B) + r_{M/(B \setminus X)}(Y \setminus B).$$

It is straightforward to verify that this is the same as the function  $\lambda_B(X, Y)$  employed in [GGK00]. Moreover, if  $X$  and  $Y$  are disjoint, then

$$\lambda_B(X, Y) = r_{M_B[X \cup Y]}(X) + r_{M_B[X \cup Y]}(Y) - r(M_B[X \cup Y]),$$

which is the usual connectivity function of  $M_B[X \cup Y]$ . In particular, if  $X$  and  $Y$  partition  $E(M)$ , then  $\lambda_B(X, Y) = r_M(X) + r_M(Y) - r(M)$ . If  $X$  and  $Y$  are disjoint, then we say that  $(X, Y)$  is a  $k$ -separation of  $M_B[X \cup Y]$  if  $|X|, |Y| \geq k$  and  $\lambda_B(X, Y) < k$ .

When  $M$  is representable the following holds:

**Lemma 5.2.** *Suppose  $A$  is an  $(X_1 \cup X_2) \times (Y_1 \cup Y_2)$   $\mathbb{P}$ -matrix (where  $X_1, X_2, Y_1$ , and  $Y_2$  are pairwise disjoint). Let  $M = M[I|A]$ . Then*

$$\lambda_{X_1 \cup X_2}(X_1 \cup Y_1, X_2 \cup Y_2) = \text{rank}(A[X_2, Y_1]) + \text{rank}(A[X_1, Y_2]).$$

Let  $M$  be a matroid on the ground set  $E$ , and let  $B$  be a basis of  $M$ . It is well-known that  $G_B(M)$  is connected if and only if  $M$  is connected. A partition  $(X, Y)$  of  $E$  is a *split* with respect to  $B$  if  $|X|, |Y| \geq 2$  and the edges in  $G_B(M)$  that join vertices in  $X$  to vertices in  $Y$  induce a complete bipartite graph. Note that this bipartite graph need not span all vertices in either  $X$  or  $Y$ .

**Proposition 5.3.** [GGK00, Proposition 4.11]. *Let  $M$  be a matroid, and suppose  $B$  is a basis of  $M$ . If  $(X, Y)$  is a 2-separation of  $M$ , then  $(X, Y)$  is a split with respect to  $B$ .*

Not every split corresponds to a 2-separation:

**Proposition 5.4.** [GGK00, Proposition 4.12]. *Let  $B$  be a basis of the matroid  $M$ , and let  $(X, Y)$  be a split with respect to  $B$ . Suppose  $x_1 y_1$  is an edge of  $G_B(M)$  with  $x_1 \in X$  and  $y_1 \in Y$ . Then  $(X, Y)$  is not a 2-separation of  $M$  if and only if there exist  $x_2 \in X$  and  $y_2 \in Y$  such that  $M_B[\{x_1, y_1, x_2, y_2\}] \cong U_{2,4}$ .*

The following definitions and lemmas are directly from [GGK00, Section 4], and will be presented here without proof. There is some overlap with

results due to Truemper [Tru86], who also gives a very detailed analysis of the structure of the resulting matrices when  $M$  is representable.

**Definition 5.5.** Let  $M$  be a matroid, and let  $B$  be a basis of  $M$ . Suppose that  $(X, Y)$  is an exact  $k$ -separation of  $M_B[X \cup Y]$ . We say that  $(X, Y)$  is *induced* if there exists a  $k$ -separation  $(X', Y')$  of  $M$  with  $X \subseteq X'$  and  $Y \subseteq Y'$ .

**Definition 5.6.** Suppose that  $M$  is a matroid, and that  $B$  is a basis of  $M$ . Let  $(X, Y)$  be a  $k$ -separation of  $M_B[X \cup Y]$ . A *blocking sequence* for  $(X, Y)$  is a sequence of elements  $v_1, \dots, v_p$  of  $E(M) \setminus (X \cup Y)$  such that

- (i)  $\lambda_B(X, Y \cup v_1) = k$ ;
- (ii)  $\lambda_B(X \cup v_i, Y \cup v_{i+1}) = k$  for  $i = 1, \dots, p-1$ ;
- (iii)  $\lambda_B(X \cup v_p, Y) = k$ ; and
- (iv) No proper subsequence of  $v_1, \dots, v_p$  satisfies the first three properties.

The following proposition shows how useful blocking sequences are:

**Proposition 5.7.** [GGK00, Theorem 4.14]. *Let  $M$  be a matroid, and suppose that  $B$  is a basis of  $M$ . Let  $(X, Y)$  be an exact  $k$ -separation of  $M_B[X \cup Y]$ . Exactly one of the following holds:*

- (i) *There exists a blocking sequence for  $(X, Y)$ .*
- (ii)  *$(X, Y)$  is induced.*

The first of the following propositions lists basic properties of blocking sequences; the next provides a means of shortening a given sequence.

**Proposition 5.8.** [GGK00, Proposition 4.15 (i, ii, iv)]. *Suppose that  $M$  is a matroid on the ground set  $E$ , and that  $B$  is a basis of  $M$ . Let  $(X, Y)$  be an exact  $k$ -separation in  $M_B[X \cup Y]$ , and suppose that  $v_1, \dots, v_p$  is a blocking sequence for  $(X, Y)$ . Then the following hold:*

- (i) *For  $1 \leq i \leq j \leq p$ ,  $v_i, \dots, v_j$  is a blocking sequence for the exact  $k$ -separation  $(X \cup \{v_1, \dots, v_{i-1}\}, Y \cup \{v_{j+1}, \dots, v_p\})$  of  $M_B[X \cup Y \cup \{v_1, \dots, v_{i-1}, v_{j+1}, \dots, v_p\}]$ .*
- (ii) *Let  $x_1, x_2 \in X \cup Y$  be such that  $x_1x_2$  is an edge of  $G_B(M)$ . Then  $v_1, \dots, v_p$  is a blocking sequence for the exact  $k$ -separation  $(X, Y)$  of  $M_{B \Delta \{x_1, x_2\}}[X \cup Y]$ .*
- (iii) *For  $i = 1, \dots, p-1$ ,  $v_i \in B$  implies  $v_{i+1} \in E \setminus B$ , and  $v_i \in E \setminus B$  implies  $v_{i+1} \in B$ .*

**Proposition 5.9.** [GGK00, Proposition 4.16]. *Let  $M$  be a matroid. Suppose that  $B$  is a basis of  $M$ , and that  $(X, Y)$  is an exact  $k$ -separation in  $M_B[X \cup Y]$ . Let  $v_1, \dots, v_p$  be a blocking sequence for  $(X, Y)$ . Then the following hold:*

- (i) *Suppose that  $Y' \subseteq Y$  contains at least  $k$  elements and that  $\lambda_B(X, Y') = k-1$ . If  $p > 1$ , then  $v_1, \dots, v_{p-1}$  is a blocking sequence for the exact  $k$ -separation  $(X, Y' \cup v_p)$  of  $M_B[X \cup Y' \cup v_p]$ .*

- (ii) Let  $y \in Y$  be such that  $v_p y$  is an edge of  $G_B(M)$ , and  $\lambda_B(X \cup y, Y) = k$ . If  $p > 1$ , then  $v_1, \dots, v_{p-1}$  is a blocking sequence for the exact  $k$ -separation  $(X, (Y \cup v_p) \setminus y)$  of  $M_{B \Delta \{v_p, y\}}[(X \cup Y \cup v_p) \setminus y]$ .
- (iii) If  $v_i$  has no neighbors in  $X \cup Y$  in  $G_B(M)$ , then  $1 < i < p$ ;  $v_{i-1} v_i$  is an edge of  $G_B(M)$ ; and  $v_1, \dots, v_{i-2}, v_{i+1}, \dots, v_p$  is a blocking sequence for the exact  $k$ -separation  $(X, Y)$  of  $M_{B \Delta \{v_{i-1}, v_i\}}[X \cup Y]$ .

For 2-separations more can be said. If  $(X_1, Y_1)$  and  $(X_2, Y_2)$  are both partitions of a set, then these partitions *cross* if  $X_i \cap Y_j \neq \emptyset$  whenever  $i, j \in \{1, 2\}$ .

**Definition 5.10.** Let  $M$  be a matroid, and suppose that  $(X_1, Y_1)$  is a 2-separation of  $M$ . We say  $(X_1, Y_1)$  is *crossed* if there exists a 2-separation  $(X_2, Y_2)$  of  $M$  such that  $(X_1, Y_1)$  and  $(X_2, Y_2)$  cross. Otherwise  $(X_1, Y_1)$  is *uncrossed*.

**Proposition 5.11.** [GGK00, Proposition 4.17]. *Let  $B$  be a basis of the matroid  $M$ . Suppose that  $(X_1, X_2)$  is an uncrossed 2-separation of  $M_B[X_1 \cup X_2]$ , and let  $v_1, \dots, v_p$  be a blocking sequence for  $(X_1, X_2)$ . Let  $(Y_1, Y_2)$  be a 2-separation of  $M_B[X_1 \cup X_2 \cup \{v_1, \dots, v_p\}]$ . Then, for some  $i, j \in \{1, 2\}$ ,  $X_i \cup \{v_1, \dots, v_p\} \subseteq Y_j$ .*

**Proposition 5.12.** [GGK00, Proposition 4.18]. *Let  $M$  be a matroid, and let  $B$  be a basis of  $M$ . Suppose that  $(X_1, X_2)$  is an uncrossed 2-separation in  $M_B[X_1 \cup X_2]$ , and let  $v \in E(M) \setminus (X_1 \cup X_2)$  be such that  $\lambda_B(X_1 \cup v, X_2) = 2$ . If  $(Y_1, Y_2)$  is a 2-separation of  $M_B[X_1 \cup X_2 \cup v]$  such that  $X_2 \subseteq Y_2$ , then  $v \in Y_2$ .*

**Proposition 5.13.** [GGK00, Corollary 4.19]. *Suppose  $B$  is a basis of the matroid  $M$ . If  $(X_1, X_2)$  is the unique 2-separation in  $M_B[X_1 \cup X_2]$ , and  $v_1, \dots, v_p$  is a blocking sequence for  $(X_1, X_2)$ , then  $M_B[X_1 \cup X_2 \cup \{v_1, \dots, v_p\}]$  is 3-connected.*

## 6. THE REDUCTION

This section contains the core of the proof of Theorem 1.2. We reduce the proof to a finite case-analysis by showing that any excluded minor for the class of near-regular matroids has at most eight elements. This part of the proof follows the arguments in [GGK00] very closely. Deviations necessarily occur when the nature of  $\text{GF}(4)$  comes into play. This happens in the case  $k = 0$  of Claim 6.1.16 (which is (15) in [GGK00]) and from Claim 6.1.21 (which is (20) in [GGK00]) to the end. All other differences are largely cosmetic: for example, rather than work with the bipartite graphs associated with matrices, we choose to work with the matrices themselves.

We denote the simplification or cosimplification of a matroid  $M$  by  $\text{si}(M)$  or  $\text{co}(M)$ . Suppose that the matroid  $M$  has  $E$  as its ground set and  $\mathcal{B}$  as its set of bases. Let  $B$  be a basis of  $M$ , and suppose that  $x \in E$ . Then  $\text{nigh}_B(x)$  denotes the set of vertices of  $G_B(M)$  that are adjacent to  $x$ . Thus

$$\text{nigh}_B(x) = \{y \in E \mid B \Delta \{x, y\} \in \mathcal{B}\}.$$

**Theorem 6.1.** *Let  $M$  be an excluded minor for the class of near-regular matroids other than  $\text{AG}(2,3)\setminus e$  or  $(\text{AG}(2,3)\setminus e)^*$ . Then  $r(M) \leq 4$  and  $r^*(M) \leq 4$ .*

*Proof.* Suppose the theorem is false. Let  $M$  be an excluded minor for the class of near-regular matroids on the ground set  $E$ , such that  $r(M) > 4$  or  $r^*(M) > 4$ , and suppose that  $M$  is isomorphic to neither  $\text{AG}(2,3)\setminus e$  nor  $(\text{AG}(2,3)\setminus e)^*$ . Lemmas 4.1 and 4.2 imply that  $M$  is ternary and 3-connected. If  $M$  is not  $\text{GF}(4)$ -representable, then it is an excluded minor for  $\text{GF}(4)$ -representability. But none of the matroids in Theorem 1.1 is a counterexample to Theorem 6.1, so this is a contradiction. Thus  $M$  is also  $\text{GF}(4)$ -representable.

Lemma 4.5 says that for some  $M' \in \{M, M^*\}$ , there is a deletion pair  $u, v$  of  $M'$ , and that  $M' \setminus \{u, v\}$  contains a 3-connected nonbinary minor of size at least  $|E| - 4$ .

**Assumption 6.1.1.**  *$M', u$ , and  $v$  have been chosen so that  $|\text{co}(M' \setminus \{u, v\})|$  is as large as possible.*

Lemma 4.3 implies that  $M$  is a counterexample to the theorem if and only if  $M^*$  is, so henceforth we relabel  $M'$  with  $M$ .

Since  $\{u, v\}$  is coindependent, there is a basis  $B$  of  $M$  that contains neither  $u$  nor  $v$ . Define  $Y := E \setminus B$ . Lemma 4.6 supplies a  $B \times Y$   $\mathbb{Q}(\alpha)$ -matrix  $A$  with entries in  $\mathbb{U}_1$  such that  $M[I \mid A - u] = M \setminus u$ ,  $M[I \mid A - v] = M \setminus v$ , and both  $A - u$  and  $A - v$  are near-unimodular. Let  $N$  be the matroid represented over  $\mathbb{Q}(\alpha)$  by  $A$ . Thus  $N \setminus u = M \setminus u$  and  $N \setminus v = M \setminus v$ . Since  $M$  is representable over both  $\text{GF}(3)$  and  $\text{GF}(4)$ , Lemma 4.10 implies that  $M$  is not  $\mathbb{Q}(\alpha)$ -representable. Hence  $M \neq N$ . There is a set  $B'$  that is a basis in exactly one of  $M$  and  $N$ , and such a basis must contain  $\{u, v\}$ . By extending  $B' \setminus \{u, v\}$  to a basis of  $M \setminus \{u, v\}$  we see that the following claim holds.

**Claim 6.1.2.** *Let  $B'$  be a set that is a basis in exactly one of  $M$  and  $N$ . There is a basis  $B''$  of  $M \setminus \{u, v\} = N \setminus \{u, v\}$  such that  $B' \setminus B'' = \{u, v\}$ .*

Let  $B'$  and  $B''$  be as in Claim 6.1.2. By Lemma 4.7, we can pivot, and assume that  $A$  is a  $B'' \times (E \setminus B'')$  matrix. Henceforth we relabel  $B''$  with  $B$  and  $E \setminus B'$  with  $Y$ . Note that, although  $M \neq M[I \mid A]$ , the fact that  $M \setminus u$  and  $M \setminus v$  are represented by  $A - u$  and  $A - v$  respectively means that  $G_B(M) = G(A)$ .

If  $B_1$  is a basis of  $M \setminus \{u, v\} = N \setminus \{u, v\}$ , and  $B_2$  is a basis of exactly one of  $M$  and  $N$ , then we say that  $B_1 \triangle B_2$  is a *distinguishing set* with respect to  $B_1$ . Define  $\{a, b\} := B \setminus B'$ . Then  $\{a, b, u, v\}$  is a distinguishing set with respect to  $B$ .

**Claim 6.1.3.**  *$B'$  is a basis of  $N$ .*

*Proof.* Suppose that the claim is false. Then the determinant of  $A[\{a, b, u, v\}]$ , evaluated over  $\mathbb{Q}(\alpha)$ , is equal to zero. Let  $\phi$  be the unique

homomorphism from  $\mathbb{U}_1$  to  $\text{GF}(3)$ . Proposition 2.6 implies that the determinant of  $\phi(A)[\{a, b, u, v\}]$ , evaluated over  $\text{GF}(3)$ , is also zero. Thus  $B'$  is not a basis of  $M[I|\phi(A)]$ . But Lemma 4.8 says that  $M[I|\phi(A)]$  is  $M$ , so we have a contradiction.  $\square$

**Claim 6.1.4.**  $G(A[\{a, b, u, v\}])$  is a cycle.

*Proof.* Suppose that the claim fails. Then there is some zero entry in  $A[\{a, b, u, v\}]$ , and hence in  $\phi(A)[\{a, b, u, v\}]$ . Since  $B'$  is not a basis of  $M$ , the determinant of  $\phi(A)[\{a, b, u, v\}]$  evaluated over  $\text{GF}(3)$  must be zero. This implies that  $\phi(A)[\{a, b, u, v\}]$  must contain a zero row or column. However, as  $\phi$  takes no nonzero element to zero, this implies that  $A[\{a, b, u, v\}]$  has a zero row or column, which is a contradiction as  $B'$  is a basis of  $N$ .  $\square$

The remainder of the proof consists of refining the choices of  $u, v, B, a$ , and  $b$ , always relabeling as necessary so that  $\{a, b, u, v\}$  remains a distinguishing set. For that, we need to restrict our pivots. A pivot over  $xy$ , where  $x \in B$  and  $y \in Y \setminus \{u, v\}$ , is *allowable* if

- (i)  $x \in \{a, b\}$ ;
- (ii)  $A_{ay} = A_{by} = 0$ ; or
- (iii)  $A_{xu} = A_{xv} = 0$ .

In the first case,  $\{a, b, u, v\} \Delta \{x, y\}$  is a distinguishing set with respect to  $B \Delta \{x, y\}$ . This is obvious, since  $(B \Delta \{x, y\}) \Delta (\{a, b, u, v\} \Delta \{x, y\}) = B \Delta \{a, b, u, v\} = B'$ . Suppose that  $A_{ay} = A_{by} = 0$ . Then the determinant of  $A[\{a, b, u, v, x, y\}]$  evaluated over  $\mathbb{Q}(\alpha)$ , is equal to  $A_{xy}$  times the determinant of  $A[\{a, b, u, v\}]$ , which is nonzero as  $B'$  is a basis of  $N$ . It follows that  $B' \Delta \{x, y\}$  is a basis of  $N$ . On the other hand, the determinant of  $\phi(A)[\{a, b, u, v, x, y\}]$  evaluated over  $\text{GF}(3)$ , is equal to  $\phi(A)_{xy}$  times the determinant of  $\phi(A)[\{a, b, u, v\}]$ , which is zero. Thus  $B' \Delta \{x, y\}$  is not a basis of  $M[I|\phi(A)] = M$ . Therefore

$$(B \Delta \{x, y\}) \Delta (B' \Delta \{x, y\}) = \{a, b, u, v\}$$

is a distinguishing set with respect to the basis  $B \Delta \{x, y\}$ . A similar argument shows that if  $A_{xu} = A_{xv} = 0$ , then  $\{a, b, u, v\}$  is a distinguishing set with respect to  $B \Delta \{x, y\}$ .

Since  $M \setminus \{u, v\} = N \setminus \{u, v\}$  is nonbinary, there is some  $C \subseteq B \cup (Y \setminus \{u, v\})$  such that  $A[C]$  is a twirl, by Lemma 2.21. If  $x$  is a vertex of  $G(A - \{u, v\})$ , then  $d(x, C)$  denotes the length of a shortest (possibly empty) path in  $G(A - \{u, v\})$  that joins  $x$  to a vertex in  $C$ .

**Assumption 6.1.5.** Subject to 6.1.1, we choose  $u, v, B, a, b$ , and  $C$  so that  $(|C|, d(a, C), d(b, C))$  is lexicographically minimal.

**Claim 6.1.6.** If  $x \in E \setminus C$ , then  $|\text{nigh}_B(x) \cap C| \leq 2$ . If  $a \notin C$ , then  $|\text{nigh}_B(a) \cap C| \leq 1$ . If  $b \notin C$  and  $|\text{nigh}_B(b) \cap C| = 2$ , then  $a \in C$ .

*Proof.* Suppose that  $x \in E \setminus C$  and that  $|\text{nigh}_B(x) \cap C| \geq 2$ . Lemma 2.22 implies that we can find a twirl  $C'$  in  $C \cup x$ . If  $|\text{nigh}_B(x) \cap C| \geq 3$ , then

$|C'| < |C|$ , and this contradicts 6.1.5, so  $|\text{nigh}_B(x) \cap C| = 2$  and  $|C'| = |C|$ . Now we suppose that  $x = a$ . Then  $0 = d(a, C') < d(a, C)$ , and we have a contradiction to 6.1.5 that proves the second statement. Finally, if  $x = b$ , then  $d(a, C') \leq d(b, C') = 0$ , and this proves the third statement.  $\square$

**Claim 6.1.7.**  $|C| = 4$ .

*Proof.* Suppose  $|C| \geq 6$ , and let  $x, y \in C$  be such that  $A_{xy} \neq 0$ . A pivot over  $xy$  is not allowable, because otherwise, by Proposition 2.15, a shorter twirl can be found, contradicting 6.1.5. It follows that  $\{a, b\} \cap C = \emptyset$ . Therefore Claim 6.1.6 implies that

$$|\text{nigh}(a) \cap C|, |\text{nigh}(b) \cap C| \leq 1.$$

Hence there is an edge  $xy$  in  $A[C]$  such that neither  $x$  nor  $y$  is adjacent to either  $a$  or  $b$ . Thus the pivot on  $xy$  is allowable, and we have a contradiction that proves the claim.  $\square$

Now we split the proof into three different cases:

- (i)  $a, b \in C$ ;
- (ii)  $a \in C$  and  $b \notin C$ ; and
- (iii)  $a, b \notin C$ .

By using Claim 6.1.6, and by scaling  $A$ , we can assume that in cases (i), (ii), and (iii) (respectively),  $A[C \cup \{a, b, u, v\}]$  is equal to  $A_1$ ,  $A_2$ , or  $A_3$  (respectively), where these matrices are shown in Table 1. Here elements in  $C \setminus \{a, b, u, v\}$  are labeled with elements from  $\{1, 2, 3, 4\}$ . A star marks an unknown entry (possibly equal to zero); entries labelled by  $g$ ,  $q$ , and  $r$  are not equal to 0 or 1. In the remainder of the proof we deal with these cases

$$A_1 = \begin{matrix} & \begin{matrix} 1 & 2 & u & v \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ q & r & 1 & g \end{bmatrix} \end{matrix}$$

$$A_2 = \begin{matrix} & \begin{matrix} 1 & 2 & u & v \end{matrix} \\ \begin{matrix} 3 \\ a \\ b \end{matrix} & \begin{bmatrix} q & 1 & * & * \\ 1 & 1 & 1 & 1 \\ * & * & 1 & g \end{bmatrix} \end{matrix}$$

$$A_3 = \begin{matrix} & \begin{matrix} 3 & 1 & u & v \end{matrix} \\ \begin{matrix} 4 \\ 2 \\ a \\ b \end{matrix} & \begin{bmatrix} q & 1 & * & * \\ 1 & 1 & * & * \\ 0 & * & 1 & 1 \\ * & * & 1 & g \end{bmatrix} \end{matrix}$$

TABLE 1.  $A[C \cup \{a, b, u, v\}]$  is one of these matrices.

one by one. Most of the work will be in the second case, which we will save for last.

**Claim 6.1.8.** *If  $A_{ay} \neq 0$  and  $A_{by} \neq 0$  for some  $y \in Y \setminus \{u, v\}$  then  $A_{by}/A_{ay} \notin \{1, g\}$ .*

*Proof.* Suppose that the claim fails. Then, after pivoting on  $ay$ , and relabeling  $y$  with  $a$ , we see that  $A[\{a, b, u, v\}]$  contains a zero entry. But pivoting on  $ay$  is allowable, so  $\{a, b, u, v\}$  remains a distinguishing set. Now we can deduce a contradiction to Claim 6.1.4.  $\square$

We dispose of the first case very easily.

**Claim 6.1.9.**  *$b \notin C$ .*

*Proof.* Suppose otherwise, so that  $a, b \in C$ , and  $A[\{a, b, 1, 2, u, v\}] = A_1$ . Claim 6.1.8 implies that  $r \notin \{1, g\}$ , and  $r \notin \{0, q\}$  as  $A[\{a, b, 1, 2\}]$  is a twirl. It follows that  $M[I|A[\{a, b, 1, 2, u\}]] \cong U_{2,5}$ , which contradicts the fact that  $M \setminus v$  is ternary.  $\square$

Note that if  $\{u, v\} \subseteq Z \subseteq E$ , then  $\{u, v\}$  is necessarily coindependent in  $M_B[Z]$ , since neither  $u$  nor  $v$  is in  $B$ . Now the following result is an easy consequence of Lemma 4.9:

**Claim 6.1.10.** *Let  $Z \subseteq E$  be such that  $\{u, v\} \subseteq Z$ ,  $M_B[Z] - u$ ,  $M_B[Z] - v$ ,  $M_B[Z] - \{u, v\}$  are stable,  $M_B[Z] - \{u, v\}$  is connected and nonbinary, and  $M_B[Z] \neq N_B[Z]$ . Then  $Z = E$ .*

Now we dispense with the third case:

**Claim 6.1.11.**  *$a \in C$ .*

*Proof.* Suppose this is false. Let  $Z := \{a, b, u, v, 1, 2, 3, 4\}$ , so  $A[Z] = A_3$ . Our first step is to recover some connectivity.

**Claim 6.1.11.1.**  *$A_{a1} \neq 0$ .*

*Proof.* Suppose otherwise. Then  $d(a, C) > 1$ . Since  $M \setminus \{u, v\}$ , and hence  $G(A - \{u, v\})$ , is connected, there is a path from  $a$  to  $C$  in  $G(A - \{u, v\})$ . Let  $x_1, \dots, x_k$  be the internal vertices of a shortest path from  $a$  to  $C$ . Then  $x_k$  has exactly one neighbor in  $C$ , because otherwise Lemma 2.22 implies the existence of a twirl  $A[C]'$ , where  $x_k \in C'$ , and  $C' \subseteq C \cup \{x_k\}$ . Then  $|C'| = 4$ , and  $d(a, C') < d(a, C)$ , contradicting 6.1.5. Let  $x$  be the unique neighbor of  $x_k$  in  $C$ . Let  $y \in C$  be a neighbor of  $x$  and let  $z \in C$  be the other neighbor of  $y$ . Since  $d(b, C) \geq d(a, C) > 1$ , pivoting on  $xy$  is allowable. But after this pivot,  $x_k$  is adjacent to both  $y$  and  $z$ , so we have reduced to a previous case and we can again derive a contradiction.  $\square$

**Claim 6.1.11.2.**  *$A_{a3} = A_{b3} = 0$ .*

*Proof.* We have already assumed that  $A_{a3} = 0$ , by virtue of Claim 6.1.6. The same claim implies that  $|\text{nigh}(b) \cap C| \leq 1$ . Suppose  $A_{b3} \neq 0$ . Then  $A_{b1} = 0$ . In this case  $M_B[Z] - \{u, v\}$  is connected (since  $G(A[Z] - \{u, v\})$  is connected), nonbinary (because it has a whirl-minor), and stable (since it is a 2-element coextension of a whirl). By examining  $G(A[Z] - \{u, 2\})$  and

using Proposition 5.3, it is easy to see that  $M_B[Z] - \{u, 2\}$  is 3-connected. Thus  $M_B[Z] - u$  is stable. A similar argument shows that  $M_B[Z] - v$  is stable. Since  $\{a, b, u, v\}$  is a distinguishing set, Claim 6.1.10 implies that  $E = Z$ , contradicting the assumption that  $r(M) \geq 5$  or  $r^*(M) \geq 5$ .  $\square$

**Claim 6.1.11.3.**  $(\{a, b, 1\}, \{2, 3, 4\})$  is an induced 2-separation of  $M_B - \{u, v\}$ .

*Proof.* Proposition 5.3 and Claim 6.1.11.2 imply that  $(\{a, b, 1\}, \{2, 3, 4\})$  is a 2-separation of  $M_B[Z] - \{u, v\}$ . Suppose that it is not induced. Then there is a blocking sequence  $v_1, \dots, v_p$ . We will assume that, subject to 6.1.1 and 6.1.5,  $u, v, B, a, b$ , and  $C$  have been chosen so that  $p$  is as small as possible.

First suppose that  $v_p$  labels a column of  $A$ . By Definition 5.6,  $(\{a, b, 1, v_p\}, \{2, 3, 4\})$  is not a 2-separation in  $M_B[(Z \setminus \{u, v\}) \cup v_p]$ . In the graph  $G(A[Z] - \{u, v\})$ , 1 is the only vertex in  $\{a, b, 1\}$  that is adjacent to a vertex in  $\{2, 3, 4\}$ . Thus it follows without difficulty from Proposition 5.4 that  $v_p$  is adjacent to either 2 or 4 in  $G(A[(Z \setminus \{u, v\}) \cup v_p])$ . Now, by pivoting on either  $A_{32}$  or  $A_{34}$  (and relabeling), we can assume  $v_p$  is adjacent to both 2 and 4. (Note that this pivot is allowable.) Thus  $(\{a, b, 1, v_p\}, \{2, 3, 4\})$  is a split, but not a 2-separation. It follows from Proposition 5.4 that  $A[\{1, 2, v_p, 4\}]$  is a twirl. We can now replace 3 with  $v_p$ . If  $p = 1$  then  $v_p$  is adjacent to  $a$  or  $b$ , contradicting Claim 6.1.11.2. If  $p > 1$ , then by taking  $Y' = \{2, 4\}$ , we see that Proposition 5.9 (i) implies  $v_1, \dots, v_{p-1}$  is a blocking sequence for  $(\{a, b, 1\}, \{2, v_p, 4\})$ . This contradicts our assumption of minimality, so we are done.

Now suppose  $v_p$  labels a row. Again,  $(\{a, b, 1, v_p\}, \{2, 3, 4\})$  is not a 2-separation in  $M_B[(Z \setminus \{u, v\}) \cup v_p]$ . Hence  $A_{v_p 3} \neq 0$ . Using an allowable pivot if necessary we also have  $A_{v_p 1} \neq 0$ . By Lemma 2.22 either  $A[\{v_p, 1, 2, 3\}]$  or  $A[\{v_p, 1, 3, 4\}]$  is a twirl. By relabeling we may assume the latter holds. We now replace 2 by  $v_p$ . Since  $(\{a, b, 1\}, \{v_p, 2, 3, 4\})$  is a 2-separation,  $p > 1$ . But Proposition 5.9 (i) implies  $v_1, \dots, v_{p-1}$  is a blocking sequence for  $(\{a, b, 1\}, \{v_p, 3, 4\})$ , and we again have a contradiction to minimality.  $\square$

**Claim 6.1.11.4.**  $M_B - \{a, b, u, v\}$  is 3-connected, and 1 is the only neighbor of  $a$  and  $b$  in  $G(A - \{u, v\})$ .

*Proof.* By the previous claim,  $M_B - \{u, v\}$  has a 2-separation  $(Z_1, Z_2)$  with  $\{a, b, 1\} \subseteq Z_1$ . By our choice of  $u$  and  $v$ ,  $M \setminus \{u, v\}$  contains a 3-connected minor on at least  $|E| - 4$  elements. This means that  $Z_1$  is equal to  $\{a, b, 1\}$ . Since  $A[\{1\}, \{2, 4\}]$  is nonzero, it follows from Lemma 5.2 that  $A[\{a, b\}, Y - \{1, u, v\}]$  must be the zero matrix. Thus  $a$  and  $b$  can have no neighbor in  $G(A - \{u, v\})$  other than 1. However,  $M_B - \{u, v\}$  is connected, so both  $a$  and  $b$  are adjacent to 1. Now we see that  $\text{co}(M \setminus \{u, v\}) = M_B - \{u, v, a, b\}$ , so we are done.  $\square$

**Claim 6.1.11.5.**  $A_{2u}$  and  $A_{2v}$  are not both equal to zero. Likewise,  $A_{4u}$  and  $A_{4v}$  are not both equal to zero.



*Proof.* If  $A_{2u} = A_{2v} = 0$ , then a pivot over 12 is allowable. But after performing this pivot, we see that  $|\text{nigh}(a) \cap C| = 2$ , and this contradicts Claim 6.1.6. The same argument shows that either  $A_{4u}$  or  $A_{4v}$  is nonzero.  $\square$

**Claim 6.1.11.6.** *If  $b' \in \{a, b\}$  and  $v' \in \{u, v\}$  then  $M_B - \{b', v'\}$  is 3-connected.*

*Proof.* Without loss of generality, we can assume that  $b' = b$  and  $v' = v$ . It follows from Claim 6.1.11.4 that  $(\{a, 1\}, E \setminus \{a, b, u, v, 1\})$  is the unique 2-separation of  $M_B - \{b, u, v\}$ . Since  $A_{au} \neq 0$ , it follows that  $(\{a, 1\}, E \setminus \{a, b, v, 1\})$  is not a 2-separation in  $M_B - \{b, v\}$ . Suppose now, that  $(\{a, u, 1\}, E \setminus \{a, b, u, v, 1\})$  is a 2-separation in  $M_B - \{b, v\}$ . Since the only neighbors of  $b$  in  $G(A)$  are  $u$ ,  $v$ , and  $1$ , we deduce that  $(\{a, b, u, 1\}, E \setminus \{a, b, u, v, 1\})$  is a 2-separation in  $M_B - v$ . But Claim 6.1.8 implies that  $A[\{a, b, u, 1\}]$  is a twirl. Since  $A[\{1, 2, 3, 4\}]$  is a twirl, this contradicts the fact that  $M \setminus v$  is stable. Thus  $(\{a, u, 1\}, E \setminus \{a, b, u, v, 1\})$  is not a 2-separation of  $M_B - \{b, v\}$ , and it follows that  $u$  is a blocking sequence for the 2-separation  $(\{a, 1\}, E \setminus \{a, b, u, v, 1\})$ . Proposition 5.13 implies that  $M_B - \{b, v\}$  is 3-connected, as desired.  $\square$

**Claim 6.1.11.7.**  *$M/a = N/a$ , and  $M/b = N/b$ .*

*Proof.* Claim 6.1.11.6 says that  $M_B - \{a, u\}$ , and  $M_B - \{a, v\}$  are 3-connected, and therefore stable. Since  $M_B - \{a, b, u, v\}$  is 3-connected by Claim 6.1.11.4, and  $b$  is adjacent to 1 in  $G(A - \{a, u, v\})$ , it follows that  $M_B - \{a, u, v\}$  is connected and stable. It is nonbinary since it contains a whirl-minor. Now, if  $M_B[E \setminus a] \neq N_B[E \setminus a]$ , then Claim 6.1.10 implies that  $E \setminus a = E$ . This contradiction shows that  $M/a = N/a$ . The same argument shows that  $M/b = N/b$ .  $\square$

**Claim 6.1.11.8.**  *$a, b$  is a deletion pair of  $M^*$ , and  $M^* \setminus \{a, b\}$  contains a 3-connected nonbinary minor on at least  $|E| - 4$  elements.*

*Proof.* Certainly  $\{a, b\}$  is independent in  $M$ . Claim 6.1.11.4 implies that  $M/\{a, b\} \setminus \{u, v\}$  is 3-connected. It follows that  $M/\{a, b\}$  is stable. Similarly, Claim 6.1.11.6 shows that  $M/a$  and  $M/b$  are stable. Moreover, Claim 6.1.11.4 asserts that both  $M/a \setminus u$  and  $M/a \setminus v$  are 3-connected. Thus both  $M/\{a, b\} \setminus u$  and  $M/\{a, b\} \setminus v$ , and hence both  $G(A - \{a, b, u\})$  and  $G(A - \{a, b, v\})$ , are connected. This means that  $G(A - \{a, b\})$  is connected, and therefore so is  $M/\{a, b\}$ . Clearly  $M/\{a, b\}$  is nonbinary, for  $A[\{1, 2, 3, 4\}]$  is a twirl. The second part of the claim follows because  $M/\{a, b\} \setminus \{u, v\}$  is 3-connected. This completes the proof.  $\square$

Claim 6.1.11.4 implies that  $\{a, b, 1\}$  is a series class in  $M \setminus \{u, v\}$ , and that  $M/\{a, b\} \setminus \{u, v\}$  is 3-connected. Therefore  $\text{co}(M \setminus \{u, v\}) \cong M/\{a, b\} \setminus \{u, v\}$ , so  $|E(\text{co}(M \setminus \{u, v\}))| = |E| - 4$ . Now 6.1.1 implies that  $|E(\text{si}(M/\{a, b\}))| \leq |E| - 4$ . The fact that  $M/\{a, b\} \setminus \{u, v\}$  is 3-connected implies that  $u$  and  $v$  are either loops or in parallel pairs in  $M/\{a, b\}$ , and that  $|E(\text{si}(M/\{a, b\}))| =$

$|E| - 4$ . Now we choose  $M^*$  instead of  $M$ , and  $a$  and  $b$  instead of  $u$  and  $v$ . The arguments of this paragraph show that 6.1.1 is still satisfied.

If we let  $B_0 = E \setminus B$ , then  $B_0$  is a basis of  $M^*$  that avoids  $\{a, b\}$ , and  $(B_0 \setminus \{u, v\}) \cup \{a, b\} = E \setminus B'$  is a basis of  $N^*$ , but not of  $M^*$ . Now  $A^T$ , the transpose of  $A$ , is a  $B_0 \times (E \setminus B_0)$   $\mathbb{Q}(\alpha)$ -matrix that represents  $N^*$ . Claim 6.1.11.7 shows that  $A^T - a$  and  $A^T - b$  represent  $M^* \setminus a = N^* \setminus a$  and  $M^* \setminus b = N^* \setminus b$  respectively. Moreover,  $G(A^T)$  is equal to  $G(A)$ , so  $A^T[C]$  is a twirl. We substitute  $u$  and  $v$  for  $a$  and  $b$ , and  $B_0$  for  $B$ . The arguments above show that 6.1.5 is still satisfied. Thus we can repeat the arguments of Claim 6.1.11.2 and show that either  $A_{u2}^T = A_{v2}^T = 0$ , or  $A_{u4}^T = A_{v4}^T = 0$ . But this contradicts Claim 6.1.11.5, and completes the proof of Claim 6.1.11.  $\square$

The remainder of the proof deals with the second case, in which  $A[C \cup \{a, b, u, v\}] = A_2$ . Let  $x_0, \dots, x_{k+1}$  be the vertices of a shortest path from  $b$  to  $C$  in  $G(A - \{u, v\})$ , with  $x_0 = b$  and  $x_{k+1} \in C$ .

**Claim 6.1.12.**  $d(b, C) = k + 1$  is odd.

*Proof.* Suppose not. Then  $x_k$  labels a column of  $A$ . Assume first that  $A_{ax_k} \neq 0$ . By pivoting over  $a1$ , if necessary, we may assume that  $A_{3x_k} \neq 0$  as well. But then Lemma 2.22 implies that  $A[\{1, 2, 3, a, x_k\}]$  contains a twirl using  $x_k$ . This contradicts the minimality of  $d(b, C)$  in 6.1.5. Therefore  $A_{ax_k} = 0$ , and hence  $A_{3x_k} \neq 0$ . Let  $Z := \{u, v, a, 1, 2, 3, x_0, \dots, x_k\}$ .

Note that  $G(A[Z] - \{u, v, 1\})$  is a path with  $x_0 = b$  and  $a$  as its end vertices. Since  $A[\{a, b\}, \{u, v\}]$  contains no zero entries, it follows that  $G(A[Z] - \{u, 1\})$  and  $G(A[Z] - \{v, 1\})$  both contain a spanning cycle. Moreover, it is not difficult to see that neither of these graphs contains a split. Proposition 5.3 implies that  $M_B[Z] - \{u, 1\}$  and  $M_B[Z] - \{v, 1\}$  are 3-connected. Hence  $M_B[Z] - u$  and  $M_B[Z] - v$  are both stable. Furthermore,  $M_B[Z] - \{u, v\}$  is clearly connected, and nonbinary, as it contains  $M_B[C]$  as a minor. Since  $G(A[Z] - \{u, v\})$  contains a single cycle, namely  $G(A[C])$ , and  $G(A[Z] - \{u, v, 1\})$  is a path, it follows that by repeatedly simplifying and cosimplifying  $M_B[Z] - \{u, v\}$ , we eventually reduce to a whirl. This implies that  $M_B[Z] - \{u, v\}$  is stable. As  $Z$  contains a distinguishing set, Claim 6.1.10 now implies that  $Z = E$ .

We wish to prove that  $u, 1$  is a deletion pair of  $M$ . Certainly  $\{u, 1\}$  is coindependent in  $M$ . We have already proved that  $M \setminus \{u, 1\}$  is 3-connected. Therefore  $M \setminus \{u, 1\}$ ,  $M \setminus u$ , and  $M \setminus 1$  are all stable. It remains to show that  $M \setminus \{u, 1\}$  is nonbinary. We noted that  $G(A[Z] - \{u, 1\}) = G(A - \{u, 1\})$  contains a spanning cycle. Thus there is an induced cycle  $C'$  in  $G(A - \{u, 1\})$  that contains the edge  $bv$ . We can assume that  $A$  has been scaled in such a way that  $A_e = 1$  for every edge  $e \in C'$  other than  $bv$ . (Note that this is compatible with our assumption that  $A[C \cup \{a, b, u, v\}] = A_2$ .) Now  $A_{bv} = g$  is not equal to one, for  $A[\{a, b, u, v\}]$  has nonzero determinant over  $\mathbb{Q}(\alpha)$ . Suppose that  $g = -1$ . Then  $\phi(A)[\{a, b, u, v\}]$  has nonzero determinant over  $\text{GF}(3)$ . But this implies that  $B'$  is a basis of  $M = M[I\phi(A)]$ , and this

is a contradiction. Therefore  $g$  is neither 1 nor  $-1$ , so  $A[C']$  has nonzero determinant. It follows that  $M_B[C'] = N_B[C']$  is a whirl. Thus  $M \setminus \{u, 1\}$  is nonbinary, as desired.

We have shown that  $u, 1$  is a deletion pair. Moreover,  $M \setminus \{u, 1\}$  is 3-connected, so  $M \setminus \{u, 1\}$  certainly contains a 3-connected nonbinary minor on at least  $|E| - 4$  elements. But  $d(b, C) > 1$ , so  $b$  is a degree-one vertex of  $G(A - \{u, v\})$ , and hence  $M \setminus \{u, v\}$  is not 3-connected. Thus

$$|E(\text{co}(M \setminus \{u, 1\}))| > |E(\text{co}(M \setminus \{u, v\}))|,$$

and we have a contradiction to 6.1.1. This completes the proof of Claim 6.1.12.  $\square$

It follows from Claim 6.1.12 that  $x_k$  labels a row, and hence either  $A_{x_k 1} \neq 0$  or  $A_{x_k 2} \neq 0$ . By pivoting over  $a1$  or  $a2$  as needed, we assume that both are nonzero. If  $k > 2$ , then the pivot over  $x_2 x_3$  is allowable, and such a pivot reduces  $d(b, C)$ , contradicting 6.1.5. Thus  $k \in \{0, 2\}$ . Likewise,  $A[\{a, 1, 2, x_k\}]$  is not a twirl, because otherwise replacing 3 by  $x_k$  would reduce  $d(b, C)$ . It follows that, by scaling, we can assume that  $A[\{a, 1, 2, 3, x_0, \dots, x_k, u, v\}]$  is one of the following matrices:

$$\begin{array}{c} \begin{array}{c} 3 \\ a \\ x_0=b \end{array} \begin{array}{c} 1 \quad 2 \quad u \quad v \\ \left[ \begin{array}{cccc} q & 1 & * & * \\ 1 & 1 & 1 & 1 \\ r & r & 1 & g \end{array} \right], \end{array} \quad \begin{array}{c} 3 \\ a \\ x_2 \\ x_0=b \end{array} \begin{array}{c} 1 \quad 2 \quad x_1 \quad u \quad v \\ \left[ \begin{array}{ccccc} q & 1 & 0 & * & * \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & * & * \\ 0 & 0 & r & 1 & g \end{array} \right]. \end{array} \end{array}$$

**Claim 6.1.13.** *For  $w \in \{u, v\}$ ,  $M_B[\{w, a, 1, 2, 3, x_0, \dots, x_k\}]$  is 3-connected if and only if  $A_{3w} \neq 0$ . Furthermore, if  $A_{3w} = 0$ , then  $(\{w, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  is the unique 2-separation of  $M_B[\{w, a, 1, 2, 3, x_0, \dots, x_k\}]$ .*

*Proof.* Let  $Z := \{w, a, 1, 2, 3, x_0, \dots, x_k\}$ . Clearly  $r \neq 0$ . It is easy to see that if  $G(A[Z \setminus \{2, 3\}])$  contains a split, then  $k = 0$ . Claim 6.1.8 implies that  $r \notin \{1, g\}$ . Now it follows from Propositions 5.3 and 5.4 that  $M_B[Z \setminus \{2, 3\}]$  is 3-connected. Hence  $(\{w, a, x_0, \dots, x_k\}, \{1, 2\})$  is the unique 2-separation in  $M_B[Z] - 3$ . Now  $A[\{a, 1, 2, 3\}]$  is a twirl, so Proposition 5.4 implies that  $(\{3, w, a, x_0, \dots, x_k\}, \{1, 2\})$  is not a 2-separation in  $M_B[Z]$ . Moreover  $(\{w, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  is a 2-separation of  $M_B[Z]$  if and only if  $A_{3w} = 0$ . If  $A_{3w} \neq 0$  then 3 is a blocking sequence for  $(\{w, a, x_0, \dots, x_k\}, \{1, 2\})$ , and Proposition 5.13 implies that  $M_B[Z]$  is 3-connected. If  $A_{3w} = 0$  then it follows without difficulty from Proposition 5.12 that  $(\{w, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  is the unique 2-separation of  $M_B[Z]$ .  $\square$

**Claim 6.1.14.** *We may assume  $A_{3v} \neq 0$ .*

*Proof.* Suppose  $A_{3v} = A_{3u} = 0$  (if  $A_{3u} \neq 0$  then we may swap  $u$  and  $v$ ). Then a pivot over  $3x$  is allowable for all  $x$  such that  $A_{3x} \neq 0$ . Claim 6.1.13

implies that

$$(3) \quad (\{v, a, x_0, \dots, x_k\}, \{1, 2, 3\})$$

is the unique 2-separation in  $M_B[\{v, a, 1, 2, 3, x_0, \dots, x_k\}]$ . If  $k = 0$  then the 2-separation in (3) is not an induced separation of  $M_B - u$ , because  $M \setminus u$  is stable, and  $A[\{a, 1, 2, 3\}]$  and  $A[\{v, a, b, 1\}]$  are twirls (since  $r \notin \{0, 1, g\}$ ). Now suppose that  $k = 2$ , and that the 2-separation in (3) is induced in  $M_B - u$ . Our choice of  $u$  and  $v$  implies that  $M_B - \{u, v\}$  contains a 3-connected nonbinary minor of size at least  $|E| - 4$ . It follows that  $(E - \{u, 1, 2, 3\}, \{1, 2, 3\})$  must be a 2-separation of  $M_B - u$ , and that  $M_B - \{u, v, 1, 3\}$  is 3-connected and nonbinary. But since  $A[\{a, 1, 2, 3\}]$  is a twirl, we now have a contradiction to the fact that  $M \setminus \{u, v\}$  is stable. Thus, in either case, the 2-separation in (3) is not induced in  $M_B - u$ . We let  $v_1, \dots, v_p$  be a blocking sequence, and we suppose that, subject to 6.1.1 and 6.1.5, we have chosen  $u, v, B, a, b$ , and  $C$  such that  $p$  is as small as possible.

First suppose  $v_p$  labels a row. Then  $(\{v, a, x_0, \dots, x_k, v_p\}, \{1, 2, 3\})$  is not a 2-separation in  $M_B[\{v, a, 1, 2, 3, x_0, \dots, x_k, v_p\}]$ , so  $A_{v_p w} \neq 0$  for some  $w \in \{1, 2\}$ . By pivoting over 31 or 32 as needed, we may assume  $A_{v_p w} \neq 0$  for all  $w \in \{1, 2\}$ . Then  $(\{v, a, x_0, \dots, x_k, v_p\}, \{1, 2, 3\})$  is a split in  $G(A[\{v, a, 1, 2, 3, x_0, \dots, x_k, v_p\}])$ . Since it is not a 2-separation, it follows without difficulty from Proposition 5.4 that  $A[\{a, v_p, 1, 2\}]$  is a twirl. If  $p = 1$  then either  $A_{v_p v} \neq 0$  or  $A_{v_p x_1} \neq 0$  (in the case that  $k = 2$ ). If  $A_{v_p v} \neq 0$ , then we can replace 3 with  $v_p$ , and we are done. Therefore we assume that  $A_{v_p v} = 0$  and that  $A_{v_p x_1} \neq 0$ . But then  $d(b, \{a, 1, 2, v_p\}) < d(b, \{a, 1, 2, 3\})$ , contradicting 6.1.5. Therefore  $p > 1$ . Now it follows from Proposition 5.9 (i) that  $v_1, \dots, v_{p-1}$  is a blocking sequence for the 2-separation  $(\{v, a, x_0, \dots, x_k\}, \{1, 2, v_p\})$  of  $M_B[\{v, a, 1, 2, v_p, x_0, \dots, x_k\}]$ . This contradicts our assumption of minimality.

Therefore we assume that  $v_p$  labels a column. It follows that  $A_{3v_p} \neq 0$  and, by pivoting on  $A_{32}$  as necessary,  $A_{av_p} \neq 0$ . Lemma 2.22 implies that  $A[\{a, 1, 3, v_p\}]$  is a twirl (we swap the labels of columns 1 and 2 as necessary). By pivoting over 13 as necessary, we can assume that  $A_{x_k v_p} \neq 0$ . Now consider replacing 2 by  $v_p$ . If  $p > 1$  then Proposition 5.9 (i) again implies that  $v_1, \dots, v_{p-1}$  is a blocking sequence for the 2-separation  $(\{v, a, x_0, \dots, x_k\}, \{1, 3, v_p\})$  of  $M_B[\{v, a, 1, 3, v_p, x_0, \dots, x_k\}]$ , contradicting our assumption of minimality. Therefore  $p = 1$ . Then  $(\{v, a, x_0, \dots, x_k\}, \{1, 2, 3, v_p\})$  is a split in  $G(A[\{v, a, 1, 2, 3, v_p, x_0, \dots, x_k\}])$ , but is not a 2-separation of  $M_B[\{v, a, 1, 2, 3, v_p, x_0, \dots, x_k\}]$ . Therefore Proposition 5.4 implies that  $A[\{a, 1, x_k, v_p\}]$  is a twirl. But  $d(b, \{a, x_k, 1, v_p\}) < d(b, \{a, 1, 2, 3\})$ , and again we have a contradiction to 6.1.5. This completes the proof of the claim.  $\square$

**Claim 6.1.15.**  $A_{3u} = 0$ .

*Proof.* Suppose  $A_{3u} \neq 0$ . Let  $Z := \{u, v, a, x_0, \dots, x_k, 1, 2, 3\}$ . By Claim 6.1.13,  $M_B[Z] - u$  and  $M_B[Z] - v$  are both 3-connected, and therefore

stable. Furthermore,  $M_B[Z] - \{u, v\}$  is certainly nonbinary. By examining  $G(A[Z] - \{u, v\})$ , we see that  $M_B[Z] - \{u, v\}$  is connected and stable, so Claim 6.1.10 implies that  $Z = E$ . Since  $r(M) > 4$  or  $r^*(M) > 4$ , this means that  $k = 2$ . Now  $b$  is in a series pair in  $M_B - \{u, v\}$ , and  $x_1$  is in a parallel pair in  $M_B - \{u, v, b\}$ . We have chosen  $u$  and  $v$  so that  $M_B - \{u, v\}$  has a 3-connected nonbinary minor of size at least  $|E| - 4$ , and this minor must be  $M_B - \{u, v, b, x_1\}$ . But  $\{a, x_2\}$  is a series pair in this matroid, so we have a contradiction.  $\square$

Although the page count suggests otherwise, we are now entering the endgame of the proof: from now on we will deal only with the 2-separation  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  of  $M_B[\{u, a, 1, 2, 3, x_0, \dots, x_k\}]$ . That this is a 2-separation follows from Claims 6.1.13 and 6.1.15. Assume that it is induced in  $M_B - v$ . If  $k = 0$  then this immediately leads to a contradiction, as  $M \setminus v$  is stable, and  $A[\{a, 1, 2, 3\}]$  and  $A[\{u, a, b, 1\}]$  are twirls. Now suppose that  $k = 2$ . There is a 3-connected nonbinary minor of size at least  $|E| - 4$  in  $M_B - \{u, v\}$ . Therefore  $(E - \{v, 1, 2, 3\}, \{1, 2, 3\})$  is a 2-separation of  $M_B - v$ , and  $M_B - \{u, v, 1, 3\}$  is 3-connected and nonbinary. Since  $A[\{a, 1, 2, 3\}]$  is a twirl, this contradicts the fact that  $M \setminus \{u, v\}$  is stable. Thus, in either case,  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  is not induced in  $M_B - u$ . Therefore there exists a blocking sequence  $v_1, \dots, v_p$  in  $M_B - v$ . Assume that, subject to 6.1.1, 6.1.5, 6.1.14, and 6.1.15,  $B, a, b, C, x_1, \dots, x_k$ , and  $v_1, \dots, v_p$  have been chosen so that  $p$  is as small as possible.

**Claim 6.1.16.**  $p \neq 1$ .

*Proof.* Suppose  $p = 1$ , and let  $Z := \{u, v, a, x_0, \dots, x_k, 1, 2, 3, v_1\}$ . Claims 6.1.13 and 6.1.14 imply that  $M_B[Z] - \{u, v_1\}$  is 3-connected, so  $M_B[Z] - u$  is stable. Claims 6.1.13 and 6.1.15 imply that  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  is the unique 2-separation in  $M_B[Z] - \{v, v_1\}$ . Since  $v_1$  is a blocking sequence for this 2-separation, Proposition 5.13 states that  $M_B[Z] - v$  is 3-connected, and therefore stable. It is easy to see that  $M_B[Z] - \{u, v\}$  is connected and nonbinary. Suppose that  $M_B[Z] - \{u, v\}$  is not stable. If  $k = 0$  then the only 2-separation in  $M_B[Z] - \{u, v, v_1\}$  is  $(\{a, x_0\}, \{1, 2, 3\})$ . If  $k = 2$ , then  $(\{1, 2, 3\}, \{a, x_0, x_1, x_2\})$  is a 2-separation. Since  $M_B[Z] - \{u, v\}$  is not stable, and  $M_B[\{a, 1, 2, 3\}]$  is nonbinary, it follows that we can create a 2-separation of  $M_B[Z] - \{u, v\}$  by adding  $v_1$  to the side of one of these separations that does not contain  $\{1, 2, 3\}$ . However,  $u$  is spanned by  $\{a, x_0\}$  (in the case that  $k = 0$ ) or  $\{a, x_0, x_2\}$  (in the case that  $k = 2$ ) in  $M_B[Z] - v$ . It follows that  $M_B[Z] - v$  contains a 2-separation, contradicting our earlier conclusion that it is 3-connected. Therefore  $M_B[Z] - \{u, v\}$  is stable. Now Claim 6.1.10 implies that  $Z = E$ .

Suppose  $k = 0$ . Since  $M$  has either rank or corank at least 5, it follows that  $v_1$  labels a column. Since  $v_1$  is a blocking sequence, neither  $(\{u, a, b, v_1\}, \{1, 2, 3\})$  nor  $(\{u, a, b\}, \{v_1, 1, 2, 3\})$  is a 2-separation of  $M_B - v$ . Now Lemma 5.2 implies that  $\text{rank}(A[\{u, v_1, 3\}]) > 0$  and  $\text{rank}(A[\{a, b, v_1, 1, 2\}]) > 1$ . Hence  $A_{3v_1} \neq 0$  and one of  $A_{av_1}$  and  $A_{bv_1}$





longer 0 or  $-1$ , then  $A'[\{u, a, x_2, 3\}]$  is a twirl, and we are done. Therefore we assume that after this pivot,  $A'_{x_2u}$  is still either 0 or  $-1$ , so  $r \in \{1, -1\}$ . Similarly, we assume that after the pivot,  $A'_{v_1u}$  is still either 0 or  $1/(1-q)$ . This means that  $r$  is either  $q-1$  or  $1-q$ . We deduce that  $q-1$  is equal to either 1 or  $-1$ . But  $q$  is an element of  $\mathbb{U}_1$ , and is therefore not equal to 2. Thus  $q=0$ , which contradicts the fact that  $A[\{a, 1, 2, 3\}]$  is a twirl. This completes the proof of the claim.  $\square$

Now we let  $C'$  be either  $\{u, a, x_2, 3\}$  or  $\{u, v_1, 1, 3\}$ , so that  $A'[C']$  is a twirl.

**Claim 6.1.16.4.**  *$M/b = N/b$  and  $M/2 = N/2$ . Moreover  $b, 2$  is a deletion pair of  $M^*$ ,  $M_{B'} - \{b, 2\}$  contains a 3-connected nonbinary minor of size at least  $|E| - 4$ , and  $|\text{co}(M^* \setminus \{b, 2\})| \geq |\text{co}(M \setminus \{u, v\})|$ .*

*Proof.* Note that  $A'[\{a, 1, 2, 3\}]$  is a twirl by Proposition 2.15. Therefore  $M_{B'} - \{b, u, v\}$  is nonbinary. By examining  $G(A' - \{b, u, v\})$  and applying Propositions 5.3 and 5.4 we see that  $M_{B'} - \{b, u, v\}$  is 3-connected. Therefore  $M_{B'} - \{b, u\}$  and  $M_{B'} - \{b, v\}$  are both stable. If  $M/b \neq N/b$ , then Claim 6.1.10 implies that  $E \setminus b = E$ . Thus  $M/b = N/b$ .

By examining  $G(A - \{u, v, 2\})$ , we see that  $(\{v_1, 2, 3, a, x_2\}, \{b, x_1\})$  is the only 2-separation of  $M_{B'} - \{u, v, 2\}$ . Moreover, since  $s-1 \neq 0$ , both  $u$  and  $v$  are length-one blocking sequences for this 2-separation. It now follows from Proposition 5.13 that  $M_{B'} - \{u, 2\}$  and  $M_{B'} - \{v, 2\}$  are both 3-connected. Therefore  $M_{B'} - \{u, v, 2\}$ ,  $M_{B'} - \{u, 2\}$ , and  $M_{B'} - \{v, 2\}$  are all stable. Moreover  $M_{B'} - \{u, v, 2\}$  is connected and nonbinary. Now it follows from Claim 6.1.10 that  $M/2 = N/2$ .

As  $A'[C']$  is a twirl, it follows without difficulty from Propositions 5.3 and 5.4 that  $M_{B'} - \{b, v, 2\}$  is 3-connected. Hence  $M/\{b, 2\}$  is stable. It is certainly nonbinary and connected. We have noted that  $M_{B'} - \{u, 2\}$  is 3-connected, so  $M/2$  is stable. Note that  $A'[\{u, 1, 2, 3\}]$  is a twirl, as  $q \neq 0$ . Now it follows from Proposition 5.4 that  $M_{B'} - \{b, v\}$  is 3-connected. Thus  $M/b$  is stable. Certainly  $\{b, 2\}$  is independent, so  $b, 2$  is a deletion pair of  $M^*$ .

Since  $M_{B'} - \{b, v, 2\}$  is 3-connected, it follows that  $M_{B'} - \{b, 2\}$  contains a 3-connected nonbinary minor on at least  $|E| - 3$  elements. Either  $M_{B'} - \{b, 2\}$  is 3-connected, or it contains a single parallel pair, and this pair contains  $v$ . In either case  $|\text{co}(M^* \setminus \{b, 2\})| \geq |E| - 3$ . As  $b$  is in a series pair in  $M \setminus \{u, v\}$ , we see that  $|\text{co}(M \setminus \{u, v\})| \leq |E| - 3$ , so we are done.  $\square$

By Lemma 4.6 and Claim 6.1.16.4,  $A'$  is the unique matrix over  $\mathbb{Q}(\alpha)$  such that  $M/b = M[I|A' - b]$  and  $M/2 = M[I|A' - 2]$ . Now  $\{2, b, u, v\}$  distinguishes  $M$  from  $N = M[I|A']$ , so if we replace  $M$  by  $M^*$ ,  $u$  and  $v$  with 2 and  $b$ , replace  $a$  and  $b$  with  $u$  and  $v$ ,  $B$  with  $B'$ , and  $C$  with  $C'$ , then we have not violated 6.1.1. However in  $G(A' - \{b, 2\})$ , the distance between  $v$  and  $C'$  is 1, which is less than  $d(b, C)$ . Thus we have a contradiction to 6.1.5, and this completes the proof of Claim 6.1.16.  $\square$



**Claim 6.1.17.**  $v_p$  labels a row.

*Proof.* Suppose  $v_p$  labels a column. Since  $(\{u, a, x_0, \dots, x_k, v_p\}, \{1, 2, 3\})$  is not a 2-separation in  $M_B[\{u, a, x_0, \dots, x_k, 1, 2, 3, v_p\}]$ , it follows that  $A_{3v_p} \neq 0$ . Claim 6.1.16 says that  $p > 1$ , so the definition of blocking sequences implies that  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3, v_p\})$  is a 2-separation. Then  $\text{rank}(A[\{a, x_0, 1, 2, v_p\}]) = 1$  (in the case that  $k = 0$ ), or  $\text{rank}(A[\{a, x_0, x_2, 1, 2, v_p\}]) = 1$  (if  $k = 2$ ). It follows from this that either  $A_{av_p} = A_{x_kv_p} = 0$ , or both  $A_{av_p}$  and  $A_{x_kv_p}$  are nonzero. Moreover, if  $k = 2$ , then  $A_{bv_p} = 0$ . Suppose that  $A_{av_p}$  and  $A_{x_kv_p}$  are nonzero. Lemma 2.22 and Claim 6.1.14 imply that one of  $A[\{v_p, a, 1, 3\}]$  and  $A[\{v_p, a, 2, 3\}]$  is a twirl. By swapping the labels of columns 1 and 2 as needed, assume  $A[\{v_p, a, 1, 3\}]$  is a twirl. By taking  $Y' = \{1, 3\}$  and applying Proposition 5.9 (i) we see that  $v_1, \dots, v_{p-1}$  is a blocking sequence for  $(\{u, a, x_0, \dots, x_k\}, \{v_p, 1, 3\})$  in  $M_B[\{u, a, x_0, \dots, x_k, 1, 3, v_p\}]$ . Now we can replace 2 with  $v_p$ , and we obtain a contradiction to the minimality of  $p$ .

It follows that  $A_{av_p} = A_{x_kv_p} = 0$ . Since  $A_{bv_p} = 0$  if  $k = 2$ , this means that 3 is the only neighbor of  $v_p$  in  $G(A[\{u, v, a, x_0, \dots, x_k, 1, 2, 3, v_p\}])$ , so  $\{3, v_p\}$  is a parallel pair in  $M_B[\{u, v, a, x_0, \dots, x_k, 1, 2, 3, v_p\}]$ . Therefore  $M_{B\Delta\{3, v_p\}}[\{u, v, a, x_0, \dots, x_k, 1, 2, v_p\}]$  is isomorphic to  $M_B[\{u, v, a, x_0, \dots, x_k, 1, 2, 3\}]$ . It is very easy to verify that

$$\lambda_B(\{u, a, x_0, \dots, x_k, 3\}, \{1, 2, 3\}) = \lambda_B(\{u, a, x_0, \dots, x_k, 3\}, \{1, 2\}) = 2.$$

Proposition 5.9 (ii) implies that  $v_1, \dots, v_{p-1}$  is a blocking sequence for  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, v_p\})$  of  $M_{B\Delta\{3, v_p\}}[\{u, a, x_0, \dots, x_k, 1, 2, v_p\}]$ . By replacing 3 with  $v_p$  we obtain a contradiction to the minimality of  $p$ .  $\square$

**Claim 6.1.18.**  $p \neq 2$ .

*Proof.* Suppose  $p = 2$ . Then  $v_1$  labels a column and  $v_2$  labels a row by Claim 6.1.17. As  $(\{u, a, x_0, \dots, x_2, v_1\}, \{1, 2, 3\})$  is a 2-separation of  $M_B[\{u, a, x_0, \dots, x_2, v_1\}]$  it follows that  $A_{3v_1} = 0$ . On the other hand,  $A[\{a, x_0, x_k\}, \{v_1\}]$  is not the zero matrix. Suppose  $A_{zv_1} \neq 0$  for exactly one  $z \in \{x_0, x_k, a\}$ . Then a pivot over  $zv_1$  is allowable, and  $v_1$  is parallel to  $z$  in  $M_B[\{u, v, a, x_0, \dots, x_k, 1, 2, 3, v_1\}]$ . Therefore

$$\begin{aligned} M_{B\Delta\{z, v_1\}}[\{u, v, a, x_0, \dots, x_k, 1, 2, 3\} \Delta \{z, v_1\}] \\ \cong M_B[\{u, v, a, x_0, \dots, x_k, 1, 2, 3\}]. \end{aligned}$$

Moreover  $\lambda_B(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3, z\}) = 2$ , so Proposition 5.9 (ii), and symmetry, implies that  $v_2$  is a blocking sequence for  $(\{u, a, x_0, \dots, x_k\} \Delta \{z, v_1\}, \{1, 2, 3\})$  in  $M_B[\{u, a, x_0, \dots, x_k, 1, 2, 3\} \Delta \{z, v_1\}]$ . Now we can replace  $z$  with  $v_1$ , and derive a contradiction to the minimality of  $p$ .

Hence  $A_{zv_1} \neq 0$  for at least two elements  $z \in \{a, x_0, x_2\}$ . Suppose  $k = 2$  and  $A_{x_0v_1} \neq 0$ . Since  $d(b, C) = 3$  we have that  $A_{av_1} = 0$  and hence  $A_{x_2v_1} \neq 0$ . We consider replacing  $x_1$  by  $v_1$ . By using symmetry

and Proposition 5.9 (i), with  $Y' = \{u, a, x_0, x_2\}$ , we see that  $v_2$  is a blocking sequence for  $(\{u, a, x_0, v_1, x_2\}, \{1, 2, 3\})$  in  $M_B[\{u, a, x_0, v_1, x_2, 1, 2, 3\}]$ . But this leads to a contradiction, as the minimality of  $p$  is violated. It follows that, for  $k = 0$  and for  $k = 2$ , both  $A_{av_1}$  and  $A_{x_kv_1}$  are nonzero. Since  $(\{u, a, x_0, \dots, x_2\}, \{v_1, 1, 2, 3\})$  is not a 2-separation, it follows that  $\text{rank}(A[\{a, x_k\}, \{v_1, 1, 2\}]) > 1$ , and therefore  $A[\{a, x_k, 1, v_1\}]$  is a twirl. But  $d(b, \{a, x_k, 1, v_1\}) < d(b, C)$ , contradicting 6.1.5.  $\square$

Define  $Z := \{u, v, a, x_0, \dots, x_k, 1, 2, 3, v_{p-1}, v_p\}$ . By Claim 6.1.17,  $v_p$  labels a row, and hence  $v_{p-1}$  labels a column. From the definition of blocking sequence we find that both  $(\{u, a, x_0, \dots, x_k, v_{p-1}\}, \{1, 2, 3\})$  and  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3, v_{p-1}\})$  are 2-separations in  $M_B[Z \setminus \{v, v_p\}]$ . It follows from Lemma 5.2 that  $A_{3v_{p-1}} = 0$ . As  $(\{u, a, x_0, \dots, x_k\}, \{v_p, 1, 2, 3\})$  is a 2-separation, but  $(\{u, a, x_0, \dots, x_k, v_{p-1}\}, \{v_p, 1, 2, 3\})$  is not, it follows that  $A_{v_p v_{p-1}} \neq 0$ . Furthermore, either  $A_{av_{p-1}} = A_{x_kv_{p-1}} = 0$  or both  $A_{av_{p-1}}$  and  $A_{x_kv_{p-1}}$  are nonzero. Suppose  $A_{av_{p-1}} = A_{x_kv_{p-1}} = 0$ . As  $(\{u, a, x_0, \dots, x_k\}, \{v_{p-1}, 1, 2, 3\})$  is a 2-separation, it follows that  $v_p$  is the only neighbor of  $v_{p-1}$  in  $G(A[Z \setminus v_p])$ . Thus  $M_{B\Delta\{v_p, v_{p-1}\}}[Z] - \{v_{p-1}, v_p\}$  is isomorphic to  $M_B[Z] - \{v_{p-1}, v_p\}$ . Proposition 5.9 (iii) implies that  $v_1, \dots, v_{p-2}$  is a blocking sequence for the 2-separation  $(\{u, a, x_0, \dots, x_k\}, \{1, 2, 3\})$  in  $M_{B\Delta\{v_{p-1}, v_p\}}$ , and we have contradicted the minimality of  $p$ . Therefore  $A_{av_{p-1}}$  and  $A_{x_kv_{p-1}}$  are both nonzero.

At least one of  $A_{v_p 1}$  and  $A_{v_p 2}$  needs to be nonzero. Assume, exchanging 1 and 2 if necessary, that  $A_{v_p 1} \neq 0$ . We deduce that, up to scaling,  $A[Z]$  is equal to one of the following matrices:

$$\begin{array}{c} \begin{array}{c} 3 \\ v_p \\ a \\ x_0=b \end{array} \begin{array}{c} 1 \quad 2 \quad v_{p-1} \quad u \quad v \\ \left[ \begin{array}{ccccc} q & 1 & 0 & 0 & s \\ 1 & * & t & 0 & * \\ 1 & 1 & 1 & 1 & 1 \\ r & r & r & 1 & g \end{array} \right] \end{array}, \quad \begin{array}{c} 3 \\ v_p \\ a \\ x_2 \\ x_0=b \end{array} \begin{array}{c} 1 \quad 2 \quad v_{p-1} \quad x_1 \quad u \quad v \\ \left[ \begin{array}{ccccc} q & 1 & 0 & 0 & 0 & s \\ 1 & * & t & 0 & 0 & * \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & * & * \\ 0 & 0 & 0 & r & 1 & g \end{array} \right] \end{array} \end{array}.$$

**Claim 6.1.19.**  $A[\{a, v_{p-1}, v_p, 1\}]$  is not a twirl.

*Proof.* Suppose  $A[\{a, v_{p-1}, v_p, 1\}]$  is a twirl. By applying Proposition 5.9 (i) twice, we see that  $v_1, \dots, v_{p-2}$  is a blocking sequence for the 2-separation  $(\{u, a, x_0, \dots, x_k\}, \{1, v_{p-1}, v_p\})$  of  $M_B[Z \setminus \{v, 2, 3\}]$ . If  $A_{v_p v} \neq 0$  then we can replace 3 with  $v_p$  and 2 with  $v_{p-1}$ , and derive a contradiction to the minimality of  $p$ . Therefore  $A_{v_p v} = 0$ . Now a pivot over  $v_p 1$  is allowable, and by scaling, we can assume that  $A^{v_p 1}[Z] - 2$  is one of the following matrices:

$$\begin{array}{c} \begin{array}{c} 3 \\ 1 \\ a \\ x_0=b \end{array} \begin{array}{c} v_p \quad v_{p-1} \quad u \quad v \\ \left[ \begin{array}{cccc} q & q & 0 & s' \\ 1 & 1 & 0 & * \\ 1 & \beta & 1 & 1 \\ r & r\beta & 1 & g' \end{array} \right] \end{array}, \quad \begin{array}{c} 3 \\ 1 \\ a \\ x_2 \\ x_0=b \end{array} \begin{array}{c} v_p \quad v_{p-1} \quad x_1 \quad u \quad v \\ \left[ \begin{array}{ccccc} q & q & 0 & 0 & s' \\ 1 & 1 & 0 & 0 & * \\ 1 & \beta & 0 & 1 & 1 \\ 1 & \beta & 1 & * & * \\ 0 & 0 & r & 1 & g \end{array} \right] \end{array} \end{array}.$$

Here,  $\beta = (t-1)/t$ ,  $s' = s - qA_{v_p v}$ , and  $g' = g - qA_{v_p v}$ . Our assumption that  $A[\{a, v_{p-1}, v_p, 1\}]$  is a twirl implies  $t$  is a fundamental element of  $\mathbb{U}_1$  other than zero or one, so  $\beta$  is defined, and is equal to neither 0 nor 1. Hence  $A^{v_p 1}[\{a, v_{p-1}, v_p, 3\}]$  is a twirl.

Proposition 5.8 (i) and (ii) implies that  $v_1, \dots, v_{p-1}$  is a blocking sequence for the 2-separation  $(\{u, a, x_0, \dots, x_k\}, \{v_p, 1, 2, 3\})$  in  $M_{B\Delta\{1, v_p\}}[Z \setminus \{v, v_{p-1}\}]$ . Proposition 5.9 (i) implies that  $v_1, \dots, v_{p-2}$  is a blocking sequence for  $(\{u, a, x_0, \dots, x_k\}, \{v_{p-1}, v_p, 3\})$  in  $M_{B\Delta\{v_p, 1\}}[Z] - \{v, 1, 2\}$ . It follows that we can replace  $B$  by  $B\Delta\{v_p, 1\}$  and  $C$  by  $\{a, v_p, v_{p-1}, 3\}$ , which contradicts the minimality of  $p$ .  $\square$

**Claim 6.1.20.**  $A_{v_p 2} = 0$ .

*Proof.* Suppose  $A_{v_p 2} \neq 0$ . As  $(\{u, a, x_0, \dots, x_k, v_p\}, \{1, 2, 3\})$  is not a 2-separation of  $M_B[Z] - \{v, v_{p-1}\}$  it follows that  $A[\{a, v_p, 1, 2\}]$  must be a twirl. Hence either  $A[\{a, v_{p-1}, v_p, 1\}]$  or  $A[\{a, v_{p-1}, v_p, 2\}]$  is a twirl, by Lemma 2.22. But, possibly after exchanging 1 and 2, this contradicts Claim 6.1.19.  $\square$

**Claim 6.1.21.**  $A_{v_p v} \neq 0$ .

*Proof.* Suppose  $A_{v_p v} = 0$ . Then  $v_p 1$  is an allowable pivot, and  $v_p$  is adjacent only to 1 in  $G(A[Z] - v_{p-1})$ . Thus  $M_{B\Delta\{v_p, 1\}}[Z] - \{v_{p-1}, 1\}$  is isomorphic to  $M_B[Z] - \{v_{p-1}, v_p\}$ . Furthermore

$$\lambda_B(\{u, a, x_0, \dots, x_k, 1\}, \{1, 2, 3\}) = \lambda_B(\{u, a, x_0, \dots, x_k, 1\}, \{2, 3\}) = 2,$$

so Proposition 5.9 (ii) then implies that  $v_1, \dots, v_{p-1}$  is a blocking sequence for  $(\{u, a, x_0, \dots, x_k\}, \{v_p, 2, 3\})$  of  $M_{B\Delta\{1, v_p\}}[Z \setminus \{v_{p-1}, 1\}]$ . This contradicts the minimality of  $p$ .  $\square$

Let  $\psi$  be an automorphism of the near-regular partial field. Then both  $A - u$  and  $\psi(A) - u$  represent  $M \setminus u$  over  $\mathbb{U}_1$ . Similarly,  $A - v$  and  $\psi(A) - v$  represent  $M \setminus v$  over  $\mathbb{U}_1$ . Obviously  $\psi(A) - u$  and  $\psi(A) - v$  are both near-unimodular. Thus  $\psi(A)$  satisfies the conditions of Lemma 4.6, so Lemma 4.8 implies that  $M = M[I|\phi(\psi(A))]$ , where  $\phi$  is the homomorphism from  $\mathbb{U}_1$  to  $\text{GF}(3)$ .

Consider  $A[\{a, 3\}, \{1, 2\}]$ . It is a submatrix of the near-unimodular matrix  $A - u$ , and its determinant is  $q - 1$ . We deduce that  $q$  is a fundamental element of  $\mathbb{U}_1$  other than 0 and 1. By Proposition 2.19, and the discussion in the previous paragraph, we can assume that  $q = \alpha$ .

Since  $B' = B\Delta\{a, b, u, v\}$  is a dependent set in  $M = M[I|\phi(A)]$ , the determinant of  $\phi(A)[\{a, b, u, v\}]$  must be zero evaluated over  $\text{GF}(3)$ . It follows that  $\phi(g) = 1$ . Claim 6.1.8 implies that  $B\Delta\{a, b, u, 1\}$  is independent in  $N \setminus v = M \setminus v$ . Hence  $\phi(A)[\{a, b, u, 1\}]$  has a nonzero determinant over  $\text{GF}(3)$ . Since  $r \neq 0$ , and the only element of  $\mathbb{U}_1$  taken to zero by  $\phi$  is zero itself, it follows that  $\phi(r) = -1$ . It follows from Claim 6.1.19 that  $\phi(A)[\{a, v_{p-1}, v_p, 1\}]$  has zero determinant, so  $\phi(t) = 1$ .

Suppose  $k = 0$ . By the preceding discussion we see that

$$\phi(A)[Z] = \begin{array}{c} 3 \\ v_p \\ a \\ x_0=b \end{array} \begin{array}{c} 1 \quad 2 \quad v_{p-1} \quad u \quad v \\ \left[ \begin{array}{ccccc} -1 & 1 & 0 & 0 & s \\ 1 & 0 & 1 & 0 & w \\ 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{array} \right] \end{array},$$

where  $s$  and  $w$  are nonzero. By scaling the column labeled with  $v$  and, if necessary, scaling the column labeled by  $u$ , the rows labeled by  $a$  and  $b$ , and swapping the labels on the last two rows, we can assume that  $s = 1$ . Thus there are two cases to consider, according to whether  $w$  is equal to 1 or  $-1$ .

If  $w = 1$  then  $M_{B\Delta\{v,3\}}[Z] - \{v, 3\} \cong F_7^-$ , which contradicts the fact that  $M$  is  $\text{GF}(4)$ -representable. Similarly, in the case that  $w = -1$ , then it is easy to check that  $M_{B\Delta\{a,u\}}[Z] - \{a, u\} \cong F_7^-$ .

Now we assume  $k = 2$ . Let  $Z' = Z \setminus \{x_1, x_2\}$ , so that

$$\phi(A[Z']) = \begin{array}{c} 3 \\ v_p \\ a \\ x_0=b \end{array} \begin{array}{c} 1 \quad 2 \quad v_{p-1} \quad u \quad v \\ \left[ \begin{array}{ccccc} -1 & 1 & 0 & 0 & s \\ 1 & 0 & 1 & 0 & w \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array}.$$

We consider four cases. If  $(s, w) = (1, 1)$  then  $M_{B\Delta\{v,3\}}[Z'] - \{v, 1\} \cong F_7^-$ . If  $(s, w)$  is equal to  $(1, -1)$  or  $(-1, 1)$  then  $M_{B\Delta\{a,u\}}[Z'] - \{a, u\} \cong F_7^-$ . Finally, if  $(s, w) = (-1, -1)$ , then  $M_{B\Delta\{3,v\}}[Z'] - v \cong \text{AG}(2, 3) \setminus e$ . Thus  $M$  has a minor isomorphic to  $\text{AG}(2, 3) \setminus e$ . But we will show in Proposition 7.3 that  $\text{AG}(2, 3) \setminus e$  is an excluded minor for the class of near-regular matroids. Thus  $M \cong \text{AG}(2, 3) \setminus e$ , which contradicts our assumption, and completes the proof of Theorem 6.1.  $\square$

## 7. CONCLUSION

In this section we complete the proof of the excluded-minor characterization. We start by describing in detail the matroids listed in Theorems 1.1 and 1.2, and proving that they are indeed excluded minors for near-regularity. Theorem 6.1 means that to prove this list is complete, we need only perform a finite case-analysis. That analysis is carried out in the second half of the section.

**7.1. The excluded minors.** The next result follows easily from Proposition 6.5.2 of [Oxl92].

**Proposition 7.1.** *Both  $U_{2,5}$  and  $U_{3,5}$  are excluded minors for the class of near-regular matroids.*

Recall that  $F_7$ , the Fano plane, and  $F_7^-$ , the non-Fano matroid, are the rank-3 matroids shown in Figure 1.



FIGURE 1. The Fano plane, and the non-Fano matroid.

The Fano plane is representable only over fields of characteristic two, and  $F_7^-$  is representable only over fields of characteristic other than two [Oxl92, p. 505]. Moreover, any proper minor of  $F_7$  or  $F_7^-$  is either regular, or a whirl (up to the addition of parallel points). The next result follows immediately.

**Proposition 7.2.** *The matroids  $F_7$ ,  $F_7^-$ , and their duals, are excluded minors for the class of near-regular matroids.*

The affine geometry  $AG(2, 3)$  is produced by deleting a hyperplane from the projective geometry  $PG(2, 3)$ . Figure 2 shows a geometric representation of  $AG(2, 3)$ . Up to isomorphism there is a unique matroid produced by deleting an element from  $AG(2, 3)$ . We denote this matroid by  $AG(2, 3)\setminus e$ . It is not difficult to see that the automorphism group of  $AG(2, 3)\setminus e$  acts transitively upon the triangles of  $AG(2, 3)\setminus e$ . It follows that up to isomorphism there is a unique matroid produced by performing a  $\Delta$ - $Y$  operation on  $AG(2, 3)\setminus e$ . We shall denote this matroid by  $\Delta_T(AG(2, 3)\setminus e)$ . Then  $\Delta_T(AG(2, 3)\setminus e)$  is represented over  $GF(3)$  by the following matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Obviously  $\Delta_T(AG(2, 3)\setminus e)$  is self-dual.

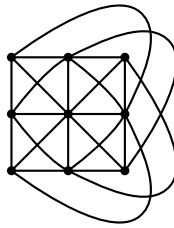


FIGURE 2.  $AG(2, 3)$ .

**Proposition 7.3.** *The matroids  $AG(2, 3)\setminus e$ ,  $(AG(2, 3)\setminus e)^*$ , and  $\Delta_T(AG(2, 3)\setminus e)$  are excluded minors for the class of near-regular matroids.*

*Proof.* Suppose that we obtain a representation of  $\text{AG}(2, 3)$  by deleting from  $\text{PG}(2, 3)$  all points  $[x_1, x_2, x_3]^T$  on the hyperplane defined by  $x_1 + x_2 + x_3 = 0$ . We then obtain a representation of  $\text{AG}(2, 3)\setminus e$  by deleting the point  $[1, 1, -1]^T$ . Thus  $\text{AG}(2, 3)\setminus e$  is represented over  $\text{GF}(3)$  by the following matrix.

$$\begin{bmatrix} 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

If  $\text{AG}(2, 3)\setminus e$  is  $\text{GF}(5)$ -representable, then by normalizing, we can assume that it is represented over  $\text{GF}(5)$  by the following matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c \\ 1 & 1 & 0 & d & e \end{bmatrix}$$

Here,  $\{a, b, c, d, e\}$  are nonzero elements of  $\text{GF}(5)$ . By comparing subdeterminants, we see that  $e = 1$ , and that  $c - a - e = 0$ , so that  $c = a + 1$ . Moreover,  $b = d = c$ , so  $b$  and  $d$  are also equal to  $a + 1$ . Finally  $ad + b - a = 0$ . This means that  $a$  is a root of the polynomial  $x^2 + x + 1$ . But there is no such root in  $\text{GF}(5)$ , so we have a contradiction. Therefore  $\text{AG}(2, 3)\setminus e$  is certainly not near-regular.

The automorphism group of  $\text{AG}(2, 3)$  is transitive on pairs of elements. It follows that the automorphism group of  $\text{AG}(2, 3)\setminus e$  is transitive on points. Using this fact, it is not difficult to see that any single-element deletion of  $\text{AG}(2, 3)\setminus e$  is isomorphic to  $P_7$  (illustrated in Figure 3). Now  $P_7$  is representable over every field of cardinality at least three [Oxl92, Lemma 6.4.13], and is therefore near-regular.

On the other hand, by again using the transitivity of  $\text{AG}(2, 3)\setminus e$  we can see that contracting any element from  $\text{AG}(2, 3)\setminus e$  produces a matroid that is obtained from  $U_{2,4}$  by adding parallel elements. Thus every proper minor of  $\text{AG}(2, 3)\setminus e$  is near-regular, so  $\text{AG}(2, 3)\setminus e$  is indeed an excluded minor for the class of near-regular matroids. It follows immediately that  $(\text{AG}(2, 3)\setminus e)^*$  is an excluded minor for the same class, and Lemma 2.27 implies that  $\Delta_T(\text{AG}(2, 3)\setminus e)$  is also an excluded minor for near-regularity.  $\square$

The matroid  $P_8$  is represented over  $\text{GF}(3)$  by the following matrix:

$$\begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

The matroid  $P_8''$  is obtained from  $P_8$  by relaxing its two circuit-hyperplanes. Lemma 6.4.14 in [Oxl92] says that  $P_8$  is representable over a field if and only if its characteristic is not two. Thus  $P_8$  is not near-regular. However, every single-element deletion or contraction of  $P_8$  is isomorphic to either  $P_7$  or  $P_7^*$  [Oxl92, p. 513], and  $P_7$  is representable over every field containing at least three elements. The next result follows.

**Proposition 7.4.** *The matroid  $P_8$  is an excluded minor for the class of near-regular matroids.*

**7.2. Case-analysis.** Next we show that the list of excluded minors in Theorem 1.2 is complete. The matroids  $P_7$  and  $O_7$  are shown in Figure 3.



FIGURE 3.  $P_7$  and  $O_7$ .

The following matrix represents  $O_7$  over any field  $\mathbb{F}$  such that  $|\mathbb{F}| \geq 3$ . Here,  $\beta \in \mathbb{F} \setminus \{0, 1\}$  if  $\mathbb{F}$  has characteristic equal to two, and  $\beta = -1$  otherwise.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & \beta \end{bmatrix}$$

It follows that  $O_7$  is near-regular. We have already noted that  $P_7$  is near-regular.

**Proposition 7.5.** *Let  $M$  be a 3-connected single-element extension of  $\mathcal{W}^3$ , the rank-3 whirl, such that  $M$  has no  $U_{2,5}$ -minor. Then  $M$  is isomorphic to one of  $F_7^-$ ,  $P_7$ , or  $O_7$ .*

*Proof.* Suppose that  $M \setminus e$  is isomorphic to  $\mathcal{W}^3$ . Let  $E(M \setminus e) = \{r_1, r_2, r_3, s_1, s_2, s_3\}$  and suppose that the triangles of  $M \setminus e$  are  $\{r_1, s_2, s_3\}$ ,  $\{r_2, s_1, s_3\}$ , and  $\{r_3, s_1, s_2\}$ . It is easy to see that if  $e$  is contained in a four-point line of  $M$ , then  $M \cong O_7$ . Thus we assume  $M$  contains no four-point lines. But  $e$  must be contained in a triangle with each of  $r_1$ ,  $r_2$ , and  $r_3$ , for otherwise  $M$  has a  $U_{2,5}$ -minor. Now the result follows easily.  $\square$

**Lemma 7.6.** *Suppose that  $M$  is an excluded minor for the class of near-regular matroids, and that  $r(M) = 3$ . Then  $M$  is isomorphic to one of  $U_{3,5}$ ,  $F_7$ ,  $F_7^-$ , or  $\text{AG}(2, 3) \setminus e$ .*

*Proof.* Suppose that  $M$  is a rank-3 excluded minor other than those listed in the statement of the lemma. Then  $M$  must be ternary, for otherwise it contains  $U_{2,5}$ ,  $U_{3,5}$ , or  $F_7$  as a minor [Bix79, Sey79]. Since  $M$  is not near-regular, and hence not regular, it is nonbinary. Certainly  $M$  has at least six elements, and hence corank at least three, for otherwise the fact that  $M \not\cong U_{3,5}$  means that  $M$  is not 3-connected. Now Corollary 11.2.19 in [Oxl92], and the fact that  $M$  has no  $U_{2,5}$ -minor, means that  $M$  has a  $\mathcal{W}^3$ -minor. Since  $M$  is not isomorphic to  $\text{AG}(2, 3) \setminus e$  or its dual, Theorem 6.1 implies that  $r^*(M) \leq 4$ , and that therefore,  $|E(M)| \leq 7$ . As  $M$  is not isomorphic to  $\mathcal{W}^3$ , it follows that  $M$  is a single-element extension of  $\mathcal{W}^3$ .

Proposition 7.5 implies that  $M$  is isomorphic to either  $P_7$  or  $O_7$ . As these are both near-regular we have a contradiction.  $\square$

Now we complete the proof of Theorem 1.2.

**Theorem 1.2.** *The excluded minors for the class of near-regular matroids are  $U_{2,5}$ ,  $U_{3,5}$ ,  $F_7$ ,  $F_7^*$ ,  $F_7^-$ ,  $(F_7^-)^*$ ,  $\text{AG}(2,3)\setminus e$ ,  $(\text{AG}(2,3)\setminus e)^*$ ,  $\Delta_T(\text{AG}(2,3)\setminus e)$ , and  $P_8$ .*

*Proof.* The results in Section 7.1 certify that the matroids listed in the theorem are indeed excluded minors for near-regularity. Now we suppose that  $M$  is an excluded minor and that  $M$  is not listed in the statement of the theorem. Clearly the rank and corank of  $M$  both exceed two. Lemma 7.6 implies that they both exceed three. It now follows from Theorem 6.1 that both are exactly equal to four, so  $M$  has precisely eight elements.

Suppose that  $M$  contains a triangle  $T$ . As  $M$  is 3-connected,  $T$  is coindependent. Lemmas 2.26 and 2.27 imply that  $\Delta_T(M)$  is an excluded minor for near-regularity with corank three. Now Lemma 7.6 implies that  $\Delta_T(M)$  is either  $U_{2,5}$ ,  $F_7^*$ ,  $(F_7^-)^*$ , or  $(\text{AG}(2,3)\setminus e)^*$ . As  $M$  contains eight elements, we conclude that  $\Delta_T(M) \cong (\text{AG}(2,3)\setminus e)^*$ . But  $T$  is an independent triad in  $\Delta_T(M)$ , by Lemma 2.26, and

$$M = \nabla_T(\Delta_T(M)) \cong \nabla_T((\text{AG}(2,3)\setminus e)^*) = (\Delta_T(\text{AG}(2,3)\setminus e))^* \cong \Delta_T(\text{AG}(2,3)\setminus e).$$

This contradiction means that  $M$  has no triangles. The dual argument shows that  $M$  has no triads.

As in the proof of Lemma 7.6, we see that  $M$  is ternary and nonbinary, and that therefore  $M$  has a  $\mathcal{W}^3$ -minor. Since  $M$  does not have a  $\mathcal{W}^4$ -minor, we may apply the Splitter Theorem. By exploiting duality, we can assume that there are elements  $e, f \in E(M)$ , such that  $M/e$  is 3-connected, and  $M/e\setminus f$  is isomorphic to  $\mathcal{W}^3$ . Therefore  $M/e$  is isomorphic to  $P_7$  or  $O_7$ , by Lemma 7.5.

Assume that  $M/e \cong O_7$ . Since  $M/e$  contains a four-point line, and  $M$  contains no triangles, it follows that  $M$  contains a  $U_{3,5}$ -restriction. This is a contradiction, so we assume that  $M/e \cong P_7$ . By scaling, and uniqueness of representations, we can assume that  $M/e$  is represented over  $\text{GF}(3)$  by the following matrix.

$$\begin{array}{c} x \\ y \\ z \\ e \end{array} \begin{bmatrix} a & b & c & d \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & \alpha & \beta & \gamma \end{bmatrix}$$

The fact that  $M$  contains no triangles means that  $\alpha$  and  $\beta$  are nonzero, and that  $\gamma \neq -1$ . Moreover,  $\alpha + \beta - \gamma \neq 0$ . This leaves us with five cases to check:

- (i)  $\alpha = 1, \beta = 1, \gamma = 0$ ;



- (ii)  $\alpha = 1, \beta = 1, \gamma = 1$ ;
- (iii)  $\alpha = 1, \beta = -1, \gamma = 1$ ;
- (iv)  $\alpha = -1, \beta = 1, \gamma = 1$ ; and
- (v)  $\alpha = -1, \beta = -1, \gamma = 0$ .

If case (i) holds, then we immediately see that  $M$  is isomorphic to  $P_8$ , a contradiction. Suppose that (ii) holds. Then  $M/y \cong F_7^-$ , a contradiction. If (iii) or (iv) holds, then  $M \cong P_8$ . Finally, if (v) holds, then  $M/x \cong F_7^-$ . This contradiction completes the proof.  $\square$

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