EXCLUDED MINORS FOR THE CLASS OF SPLIT MATROIDS

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Abstract. Split matroids form a minor-closed class of matroids, and are defined by placing conditions on the system of split hyperplanes in the matroid base polytope. They can equivalently be defined in terms of structural properties involving cyclic flats. We confirm a conjecture of Joswig and Schröter by proving an excluded-minor characterisation of the class of split matroids.

1. Introduction

The class of split matroids was recently introduced by Joswig and Schröter [5], who successfully deployed them as a tool in tropical linear geometry. The definition arises from natural considerations in the polyhedral view of matroids. Let M be a matroid on the ground set $\{1,\ldots,n\}$. Any subset of $\{1,\ldots,n\}$ is identified with its characteristic vector in $\mathbb{R}^n$. The *matroid polytope*, $P(M)$, is the convex hull of the characteristic vectors of the bases of M. Roughly speaking, a split of a polytope is a division into two polytopes by a hyperplane, called a split hyperplane. If all pairs of split hyperplanes in a matroid polytope satisfy a certain compatibility condition, then the matroid is split. Although the motivation for split matroids arises from tropical linear geometry, natural questions also arise in the area of structural matroid theory, and it is one of these questions that we address here.

First we provide more detail on the polyhedral background. Let X be a finite set of points in $\mathbb{R}^n$. The *convex hull* of X is the intersection of all closed half-spaces that contain X. A *polytope* is the convex hull of a finite set of points. The intersection of two polytopes is also a polytope. If X is empty, then so is its convex hull. Let P be the convex hull of the non-empty finite set X. Let A be the affine subspace of $\mathbb{R}^n$ spanned by P, and let H be any hyperplane of A. Thus $A-H$ is partitioned into two open half-spaces of A. If one of these has an empty intersection with P, and yet $H\cap P$ is non-empty, then $H\cap P$ is a *face* of P. In addition, we declare the empty set and P itself to be faces of P. A face is a *facet* if the only face that properly contains it is P. A *vertex* is a minimal non-empty face. A point in P that is in no face other than P itself is in the *relative interior* of P. Every vertex of P is a point in X, but the converse need not be true, as P may contain points of X in its interior.

The following definition comes from [4]. We let P be a polytope. A *split* of P is a collection, $\mathcal{C}$, of polytopes such that:

(i) the empty polytope is in $\mathcal{C}$,
(ii) if Q is in $\mathcal{C}$, then all the vertices of Q are also vertices of P,
(iii) if Q is in $\mathcal{C}$, so are all the faces of Q,
(iv) the intersection of any two distinct polytopes $Q_1, Q_2 \in \mathcal{C}$ is a face of both Q_1 and Q_2,
(v) $\bigcup_{C\in\mathcal{C}} C = P$, and
(vi) there are exactly two maximal polytopes in $\mathcal{C}$.

The members of $\mathcal{C}$ are called the *cells* of the split. The affine subspace spanned by the intersection of the two maximal cells is called a *split hyperplane*.

Let $\Delta(r,n)$ be the $(n-1)$-dimensional *hypersimplex*: that is, the convex hull of those $0,1$-vectors with exactly r ones. Hence $\Delta(r,n)$ is the polytope of the uniform matroid $U_{r,n}$. Note that the polytope of any rank-r matroid on n elements is contained in $\Delta(r,n)$. Let M be a rank-r matroid with ground set $\{1,\ldots,n\}$. If x is in $\mathbb{R}^n$, then x_i stands for the entry of x indexed by $i \in E(M)$. Edmonds [2] proved that

$$P(M) = \left\{ x \in \Delta(r,n) \,:\, \sum_{i\in F} x_i \leq r(F) \text{ for all flats } F \text{ of } M \right\}.$$

Let F be a flat of M. Then $H(F)$ is the set

$$\left\{ x \in \mathbb{R}^n \,:\, \sum_{i\in F} x_i = r(F) \right\}.$$

If F is minimal under inclusion with respect to $H(F)$ intersecting $P(M)$ in a facet of $P(M)$, then we say that F is a *flacet* of M. If, in addition, $H(F) \cap \Delta(r,n)$ spans a split hyperplane of $\Delta(r,n)$, then we say that F is a *split flacet* of M. In this case, we can think of $H(F)$ as separating $P(M)$ from a portion of $\Delta(r,n)$ that does not intersect $P(M)$. Roughly speaking, the split flacets are the hyperplanes we use when carving off portions of $\Delta(r,n)$ to obtain $P(M)$.

We say that a flat, F, of the matroid, M, is *proper* if $0 < r(F) < r(M)$. A set is *cyclic* if it is a union of circuits or if it is the empty set. The next result is [5, Proposition 1].

Proposition 1.1. *Let F be a flat of the connected matroid M. Then F is a flacet if and only if it is proper, and both $M|F$ and M/F are connected.*

Definition 1.2 ([5])**.** Assume that M is a rank-r matroid with ground set $\{1,\ldots,n\}$. Let A be the affine subspace of $\mathbb{R}^n$ spanned by $P(M)$. We use $[0,1]^n$ to denote the closed unit cube. Assume that the following holds for any distinct split flacets, F_1 and F_2, of M: no point in $H(F_1) \cap H(F_2)$ is in the relative interior of $A \cap [0,1]^n$. Then we say that M is a *split matroid*.

Joswig and Schröter observe that the matroid polytopes of split matroids are exactly those polytopes whose faces of codimension at least two are contained in the boundary of $\Delta(r, n)$. They use the notion of split matroids to resolve some open questions concerning tropical Grassmanians and Dressians. A tropical linear space is a polytopal subdivision of a hypersimplex (or a regular subdivision of a matroid polytope) into matroid polytope cells, and is cryptomorphic to a valuated matroid. Representability of a tropical linear space is thus representability of valuated matroids [1]. Joswig and Schröter use split matroids and the Dressian to construct a number of non-representable tropical linear spaces, and give a characterisation of matroid representability in terms of these spaces. In addition, they prove that the class of split matroids contains the (possibly dominating) class of sparse paving matroids.

The following property is [5, Proposition 44].

Proposition 1.3. *The class of split matroids is closed under duality and under taking minors.*

Therefore we naturally ask what the excluded minors are for the class of split matroids. Joswig and Schröter identify five excluded minors. We show here that their list of excluded minors is complete. Figure 1 shows geometric representations of four connected rank-3 matroids, each with six elements. Note that $S_1^* \cong S_2$, whereas S_3 and S_4 are both self-dual matroids. In addition, we define S_0 to be the matroid constructed from the direct sum $U_{2,3} \oplus U_{2,3}$ by adding one parallel point to each of the two connected components. Thus S_0 is the direct sum of two copies of $M(\mathcal{W}_2)$, where $\mathcal{W}_2$ is the graph obtained by adding a parallel edge to a triangle. We will make use of the fact that $M(\mathcal{W}_2)$ is self-dual.

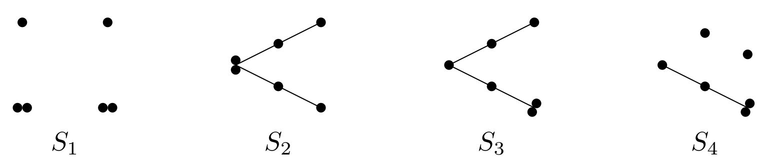

FIGURE 1. Connected excluded minors for split matroids.

Theorem 1.4. *The excluded minors for the class of split matroids are S_0, S_1, S_2, S_3, and S_4.*

To prove Theorem 1.4, we employ Joswig and Schröter's equivalent formulation of Definition 1.2 that relies entirely on structural concepts. The following result (which combines Lemma 10 and Proposition 15 of [5]) gives us a simple characterisation of disconnected split matroids in terms of connected matroids.

Proposition 1.5. *Let $U_1, \ldots, U_t$ be the connected components of the matroid M, where $t > 1$. Then M is a split matroid if and only if each connected matroid, $M|U_i$, is a split matroid, and at most one of these matroids is non-uniform.*

Now we can concentrate on connected split matroids.

Definition 1.6. Let M be a matroid, and let Z be a proper cyclic flat of M. If both $M|Z$ and M/Z are connected matroids, but at least one of them is a non-uniform matroid, then we say that Z is a *certificate* for *non-splitting.*

Lemma 1.7. *Let M be a connected matroid. Then M is split if and only if it has no certificate for non-splitting.*

Proof. This will follow immediately from Theorem 11 in [5] provided that we can demonstrate that the flat Z is a split flacet if and only if it is a proper cyclic flat such that $M|Z$ and M/Z are connected. Let r be the rank of M, and assume that $\{1, \ldots, n\}$ is the ground set of M.

Assume that Z is a proper cyclic flat of M such that $M|Z$ and M/Z are connected. Then Z is a flacet by Proposition 1.1. Furthermore $0 < r(Z) < |Z|$, since Z is a proper flat and is not independent. As Z and $E(M) - Z$ are non-empty, we can find an element in $E(M) - Z$ that is not a coloop (since M is connected). Now Lemma 6 of [5] implies that Z is a split flacet.

For the converse, we let Z be a split flacet. Then Z is a proper flat, and $M|Z$ and M/Z are both connected by Proposition 1.1. Assume $|Z| \leq 1$. Now Proposition 4 of [5] asserts that there must be a positive integer, μ, which satisfies both $\mu < r$ and $\mu > r - |Z| \geq r - 1$. Since this is impossible, $|Z| \geq 2$, so Z is a cyclic flat by Proposition 13 in [5]. We have already observed it is a proper flat, so we are done. □

This allows us to reformulate the definition of split connected matroids. The following is a consequence of Proposition 2.5.

Corollary 1.8. *Let M be a connected matroid. Then M is split if and only if, for every $M' \in \{M, M^*\}$, and every proper cyclic flat, Z, of M', whenever $M'|Z$ is connected, it is uniform.*

Any unexplained matroid terms can be found in [6].

2. Proof of the main theorem

We can easily confirm that S_0 is not split, using Proposition 1.5. It is also easy to check that S_0 is an excluded minor for the class of split matroids. The connected matroids S_1, S_2, S_3, and S_4 all contain certificates for non-splitting. It is routine to verify that they are all excluded minors.

We now show that there is only one disconnected excluded minor. The following result is a consequence of [3, Theorem 4.1].

Proposition 2.1. *Every connected non-uniform matroid M has an $M(\mathcal{W}_2)$-minor.*

Proposition 2.2. *The only disconnected excluded minor for the class of split matroids is S_0.*

Proof. Suppose M is a disconnected excluded minor, so M is not a split matroid, but every proper minor of M is. Let the connected components of M be $U_1, \ldots, U_t$, where $t > 1$. As each $M|U_i$ is a proper minor of M, we see that $M|U_i$ is a split matroid for each i. If at most one component of M is non-uniform, then M is split, which is a contradiction. So let $M|U_i$ and $M|U_j$ be non-uniform, where $1 \leq i < j \leq t$. Now $M|(U_i \cup U_j)$ has two components, U_i and U_j. Both $M|U_i$ and $M|U_j$ are split but non-uniform, so $M|(U_i \cup U_j)$ is not split. Therefore it cannot be a proper minor of M. From this we deduce that $i = 1$ and $j = t = 2$. By Proposition 2.1, each of the two components of M contains $M(\mathcal{W}_2)$ as a minor. Hence M contains a minor isomorphic to $S_0 \cong M(\mathcal{W}_2) \oplus M(\mathcal{W}_2)$. As S_0 is an excluded minor, and no excluded minor can properly contain another, we now see that M is isomorphic to S_0, as desired. □

Lemma 2.3. *Let M be a connected matroid. If M has a proper cyclic flat, Z, such that $M|Z$ is connected and has an $M(\mathcal{W}_2)$-minor, then M has a minor isomorphic to S_2, S_3, or S_4.*

Proof. Let M be a counterexample chosen to be as small as possible. We let Z be a proper cyclic flat of M such that $M|Z$ is connected with an $M(\mathcal{W}_2)$-minor. Amongst all such flats, we assume that we have chosen Z to be as small as possible. Since M is a counterexample, it has no minor isomorphic to S_2, S_3, or S_4.

2.3.1. *If e is any element of Z, then $(M|Z)\backslash e$ has no $M(\mathcal{W}_2)$-minor.*

Proof. We assume otherwise. It is well-known and easy to verify that $Z - e$ is a flat of $M\backslash e$. First we consider the case that $(M|Z)\backslash e = M|(Z - e)$ is connected. Since $M|(Z - e)$ is a connected, non-empty matroid, it contains no coloops. This shows that $Z - e$ is a cyclic flat of $M\backslash e$. Since $M|(Z - e)$ has an $M(\mathcal{W}_2)$-minor, it has rank greater than zero. As e is not a coloop of M, or of $M|Z$, we also have $r_{M\backslash e}(Z - e) = r_M(Z) < r(M) = r(M\backslash e)$. This establishes that $Z - e$ is a proper cyclic flat of $M\backslash e$. Assume that $M\backslash e$ is not connected, and let (U, V) be a separation. Since $M|(Z - e)$ is connected, we can assume that $Z - e$ is a subset of U. As Z is a cyclic flat, e is spanned by $Z - e$ in M. From this it follows that $(U \cup e, V)$ is a separation of M, which is impossible. Therefore $M\backslash e$ is a connected matroid, and $Z - e$ is a proper cyclic flat of $M\backslash e$ such that $(M\backslash e)|(Z - e) = (M|Z)\backslash e$ is connected and has an $M(\mathcal{W}_2)$-minor. We have shown that $M\backslash e$ is a smaller counterexample to the lemma, and from this contradiction we deduce that $(M|Z)\backslash e$ is not connected.

Let $(U_1, \ldots, U_t)$ be the partition of $Z - e$ into connected components of $(M|Z)\backslash e$, where $t > 1$. Thus $(M|Z)\backslash e = (M|U_1) \oplus \cdots \oplus (M|U_t)$. Since $M(\mathcal{W}_2)$ is a connected matroid, we can assume that $M|U_1$ has an $M(\mathcal{W}_2)$-minor [6, Proposition 4.2.20]. As U_1 is a connected component of $(M|Z)\backslash e$ with at least four elements there are no coloops in $M|U_1$. It follows that U_1 is a cyclic flat of $(M|Z)\backslash e$. Assume that U_1 is not a flat of M, and let z be an element in $\mathrm{cl}_M(U_1) - U_1$. Note that $\mathrm{cl}_M(U_1) \subseteq \mathrm{cl}_M(Z) = Z$, so z is in Z. If $z = e$, then $(U_1 \cup e, U_2 \cup \cdots \cup U_t)$ is a separation of the connected matroid $M|Z$, so $z \neq e$. Let C be a circuit containing z such that $C \subseteq U_1 \cup z$. Then C contains elements from both U_1 and $U_2 \cup \cdots \cup U_t$, and as $(U_1, U_2 \cup \cdots \cup U_t)$ is a separation of $(M|Z)\backslash e$, we have a contradiction. Therefore U_1 is a cyclic flat of M. Now $r_M(U_1) \leq r_M(Z) < r(M)$, and obviously $r_M(U_1) > 0$, so U_1 is a proper cyclic flat of M. Moreover $M|U_1$ is connected and has an $M(\mathcal{W}_2)$-minor. But we chose Z to be the smallest possible cyclic flat with these properties, and U_1 does not contain any element of $U_2 \cup \cdots \cup U_t$ so it is strictly smaller than Z. This contradiction completes the proof. □

2.3.2. *If x is an element in the complement of Z, then $M\backslash x$ is not connected.*

Proof. Assume otherwise. Note that $r_{M\backslash x}(Z) = r_M(Z) < r(M) = r(M\backslash x)$, so it is obvious that Z is a proper cyclic flat of $M\backslash x$. Moreover $(M\backslash x)|Z = M|Z$ is connected and has an $M(\mathcal{W}_2)$-minor. This contradicts the minimality of M, so $M\backslash x$ is not connected. □

2.3.3. *The complement of Z is a series pair of M.*

Proof. Choose an arbitrary element, x, in the complement of Z. Using 2.3.2, we let $(U_1, \ldots, U_t)$ be the partition of $E(M) - x$ into connected components of $M\backslash x$, where $t > 1$. As $M|Z$ is connected, we can assume that $Z \subseteq U_1$. Then Z is a cyclic flat of $M|U_1$. If it is a proper cyclic flat of $M|U_1$, then $M|U_1$ is a connected matroid with a proper cyclic flat such that the restriction to this cyclic flat is connected with an $M(\mathcal{W}_2)$-minor. This contradicts the minimality of M, so Z spans U_1. It is straightforward to verify that U_1 is a flat of M, using some of the same arguments as in 2.3.1. Hence $Z = U_1$.

Let y be an element of U_2. Again using 2.3.2, we see that $M\backslash y$ is not connected. Therefore M/y is connected [6, Theorem 4.3.1]. We can easily check that $\mathrm{cl}_{M/y}(Z)$ is a cyclic flat of M/y, and that $(M/y)|(\mathrm{cl}_{M/y}(Z))$ is connected with an $M(\mathcal{W}_2)$-minor. So if $\mathrm{cl}_{M/y}(Z)$ is a proper cyclic flat of M/y, we have contradicted the minimality of M. Therefore Z is not a proper cyclic flat of M/y, meaning that $r(Z) = r(M) - 1$. Hence Z is a hyperplane of M, and its complement is a cocircuit. However,

$$r(M) = r(M\backslash e) = r(U_1) + \cdots + r(U_t) = r(Z) + r(U_2) + \cdots + r(U_t).$$

From this, and the fact that M has no loops, we deduce that $t = 2$, and that $r(U_2) = 1$. Assume that $|U_2| > 1$, and let z be an element in $U_2 - y$. Then $\{y, z\}$ is a parallel pair. But deleting an element from a parallel pair in a connected matroid always produces another connected matroid, so we are led to a violation of 2.3.2. Thus $U_2 = \{y\}$, and we conclude that the complement of Z is the series pair $\{x, y\}$. □

Let $\{x, y\}$ be the complement of Z, so that $\{x, y\}$ is a parallel pair of M^*. By 2.3.1, we see that there is a subset $I \subseteq Z$, such that $(M|Z)/I$ is isomorphic to $M(\mathcal{W}_2)$. Assume I is not independent, and let e be an element contained in a circuit of $M|I$. Then $(M|Z)/I = (M|Z)/(I-e)\backslash e$, so we have a contradiction to 2.3.1. Therefore I is an independent set. Dualising, we see that $(M|Z)^* = (M\backslash\{x,y\})^* = M^*/\{x,y\}$ has a coindependent set, I, such that $M^*/\{x,y\}\backslash I$ is isomorphic to $M(\mathcal{W}_2)$. As I is coindependent, $M^*/\{x,y\}$ has rank two, and hence $r(M^*) = 3$.

We choose elements a, b, c, and d, so that $(M^*/\{x,y\})|\{a,b,c,d\}$ is isomorphic to $M(\mathcal{W}_2)$, where $\{a,b\}$ is a parallel pair in $M^*/\{x,y\}$. Note that $\{a,b,x\}$ has rank two in M^*, that $\{c,d,x\}$ is independent, and that neither c nor d is on the line spanned by $\{a,b,x\}$. We divide into two cases, according to whether $\{a,b\}$ is a parallel pair in M^*.

First assume that $\{a,b\}$ is independent. Note that $M^*|\{a,b,x,y\}$ is isomorphic to $M(\mathcal{W}_2)$. The lines $\mathrm{cl}^*_M(\{c,d\})$ and $\mathrm{cl}^*_M(\{a,b,x,y\})$ intersect in a flat of rank at most one, and this flat cannot contain x. Hence the intersection of $\mathrm{cl}^*_M(\{c,d\})$ and $\{a,b,x,y\}$ is either empty, or it contains a (up to symmetry between a and b). In the first case, the restriction $M^*|\{a,b,c,d,x,y\}$ is isomorphic to S_4, and in the second it is isomorphic to S_3. In these cases, M also has a minor isomorphic to S_3 or S_4. Since this is a contradiction, we assume that $\{a,b\}$ is a parallel pair of M^*.

If $\{a,c,d\}$ is independent, then $M^*|\{a,b,c,d,x,y\}$ is isomorphic to S_1, which implies that M has a minor isomorphic to $S_1^* \cong S_2$. This is a contradiction, so $\{a,c,d\}$ has rank two. Note that the restriction to $\{a,b,c,d\}$ is isomorphic to $M(\mathcal{W}_2)$. As M^* is a connected rank-3 matroid, the complement of the line $\mathrm{cl}^*_M(\{a,b,c,d\})$ has rank at least two. We let z be an element in this complement chosen so that $\{x,z\}$ is independent. The intersection of $\mathrm{cl}^*_M(\{x,y,z\})$ and $\{a,b,c,d\}$ is either $\emptyset$, $\{a,b\}$, or $\{c\}$ (up to symmetry between c and d). In the first case, $M^*|\{a,b,c,d,x,z\}$ is isomorphic to S_4. In the second and third cases, $M^*|\{a,c,d,x,y,z\}$ is isomorphic to S_3. Thus we have a contradiction in any case, and this completes the proof of the lemma. □

Proposition 2.4. *Let Z be a proper cyclic flat of the matroid M. If $E(M)-Z$ is not a proper cyclic flat of M^*, then every element in $E(M)-Z$ is a coloop of M.*

Proof. Let E be the ground set of M. The fact that $E-Z$ is a cyclic flat of M^* is well-known and easy to verify. Suppose it is not proper; that is, $r^*(E-Z) = r(M^*)$ or $r^*(E-Z) = 0$. First, consider the case where $r^*(E-Z) = r(M^*) = |E| - r(M)$. Then the corank function gives

$$|E| - r(M) = r(Z) + |E-Z| - r(M).$$

This implies that $r(Z) = |Z|$, so Z is an independent set in M. The only independent cyclic flat is the empty set, and Z is non-empty since it is a proper flat of M. So if $E-Z$ is not a proper flat, then $r^*(E-Z) = 0$, and this implies that every element in $E-Z$ is a coloop of M. □

Proposition 2.5. *Let M be a connected matroid that is not split. There exists $M' \in \{M, M^*\}$ such that the following holds: M' has a proper cyclic flat, Z, where $M'|Z$ is connected and non-uniform.*

Proof. Let E be the ground set of M. As M is connected and not split, it contains a certificate, Z, for non-splitting, by Lemma 1.7. Thus Z is a proper cyclic flat such that both $M|Z$ and M/Z are connected matroids and either $M|Z$ or M/Z is non-uniform. If $M|Z$ is non-uniform, then we set M' to be M and we are done. So we assume that M/Z is non-uniform. If M contains a coloop, then it is isomorphic to the uniform matroid $U_{1,1}$, and is therefore a split matroid. This is impossible, so M has no coloops. We apply Proposition 2.4 and deduce that $E - Z$ is a proper cyclic flat of M^*. Note that $M^*|(E - Z) = (M/Z)^*$ and $M^*/(E - Z) = (M|Z)^*$. Both of these matroids are connected, and $M^*|(E - Z) = (M/Z)^*$ is non-uniform. Therefore we set M' to be M^* and relabel $E - Z$ as Z. □

We can now easily prove our main result.

Proof of Theorem 1.4. Let M be an excluded minor for the class of split matroids. If M is not connected, then it is isomorphic to S_0 by Proposition 2.2. Therefore we assume that M is connected. By using Proposition 2.5 and duality, we can assume that M has a proper cyclic flat, Z, such that $M|Z$ is connected and non-uniform. Proposition 2.1 implies that $M|Z$ has an $M(\mathcal{W}_2)$-minor. Lemma 2.3, and the fact that no excluded minor properly contains another, implies that M is isomorphic to S_2, S_3, or S_4. (Note that S_1 does not appear in this analysis because of our duality assumption.) □

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