EXCLUDED MINORS FOR THE CLASS OF SPLIT MATROIDS

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Abstract. The class of split matroids arises by putting conditions on the system of split hyperplanes of the matroid base polytope. It can alternatively be defined in terms of structural properties of the matroid. We use this structural description to give an excluded minor characterisation of the class.

1. Introduction

Our aim is to give a excluded-minor characterisation of the class of split matroids, defined by Joswig and Schröter, and motivated by natural considerations from the polyhedral view of matroids. Let M be a matroid on the ground set $\{1, \ldots, n\}$. Any subset of $\{1, \ldots, n\}$ is identified with its characteristic vector in $\mathbb{R}^n$. The *matroid polytope*, $P(M)$, is the convex hull of the characteristic vectors of the bases of M. If M is connected, then the affine dimension of $P(M)$ is $|E(M)| - 1$. Roughly speaking, a split of a polytope is a division into two polytopes by a hyperplane, called a split hyperplane. If all pairs of split hyperplanes in a matroid polytope satisfy a certain compatibility condition, then the matroid is split. We provide more details below.

Let X be a finite set of points in $\mathbb{R}^n$. The *convex hull* of X is the intersection of all closed half-spaces that contain X. A *polytope* is the convex hull of a finite set of points. If X is empty, then so is its convex hull. The intersection of any two polytopes is another polytope. Let P be the convex hull of the finite set X. Let A be the affine subspace of $\mathbb{R}^n$ spanned by P, and let H be a hyperplane of A. Thus $A - H$ is partitioned into two open half-spaces of A. If one of these has an empty intersection with P, and yet $H \cap P$ is non-empty, then $H \cap P$ is a *face* of P. A *facet* is a maximal face, and a *vertex* is a minimal face. A point in P that is in no face is an *interior* point of P. Every vertex of P is a point in X, but the converse need not be true, as P may contain points of X in its interior.

The following definition comes from [3]. We let P be a polytope. A *split* of P is a collection, $\mathcal{C}$, of polytopes such that

(i) the empty polytope is in $\mathcal{C}$,
(ii) if Q is in $\mathcal{C}$, then all the vertices of Q are also vertices of P,
(iii) if Q is in $\mathcal{C}$, so are all the faces of Q,

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(iv) the intersection of any two distinct polytopes $Q_1, Q_2 \in \mathcal{C}$ is a face of both Q_1 and Q_2,
(v) $\bigcup_{C \in \mathcal{C}} C = P$, and
(vi) there are exactly two maximal polytopes in $\mathcal{C}$.

The elements of $\mathcal{C}$ are called the *cells*. The affine subspace spanned by the intersection of the two maximal cells is called a *split hyperplane*.

The base polytope of a matroid can be written in terms of flats, as well as in terms of bases as described earlier. Let $\Delta(r,n)$ be the $(n-1)$-dimensional hypersimplex: that is, the convex hull of those $0,1$-vectors with exactly r ones. Hence $\Delta(r,n)$ is the polytope of the uniform matroid $U_{r,n}$. Note that the polytope of any matroid on n elements is a subpolytope of the hypersimplex. Let M be a rank-r matroid with $E(M) = \{1,\ldots,n\}$. If x is in $\mathbb{R}^n$, then x_i stands for the entry of x indexed by $i \in E(M)$. Edmonds [1] proved that

$$P(M) = \{x \in \Delta(r,n) \mid \sum_{i \in F} x_i \leq r(F) \text{ for all flats } F \text{ of } M\}.$$

Let F be a flat of M. Then the *F-hyperplane*, $H(F)$, is the set

$$\{x \in \mathbb{R}^n \mid \sum_{i \in F} x_i = r(F)\}.$$

Let A be the affine subspace spanned by $P(M)$. Then $H(F) \cap A$ is a hyperplane of A. If F is minimal under inclusion with respect to $H(F)$ intersecting $P(M)$ in a facet of $P(M)$, then we say that F is a *flacet* of M. If, in addition, $H(F) \cap \Delta(r,n)$ spans a split hyperplane of $\Delta(r,n)$, then we say that F is a *split flacet* of M.

Say that two elements in the matroid M are equivalent if they are equal, or if they are contained in a common circuit. Then this is an equivalence relation on $E(M)$, and the equivalence classes are called *connected components*. A matroid is *connected* if it has only one connected component. A matroid is connected if and only if its dual is. The next result is [2, Proposition 2.6].

Proposition 1.1. *Let F be a flat of the connected matroid M. Then F is a flacet of M if and only if both $M|F$ and M/F are connected.*

Definition 1.2 ([4])**.** Assume that M is a rank-r matroid with $E(M) = \{1,\ldots,n\}$. Let A be the affine subspace spanned by $P(M)$. We use $[0,1]^n$ to denote the closed unit cube. Assume that the following holds for any distinct split flacets, F_1 and F_2, of M: no point in $H(F_1) \cap H(F_2)$ is in the interior of $A \cap [0,1]^n$. Then we say that M is a *split matroid*.

Joswig and Schröter prove that every sparse paving matroid is a split matroid, so it is possible that asymptotically every matroid is split. The following property is proved in [4, Proposition 44].

Proposition 1.3. *The class of split matroids is closed under duality and under taking minors.*

Therefore we can reasonably ask what the excluded minors are for the class of split matroids. Joswig and Schröter identify five excluded minors. The main result of this paper shows that that their list of excluded minors is complete. Figure 1 shows geometric representations of four connected rank-3 matroids, each with six elements. Note that $S_1^* \cong S_2$, whereas S_3 and S_4 are both self-dual matroids. In addition, S_0 is constructed from the direct sum $U_{2,3} \oplus U_{2,3}$ by adding one parallel point to each of the two connected components.

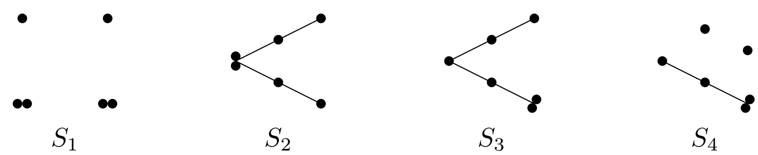

FIGURE 1. Connected excluded minors for split matroids.

Theorem 1.4. *The excluded minors for the class of split matroids are S_0, S_1, S_2, S_3, and S_4.*

In order to prove this theorem, we rely on Joswig and Schröter's equivalent formulation of Definition 1.2 that relies entirely on matroidal structural concepts.

We say that a flat, F, of the matroid, M, is *proper* if $0 < r(F) < r(M)$. A set is *cyclic* if it is a union of circuits or it is the empty set. We first note that the following result (which combines Lemma 10 and Proposition 15 of [4]) means that we need only concern ourselves with characterising connected split matroids.

Proposition 1.5. *Let M be a disconnected matroid, with connected components $X_1, \ldots, X_t$. Then M is a split matroid if and only if each connected matroid, $M|X_i$, is a split matroid, and at most one of these matroids is non-uniform.*

Definition 1.6. Let M be a matroid, and let Z be a proper cyclic flat of M. If both $M|Z$ and M/Z are connected matroids, but at least one of them is a non-uniform matroid, then we say that Z is a *certificate* for *non-splitting.*

Lemma 1.7. *A connected matroid is a split matroid if and only if it has no certificate for non-splitting.*

Proof. This will follow immediately from Theorem 11 in [4] provided that we can demonstrate that the flat Z is a split flacet if and only if it is a proper cyclic flat such that $M|Z$ and M/Z are connected.

Assume that Z is a proper cyclic flat of M such that $M|Z$ and M/Z are connected. Then Z is a flacet by Proposition 1.1. Furthermore $0 < r(Z) < |Z|$, since Z is a proper flat and is not independent. As Z and $E(M) - Z$

are non-empty, we can find an element in $E(M) - Z$ that is not a coloop, since M is connected. Now Lemma 6 of [4] implies that Z is a split flacet.

For the converse, we let Z be a split flacet. Then $M|Z$ and M/Z are connected by Proposition 1.1. Assume $|Z| \leq 1$. Now Proposition 4 of [4] asserts that there must be a positive integer, μ, which satisfies both $\mu < r(M)$ and $\mu > r(M) - |Z| \geq r(M) - 1$. Since this is impossible, $|Z| \geq 2$, so Z is a cyclic flat by Proposition 13 in [4]. It remains only to show that Z is a proper flat. If not, then $Z = E(M)$, as $|Z| \geq 2$. But every point, x, in $P(M)$ satisfies $\sum x_i \leq r(M)$, which means that $P(M)$ is contained in the hyperplane $H(E(M))$, so this hyperplane does not intersect $P(M)$ in a facet. This completes the proof. □

2. Proof of the main theorem

We discuss some preliminaries: Let M be a matroid on the ground set E, and let U be a subset of E. Recall that $\lambda(U)$ is defined to be $r(U) + r(E - U) - r(M)$. This is equal to $r(U) + r^*(U) - |U|$. A *k-separation* is a partition, (U, V), of E, such that $|U|, |V| \geq k$, and $\lambda(U) < k$. A matroid is *n-connected* if it has no k-separation with $k < n$. A matroid is *connected* if it is 2-connected (equivalently, if every pair of distinct elements is contained in a circuit). We refer to a 1-separation as a *separation*. We make use of the fact that if M is a connected matroid, and $e \in E(M)$, then either $M \backslash e$ or M/e is connected [5, Theorem 4.3.1]. In addition, if a single-element extension of a connected matroid by the element e is not connected, then e is a loop or a coloop in the extension [5, Proposition 8.2.7]. Other foundational material on matroids can be found in [5].

Lemma 2.1. *Let Z be a proper cyclic flat of the connected matroid M. Then $E(M) - Z$ is a proper cyclic flat of M^*.*

Proof. Let E be the ground set of M. The fact that $E - Z$ is a cyclic flat of M^* is well-known and easy to verify. Suppose it is not proper, that is $r^*(E - Z) = r(M^*)$ or $r^*(E - Z) = 0$. First, consider the case where $r^*(E - Z) = r(M^*) = |E| - r(M)$. Then the corank function gives

$$|E| - r(M) = r(Z) + |E - Z| - r(M),$$

meaning $r(Z) = |Z|$, so Z is an independent set. The only set that is cyclic and independent is the empty set, and this is impossible, as Z is a proper flat. Now suppose $r^*(E - Z) = 0$. As Z is a proper flat it cannot be equal to E. Therefore $E - Z$ contains an element, and this element is a coloop. The only connected matroid with a coloop has a ground set of size one, but this is impossible since Z and $E - Z$ are both non-empty. □

First we note that it is easy to confirm that S_0 is not split, by Proposition 1.5, and in fact it is an excluded minor for the class of split matroids. Moreover, the connected matroids S_1, S_2, S_3, and S_4 all contain certificates for non-splitting, and are indeed excluded minors.

We now show that there is only one disconnected excluded minor for split matroids. Recall that S_0 is the matroid constructed from the direct sum, $U_{2,3} \oplus U_{2,3}$, by adding a parallel point to each of the two connected components.

Proposition 2.2. *The only disconnected excluded minor for the class of split matroids is S_0.*

Proof. Suppose M is a disconnected excluded minor. This means M is not a split matroid, but every proper minor of M is. Let the connected components of M be $X_1, \ldots, X_t$, where $t > 1$. Suppose that $M|X_i$ is not split for some i. Choose an element $e \notin X_i$. Then, as deletion distributes over direct sums, $M|X_i$ is a component of $M\backslash e$. Thus $M\backslash e$ has a non-split component, and is therefore itself not split. This contradiction shows that every component of M is split. If at most one component of M is non-uniform, then M will be split, which is a contradiction. So let $M|X_i$ and $M|X_j$ be non-uniform, where $1 \leq i < j \leq t$. If there is an element $e \notin X_i \cup X_j$, then $M|X_i$ and $M|X_j$ are both non-uniform components of $M\backslash e$, which is a contradiction as $M\backslash e$ is split. So we must have $i = 1$ and $j = t = 2$.

Let e be an arbitrary element in X_1. Adding a loop to a split matroid produces another split matroid. It follows that e is not a loop in M, so $|X_1| > 1$. Since $M\backslash e$ is split and $M|X_2$ is not uniform, $(M|X_1)\backslash e$ must be either connected and uniform, or disconnected with all components uniform. Either deleting or contracting e from $M|X_1$ produces a connected matroid, and by duality, we can assume that $(M|X_1)\backslash e$ is connected, and therefore uniform. Note $r(X_1) > 1$, for otherwise $r(X_1) = 1$, and $M|X_1$ is a rank-one uniform matroid. Let C be a smallest circuit of $M|X_1$ that contains e, and note that C is not spanning, since $M|X_1$ is not uniform but $(M|X_1)\backslash e$ is. Take $c \in C - e$. If $|C| > 2$, then e is not a loop in $(M|X_1)/c$, and hence this matroid is a connected extension of the uniform matroid $((M|X_1)\backslash e)/c$. It is also non-uniform, since $C - c$ is a non-spanning circuit. Thus M/c contains two non-uniform components: $(M/c)|(X_1 - c)$ and $(M/c)|X_2$. Therefore M/c is not split and we have a contradiction. Hence $|C| = 2$. Let x be an element in $X_1 - C$. Then $(M|X_1)/x$ is a parallel extension of a uniform matroid with rank $r(X_1) - 1 \geq 1$. Since this matroid must be uniform, we conclude it actually has rank one. Thus $r(X_1) = 2$, so $M|X_1$ is a parallel extension of a rank-2 uniform matroid. If $|X_1| > 4$, then we can let x be an element not in the parallel pair, and $(M|X_1)\backslash x$ is connected and non-uniform. Thus $M|X_1$ is a parallel extension of $U_{2,3}$. Note that $M|X_1$ is self-dual. Now symmetric arguments show that $M|X_2$ is also $U_{2,3}$ plus a parallel point, and so M is isomorphic to S_0. □

Lemma 2.3. *Let (U, V) be a 2-separation in the connected matroid, M, and assume that there is no parallel pair contained in U. Then there is an element, $u \in U$, such that M/u is connected.*

Proof. Assume that the lemma fails, and that we have chosen a counterexample with $|U|$ as small as possible. We first note that if U contains a series pair, $\{u, v\}$, then $M\backslash u$ is not connected, as v is a coloop in this matroid. This implies that M/u is connected, contrary to hypothesis. Thus U does not contain a series pair. As $|U| \geq 2$, and U contains no parallel pairs, we see that $r(U) > 1$. Therefore $r(V) < r(M)$.

Assume that $|U| \leq 3$, which implies that $|U| = 2$ or $|U| = 3$, as (U, V) is a 2-separation. Note that $\lambda(U) = 1$ implies $r(U) + r^*(U) \leq 4$, which is possible only if $r(U) = r^*(U) = 2$, as U contains no parallel pair and no series pair. If $|U| = 2 = r^*(U)$, then U is coindependent, so V contains a basis, contradicting the earlier conclusion that V is not spanning. Hence $|U| = 3$, and U is both a circuit and a cocircuit. Let u be an element of U. Then M/u has a separation, (X, Y), by hypothesis. Since $U - u$ is a circuit in M/u, we can assume that $U - u \subseteq X$. But since U is a cocircuit of M, it follows that $r^*(U) = r^*(U \cup u)$, and now it is easy to verify that $(X \cup u, Y)$ is a separation of M, a contradiction. Therefore $|U| \geq 4$.

As V does not span M, we can chose an arbitrary element, u, in $U - \mathrm{cl}(V)$. If u is in a parallel pair with the element z, then by hypothesis, z is in V, implying u is in the span of V, contrary to our choice. Hence M/u contains no loops. By assumption, M/u is not connected. Let (X, Y) be a separation of M/u. Since M/u has no loops, it follows that $|X| \geq 2$ and $|Y| \geq 2$. Now standard rank calculations show that both $(X \cup u, Y)$ and $(X, Y \cup u)$ are 2-separations of M. Since $|U - u| \geq 3$, we can assume without loss of generality that $|U \cap X| \geq 2$.

The submodularity of the connectivity function [5, Lemma 8.2.9] implies that $\lambda(U \cap X) + \lambda(U \cup X) \leq \lambda(U) + \lambda(X) = 2$. Assume that $\lambda(U \cap X) \leq 1$. Then it is clear that $(U \cap X, V \cup Y \cup u)$ is a 2-separation of M, and as $U \cap X$ contains no parallel pairs, we contradict the minimality of U, since $U \cap X \subseteq U - u$. It follows that $\lambda(U \cup X) = 0$. As M is connected, this means that $V \cap Y = \emptyset$. But then Y is a proper subset of U, and as (X, Y) is a 2-separation, we again reach a contradiction to our assumption on $|U|$. □

We can now prove the rest of the characterisation.

Theorem 2.4. *Let M be a connected excluded minor for the class of split matroids. Then M is isomorphic to one of S_1, S_2, S_3, or S_4.*

Proof. Let M be a connected excluded minor. If M contains a loop, then it is isomorphic to the uniform matroid $U_{0,1}$, and is therefore a split matroid. Hence M is loopless. As M is connected and not split, it contains a certificate, Z, for non-splitting. Both $M|Z$ and M/Z are connected matroids and either $M|Z$ or M/Z is non-uniform. We would like to assume that $M|Z$ is non-uniform, so consider the case when this fails. Then M/Z is non-uniform. We will apply duality. Note that $E(M) - Z$ is a proper cyclic flat of M^* by Lemma 2.1, and that $M^*|(E(M) - Z) = (M/Z)^*$ while $M^*/(E(M) - Z) = (M|Z)^*$. Both of these matroids are connected, and $M^*|(E(M) - Z) = (M/Z)^*$ is non-uniform. Therefore we relabel M^* as M, and $E(M)-Z$ as Z. Now we can assume without loss of generality that $M|Z$ is not uniform. In the following analysis, we should expect to encounter S_2, S_3, and S_4, but not S_1, since it does not possess a proper cyclic flat of this type. Instead, we will encounter its dual, S_2.

2.5. *Let z be an element in Z such that $(M|Z)/z$ is connected and non-uniform. Then $(Z, E(M)-Z)$ is a 2-separation of M, and $z \in \mathrm{cl}_M(E(M)-Z)$.*

Proof. Note that since M is loopless and $M|Z$ is non-uniform, it follows that $r(Z) > 1$. Now it is very easy to confirm that $Z-z$ is a proper cyclic flat of M/z. Moreover, $(M/z)/(Z-z) = M/Z$ is connected, since Z is a certificate of non-splitting in M. We have assumed that $(M/z)|(Z-z) = (M|Z)/z$ is connected. Furthermore, $(M/z)|(Z-z)$ is not uniform by assumption. Thus $Z-z$ is a certificate for non-splitting in M/z. If M/z is connected, then this implies that M/z is not a split matroid, which is impossible as M is an excluded minor for the class of split matroids. Therefore we let (U, V) be a separation in M/z.

If both U and V contain elements of Z, then $(U \cap Z, V \cap Z)$ is a separation of $(M/z)|(Z-z) = (M|Z)/z$, and we have assumed this matroid is connected. Therefore we can assume without loss of generality that $Z - z \subseteq U$. If both U and V contain elements of $E(M)-Z$, then $(U-Z, V-Z)$ is a separation of the connected matroid $(M/z)/(Z-z) = M/Z$. Therefore we must have $V = E(M)-Z$, and $U = Z-z$. As M is connected, (U, V) is not a separation of M, and standard rank calculations show that $(U \cup z, V)$ is a 2-separation in M satisfying $z \in \mathrm{cl}_M(V)$. This is exactly what we set out to prove. ■

2.6. *Let z_1 and z_2 be distinct elements in Z such that $(M|Z)/z_1$ and $(M|Z)/z_2$ are both non-uniform. Then at most one of $(M|Z)/z_1$ and $(M|Z)/z_2$ is connected.*

Proof. Assume that both $(M|Z)/z_1$ and $(M|Z)/z_2$ are connected. Then 2.5 implies that $(Z, E(M)-Z)$ is a 2-separation of M, and both z_1 and z_2 are in $\mathrm{cl}_M(E(M)-Z)$. This means that $r((E(M)-Z)\cup\{z_1,z_2\}) = r(E(M)-Z)$, and as $(Z, E(M)-Z)$ is a 2-separation, we can use the submodularity of the rank function to establish that

$$\begin{aligned} r(\{z_1,z_2\}) &\le r(Z) + r((E(M)-Z)\cup\{z_1,z_2\}) - r(E(M)) \\ &= r(Z) + r(E(M)-Z) - r(M) = 1. \end{aligned}$$

Because M has no loops, this implies that $\{z_1,z_2\}$ is a parallel pair of M. Thus z_2 is a loop in $(M|Z)/z_1$. Since this matroid is connected, it must consist of the single loop, z_2. Therefore $Z = \{z_1, z_2\}$. This implies that $M|Z$ is isomorphic to the uniform matroid $U_{1,2}$, which is impossible as we have assumed that $M|Z$ is non-uniform. ■

2.7. *Z contains a parallel pair.*

Proof. We assume otherwise. Since $M|Z$ is not uniform, it contains a non-spanning circuit C. Let z be an arbitrary element in C. Then $C - z$ is a non-spanning circuit in $(M/z)|(Z - z) = (M|Z)/z$, so this matroid is non-uniform. Choose distinct elements z and z' from C. From 2.6 we see that at most one of $(M|Z)/z$ and $(M|Z)/z'$ is connected. Without loss of generality, we assume that $(M|Z)/z$ has a separation, (U, V). By assumption, z is not in a parallel pair in $M|Z$. Therefore $(M|Z)/z$ contains no loops. This implies that $|U| \geq 2$ and $|V| \geq 2$. Since $M|Z$ is connected, we deduce that both $(U, V \cup z)$ and $(U \cup z, V)$ are 2-separations of $M|Z$. As neither U nor V contains a parallel pair, we can apply Lemma 2.3, and deduce that there are elements $z_1 \in U$ and $z_2 \in V$ such that $(M|Z)/z_1$ and $(M|Z)/z_2$ are connected. If both $(M|Z)/z_1$ and $(M|Z)/z_2$ are non-uniform, then we have a violation of 2.6. Therefore we can assume without loss of generality that $(M|Z)/z_1$ is uniform.

It cannot be the case that z_1 is a coloop of $(M|Z)/z$, for then it would be a coloop in the connected matroid $M|Z$. Therefore we let C' be a circuit of $(M|Z)/z$ that contains z_1. Since (U, V) is a separation, it follows that $C' \subseteq U$. There is a circuit, C, of $M|Z$ such that C is equal to either C' or $C' \cup z$. Then $C - z_1$ is a circuit of the uniform matroid $(M|Z)/z_1$, so $C - z_1$ spans $(M|Z)/z_1$. Thus C is a spanning circuit of $M|Z$ that is contained in $U \cup z$. Therefore $(U \cup z, V)$ is a 2-separation of $M|Z$ satisfying $r(U \cup z) = r(M|Z)$. This implies that $r(V) = 1$. But $|V| \geq 2$, so $V \subseteq Z$ contains a parallel pair and we have a contradiction. ■

Henceforth we let $\{x, y\}$ be a parallel pair contained in Z.

2.8. *$\{x, y\}$ is the only parallel pair in M.*

Proof. Assume that $\{a, b\}$ is a parallel pair not equal to $\{x, y\}$. Without loss of generality, we can assume that $a \notin \{x, y\}$. It is an easy exercise to show that deleting an element from a parallel pair does not disconnect a connected matroid. Therefore $M\backslash a$ is connected.

In the first case, assume that a, and hence b, is in Z. Then $Z - a$ is a proper cyclic flat of $M\backslash a$, and $(M\backslash a)|(Z - a)$ is connected. As a is in the span of $Z - a$, it is a loop in $M/(Z - a)$. Therefore

$$M/Z = M/(Z - a)/a = M/(Z - a)\backslash a = (M\backslash a)/(Z - a)$$

so $(M\backslash a)/(Z - a)$ is also connected. Furthermore $(M\backslash a)|(Z - a)$ is non-uniform, since $r(Z - a) = r(Z) > r(\{x, y\})$, so $\{x, y\}$ is a non-spanning circuit in $(M\backslash a)|(Z - a)$. Therefore $Z - a$ is a certificate for non-splitting in the connected matroid $M\backslash a$, and we have a contradiction as $M\backslash a$ is a proper minor of M.

For the second case, we assume that a is not in Z. Therefore Z is a proper cyclic flat of $M\backslash a$, and $(M\backslash a)|Z = M|Z$ is connected and non-uniform. We know that M/Z is connected, and $\{a, b\}$ is a parallel pair in this matroid, since neither a nor b is in the span of Z. Therefore $(M\backslash a)/Z = (M/Z)\backslash a$ is obtained from a connected matroid by deleting an element from a parallel pair, and is hence connected. Thus Z is a certificate for non-splitting in the connected matroid $M\backslash a$, and we again have a contradiction. ■

2.9. $r(Z) = 2$

Proof. Assume that $r(Z) > 2$. Let z be an arbitrary element in $Z - \{x, y\}$. Then $(M|Z)/z$ is non-uniform, because it has rank at least two, but it also contains a parallel pair. From 2.6 we deduce that if z and z' are distinct elements in $Z - \{x, y\}$ then at most one of $(M|Z)/z$ and $(M|Z)/z'$ is connected. Thus we choose distinct elements z and z' in $Z - \{x, y\}$, and without loss of generality, we assume that $(M|Z)/z$ has a separation (U, V). Since z is not in a parallel pair, it follows that $(M|Z)/z$ has no loops, so $|U| \geq 2$ and $|V| \geq 2$. We deduce that $(U, V \cup z)$ and $(U \cup z, V)$ are 2-separations in $(M|Z)/z$. Since $\{x, y\}$ is a circuit in $(M|Z)/z$, and (U, V) is a separation in this matroid, we relabel as necessary and assume that $x, y \in U$. Therefore V contains no parallel pair of $M|Z$, so we can apply Lemma 2.3 to $(U \cup z, V)$ and deduce that there is an element $z_1 \in V$ such that $(M|Z)/z_1$ is connected. The earlier conclusion shows that if w is an element in $Z - \{x, y, z_1\}$, then $(M|Z)/w$ is not connected.

Let w be an arbitrary element in $Z - \{x, y, z_1\}$ and let (U_w, V_w) be a separation of $(M|Z)/w$. As $(M|Z)/w$ has no loops, we deduce that $|U_w| \geq 2$ and $|V_w| \geq 2$, and both $(U_w, V_w \cup w)$ and $(U_w \cup w, V_w)$ are 2-separations of $M|Z$. Without loss of generality, we assume that z_1 is in V_w. We claim that $x, y \in U_w$. If this is not the case, then U_w contains no parallel pair of $M|Z$. Therefore we can apply Lemma 2.3 to $(U_w, V_w \cup w)$ and deduce that there is an element $u \in U_w \subseteq Z - \{x, y, z_1\}$ such that $(M|Z)/u$ is connected. This contradicts an earlier conclusion, so $x, y \in U_w$, as claimed. Assume that we have chosen w from $Z - \{x, y, z_1\}$ in such a way that $|U_w|$ is as small as possible.

If w is not in the closure of U_w in $M|Z$, then it is in the coclosure of V_w. This implies that $(U_w, V_w \cup w)$ is a separation in the connected matroid $M|Z$. Therefore w is in the closure of U_w. The same argument shows that w is in the closure of V_w. Let $C \subseteq U_w \cup w$ be a circuit of $M|Z$ that contains w. Then $C \nsubseteq \{x, y, w\}$, as $\{x, y\}$ is a circuit in $M|Z$, and w is not parallel to x or y. Let t be an element in $C - \{x, y, w\}$, so that t belongs to U_w, and also to $Z - \{x, y, z_1\}$. Therefore $(M|Z)/t$ has a separation (U_t, V_t) where $z_1 \in V_t$ and $x, y \in U_t$. As before, we see that $|U_t| \geq 2$ and $|V_t| \geq 2$, and both $(U_t, V_t \cup t)$ and $(U_t \cup t, V_t)$ are 2-separations of $M|Z$, where t is in the closure of both U_t and V_t.

Let λ be the connectivity function of $M|Z$. Then λ is submodular, so

$$\lambda(U_w \cap U_t) + \lambda(U_w \cup U_t) \leq \lambda(U_w) + \lambda(U_t) = 2.$$

Neither $U_w \cap U_t$ nor $E(M) - (U_w \cup U_t)$ is empty (the former contains x and y, the latter contains z_1). Therefore neither $\lambda(U_w \cap U_t)$ nor $\lambda(U_w \cup U_t)$ is zero. We deduce that $\lambda(U_w \cap U_t) = 1$. We can apply the same argument to $\lambda(U_w \cap (U_t \cup t)) + \lambda(U_w \cup (U_t \cup t))$ and deduce that $\lambda(U_w \cap (U_t \cup t)) = 1$.

Since t is in the closure of V_t it follows that

$$r(Z-(U_w\cap U_t)) = r(Z-(U_w\cap(U_t\cup t))).$$

From $\lambda(U_w\cap U_t) = \lambda(U_w\cap(U_t\cup t))$ we can deduce that $r(U_w\cap U_t) = r(U_w\cap(U_t\cup t))$ and therefore t is in the closure of $U_w\cap U_t$. We have now shown that $(U_w\cap U_t, Z-(U_w\cap U_t))$ is a 2-separation of $M|Z$, and that t is in the closure of both sides. Standard rank calculations now show that $(U_w\cap U_t, Z-(U_w\cap(U_t\cup t)))$ is a separation of $(M|Z)/t$. Note that $U_w\cap U_t$ contains $\{x,y\}$. But $U_w\cap U_t$ does not contain $t\in U_w$, so $|U_w\cap U_t|<|U_w|$, and we have a contradiction to our choice of w. This contradiction completes the proof. ∎

2.10. $|Z|=4$

Proof. We now know that Z is a rank-two cyclic flat containing a parallel pair, $\{x,y\}$. Therefore $|Z|>2$. If $|Z|=3$ then the element in $Z-\{x,y\}$ would be a coloop of $M|Z$, and thus Z would not be cyclic.

Suppose that $|Z|>4$, and let a, b, and c be elements of $Z-\{x,y\}$. Then $M|\{a,b,c,x\}$ is isomorphic to $U_{2,4}$. Assume that $M\backslash a$ has a separation, (U,V). Without loss of generality, U contains two elements of $\{b,c,x\}$. Then a is in the closure of U, so $(U\cup a,V)$ is a separation of M, a contradiction. Therefore $M\backslash a$ is connected. Exactly the same argument shows that $(M|Z)\backslash a = (M\backslash a)|(Z-a)$ is connected. We also know that $(M\backslash a)/(Z-a) = M/Z$ is connected. It is clear that $Z-a$ is a proper cyclic flat of $M\backslash a$. Furthermore $(M\backslash a)|(Z-a)$ is not uniform, as it has rank two and contains a parallel pair. Therefore $Z-a$ is a certificate for non-splitting in the connected matroid $M\backslash a$. This is a contradiction, so the proof is complete. ∎

2.11. *If $r(M)=3$ then M is isomorphic to S_2, S_3, or S_4.*

Proof. Assume that $r(M)=3$. Let w be an arbitrary element not in Z. Note that $(M\backslash w)|Z = M|Z$ is connected and non-uniform. Also, $(M\backslash w)/Z$ is a rank-1 matroid. It is loopless because Z is a flat. Hence it is connected. Since Z is a proper cyclic flat of $M\backslash w$, it is a certificate for $M\backslash w$. Since $M\backslash w$ is a split matroid, $M\backslash w$ cannot be connected. Note that $M\backslash w$ contains no parallel pair other than $\{x,y\}$, by 2.8. Since $M\backslash w$ has rank three and is not connected, it now follows that it is equal to the direct sum of $M|Z$ with a coloop, w'. Hence $\{w,w'\}$ is a series pair in M, and M contains six elements. When we contract w, the element w' is projected onto the line spanned by Z. Thus, in M/w, the element w' is in a parallel class of size three, two, or one. These cases correspond to M being isomorphic to S_2, S_3, or S_4, respectively. ∎

Henceforth we assume that $r(M)>3$.

2.12. *If $w\notin Z$, then $(M/Z)/w$ is not connected.*

Proof. Assume otherwise, so that $(M/Z)/w$ is connected. Assume that Z is not a flat in M/w, and let x be an element in $\mathrm{cl}_{M/w}(Z)-Z$. Then x is a loop of $(M/Z)/w$, and since this matroid is connected, it follows that it contains a single element, x. Thus x and w are the only elements in $E(M)-Z$. As $r(M) \geq r(Z)+2$, both x and w are coloops, a contradiction. Therefore Z is a proper cyclic flat of M/w. Moreover, $(M/w)|Z = M|Z$ is connected and non-uniform and $(M/w)/Z$ is connected by assumption. Hence Z is a certificate for M/w. As M/w is a split matroid, it cannot be connected. Let (U, V) be a separation of M/w.

As $(M/w)|Z$ is connected, we can assume that $Z \subseteq U$. In fact, $U = Z$, for otherwise $(U-Z, V)$ is a separation of $(M/w)/Z$, and this matroid is connected. Let C be a circuit of M that contains an element of Z and an element of V. Then $w \notin C$, for otherwise $C-w$ is a circuit in M/w that contains elements in U and V. As C is not a circuit of M/w, it is a union of at least two circuits. Let z be an element in $Z \cap C$, and let $D \subseteq C$ be a circuit of M/w that contains z. Thus D is contained in $U = Z$. Note that D is not a circuit of M, for it is properly contained in C. Therefore $D \cup w$ is a circuit of M. This implies that $w \in \mathrm{cl}(Z)$, which is impossible as Z is a flat. ■

2.13. *If $w \notin Z$, then $M\backslash w$ is not connected.*

Proof. Because M/Z is connected, but $(M/Z)/w$ is not by 2.12, it follows that $(M/Z)\backslash w$ is connected. Certainly Z is a proper cyclic flat in $M\backslash w$. Also $(M\backslash w)|Z = M|Z$ is connected and non-uniform, and $(M\backslash w)/Z = (M/Z)\backslash w$ is connected. Thus Z is a certificate for non-splitting in $M\backslash w$. As $M\backslash w$ is a split matroid, it cannot be connected. ■

Now we choose an element $w \notin Z$, and let $X_1, \ldots, X_t$ be the connected components of $M\backslash w$. Since $(M|Z)\backslash w = M|Z$ is connected we can assume that $Z \subseteq X_1$. Because M/Z is connected, but $(M/Z)/w$ is not, by 2.12, we see that $(M/Z)\backslash w = (M\backslash w)/Z$ is connected. The only way that this can occur is if $X_1 = Z$, and $t = 2$. Since $M\backslash w$ is a split matroid, and $M|Z = M|X_1$ is not uniform, we deduce that $M|X_2$ is uniform. As $r(Z) = 2$ and $r(M) > 3$, the rank of $M|X_2$ is at least two. Now M/Z is connected and is an extension of the uniform matroid $M|X_2$ by the element w. The rank of M/Z is at least two. Thus we can choose $w' \in M/Z$ that is not equal or parallel to w. Therefore $(M/Z)/w'$ is an extension of a uniform matroid by the element w and $r((M/Z)/w') \geq 1$. As w is not a loop or coloop in $(M/Z)/w'$, it follows that $(M/Z)/w'$ is connected, contradicting 2.12. Thus we have a final contradiction that completes the proof of Theorem 2.4.

□

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