OBSTACLES TO DECOMPOSITION THEOREMS FOR SIXTH-ROOT-OF-UNITY MATROIDS

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ABSTRACT. We construct an infinite family of highly connected sixth-root-of-unity matroids that are not near-regular. This family is an obstacle to any decomposition theorem for sixth-root-of-unity matroids in terms of near-regular matroids.

1. Introduction

Seymour's decomposition theorem for matroids representable over the regular partial field [8] is one of the classical results of matroid theory. It shows that an internally 4-connected regular matroid is graphic, cographic, or sporadic.

It is only natural to hope that there may be similar decomposition results for matroids representable over other partial fields. However, internal 4-connectivity is not quite the right notion to use when considering non-regular matroids. We would like a notion of connectivity that captures when a connected matroid cannot be decomposed via a generalised parallel connection along a point or line. This notion should still allow the matroid to have long lines, unlike internal 4-connectivity. We also want our notion of connectivity to be closed under duality, so we make the following definition.

Definition 1.1. A matroid is *fused* if it is 3-connected and whenever (U, V) is a 3-separation, either $\min\{r(U), r(V)\} \leq 2$, or $\min\{r^*(U), r^*(V)\} \leq 2$.

Note that if M is 3-connected, and either M or M^* is vertically 4-connected, then M is fused. The converse does not hold: if a matroid contains both triangles and triads, then neither it, nor its dual, is vertically 4-connected, but it may still be fused. Every internally 4-connected matroid is fused, and a binary matroid is fused if and only if it is internally 4-connected.

We would very much like to obtain decomposition results giving us control over the fused matroids representable over certain partial fields. Some recent evidence shows that this may be a forlorn hope in the case of near-regular matroids [4] and dyadic matroids [5]. The current paper provides similarly negative evidence against a decomposition theorem for sixth-root-of-unity matroids in terms of near-regular matroids.

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The authors of [1, 2] conjectured the existence of a decomposition theorem showing that any fused sixth-root-of-unity matroid that fails to be near-regular is isomorphic to a restriction of $\mathrm{AG}(2, 3)$ (up to duality and Δ-Y operations). Unfortunately, the matroid N, illustrated in Figure 1, provides a counterexample. The matrix representation of N is over the complex numbers, and ξ is a primitive sixth-root of unity. It can be verified that every non-zero subdeterminant of this matrix is a power of ξ, so N is a sixth-root-of-unity matroid. It is easy to see that $N/y_1\backslash x_1$ is isomorphic to $\mathrm{AG}(2, 3)\backslash e$, so N is certainly not near-regular (see [3]). Moreover, the only 3-separating sets in N are triangles and the complements of triangles, so N is fused.

$$\begin{array}{c} \\ c \\ a \\ x_0 \\ x_1 \end{array} \begin{array}{c} \begin{array}{cccccc} u & v & w & b & y_0 & y_1 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ \hline \bar{\xi} & 0 & 1 & 0 & 0 & 1 \\ 1 & \xi & \xi & 1 & 1 & 0 \\ 1 & \xi & \xi & 0 & 1 & 1 \end{array}\right] \end{array}$$

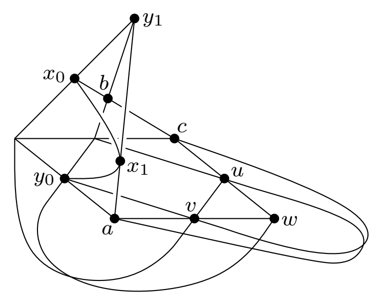

FIGURE 1. Matrix and geometric representations of N.

We might still hope that there are only finitely many fused matroids that are sixth-root-of-unity without being near-regular. In this case a decomposition theorem might need to deal with only a finite number of sporadic matroids. Our main theorem banishes this hope also.

Theorem 1.2. *There are infinitely many fused matroids that are sixth-root-of-unity without being near-regular.*

Our proof of Theorem 1.2 is not specific to the sixth-root-of-unity partial field, so we operate at a slightly higher level of generality:

Theorem 1.3. *Let $\mathbb{P}$ be a partial field and let M be a $\mathbb{P}$-representable matroid. Assume that $E(M)$ contains distinct elements a, b, c, x_0, x_1, y_0, and y_1, where $T_0 = \{x_0, x_1, y_0\}$, $T_1 = \{a, x_1, y_1\}$, and $T_2 = \{b, c, x_0\}$ are triangles, and $\{b, x_0, x_1, y_1\}$ is a cocircuit. Assume also that the following conditions hold:*

(i) $r(T_0 \cup T_1) = 3$,
(ii) $\{b, x_0, x_1, y_1\}$ *is independent,*
(iii) $r(M), r^*(M) > 3$,
(iv) $M/y_1\backslash x_1$ *is* 3*-connected,*
(v) M *is fused.*

Then there are infinitely many fused $\mathbb{P}$-representable matroids that have M as a minor.

The conditions of Theorem 1.3 apply to the sixth-root-of-unity partial field, and the matroid N described in Figure 1. Therefore Theorem 1.2 follows immediately from Theorem 1.3.

2. Preliminaries

Any undefined notation or terminology is in Oxley [6]. Our general reference for partial fields is Pendavingh and Van Zwam [7]; proofs of the results in this section can be found there. A *partial field* is a pair, (R, G), where R is a commutative ring with identity, and G is a subgroup of the group of units. We require $-1 \in G$. In particular, if $\mathbb{F}$ is a field, then $(\mathbb{F}, \mathbb{F} - \{0\})$ is a partial field. The *regular* partial field is $(\mathbb{Z}, \{1, -1\})$. The *near-regular* partial field is $(\mathbb{Q}(\alpha), \{\pm\alpha^i(1-\alpha)^j : i, j \in \mathbb{Z}\})$, where $\mathbb{Q}(\alpha)$ is the field of rationals extended by the transcendental α. The *sixth-root-of-unity* partial field is $(\mathbb{C}, \{z : z^6 = 1\})$.

Let A be a matrix with entries from the ring R, and assume that the rows of A are labeled by the set X and the columns are labeled by the set Y. If $X' \subseteq X$ and $Y' \subseteq Y$, then we use the notation $A[X', Y']$ to stand for the submatrix of A with rows and columns labeled by X' and Y'. If $Z \subseteq X \cup Y$, then $A - Z = A[X - Z, Y - Z]$. As usual, we omit the set brackets about singleton sets. We treat a 1×1 matrix as a member of R.

Assume $A[x, y]$ is non-zero, and let A^{xy} be the matrix obtained from A by *pivoting* on (x, y). This means that the labels x and y are swapped, so that the rows of A^{xy} are labeled by $(X - x) \cup y$, and the columns are labeled by $(Y - y) \cup x$. For any u labeling a row of A^{xy}, and any v labeling a column, we have:

$$A^{xy}[u,v] = \begin{cases} A[x,y]^{-1} & \text{if } (u,v) = (y,x) \\ A[x,y]^{-1}A[x,v] & \text{if } u = y,\ v \neq x \\ -A[x,y]^{-1}A[u,y] & \text{if } u \neq y,\ v = x \\ A[u,v] - A[x,y]^{-1}A[u,y]A[x,v] & \text{otherwise} \end{cases}$$

Pictorially, this means that if A has the following form

$$\begin{array}{c} \\ x \end{array}\left[\begin{array}{c|c} & y \\ D & \mathbf{d} \\ \hline \mathbf{c}^T & \alpha \end{array}\right]$$

then A^{xy} is

$$\begin{array}{c} \\ y \end{array}\left[\begin{array}{c|c} & x \\ D - \alpha^{-1}\mathbf{d}\mathbf{c}^T & -\alpha^{-1}\mathbf{d} \\ \hline \alpha^{-1}\mathbf{c}^T & \alpha^{-1} \end{array}\right]$$

Note that if we pivot on (y, x) in the matrix A^{xy}, then we recover A. If A is a square matrix, then

$$\det(A) = \pm A[x,y]\det(A^{xy} - \{x,y\}). \tag{1}$$

If $\mathbb{P} = (R, G)$ is a partial field, then a $\mathbb{P}$-*matrix* is a matrix with entries from R, such that the determinant of any square submatrix is in $G \cup \{0\}$. If X and Y label the rows and columns of a $\mathbb{P}$-matrix, A, then

$$\{X\} \cup \{Z \subseteq X \cup Y \colon |Z| = |X|,\ \det(A[X - Z, Y \cap Z]) \neq 0\}$$

is the collection of bases of a matroid. We denote this matroid $M[I|A]$. If x labels a row, then $M[I|A]/x = M[I|A - x]$, and if y labels a column, then $M[I|A]\backslash y = M[I|A - y]$. We say that $M[I|A]$ is $\mathbb{P}$-*representable*. A matroid that is representable over the near-regular (respectively sixth-root-of-unity) partial field is said to be *near-regular* (*sixth-root-of-unity*). Modifying a $\mathbb{P}$-matrix by scaling a row or column with an element in G or by pivoting on a non-zero entry produces another $\mathbb{P}$-matrix, and these two matrices represent the same matroid.

3. Proof of Theorem 1.3

Henceforth we let $\mathbb{P} = (R, G)$ and M be as described in the statement of Theorem 1.3. Let H be the complementary hyperplane to $\{b, x_0, x_1, y_1\}$. Note that $r(\{a, y_0\}) = 2$, or else $r(T_0 \cup T_1) = 2$. We observe that $\text{cl}(\{x_0, x_1, y_1\})$ contains y_0 and a, because of the triangles T_0 and T_1. If $r(\{a, y_0, c\}) = 2$, then $\text{cl}(\{x_0, x_1, y_1\})$ also contains c, and hence b, because of the triangle $\{x_0, b, c\}$. But this contradicts the fact that $\{b, x_0, x_1, y_1\}$ is independent. Therefore $r(\{a, y_0, c\}) = 3$. Let B' be a basis of H that contains $\{a, y_0, c\}$. Now $B' \cup x_1$ is a basis of M. The fundamental circuit of x_0 with respect to this basis is $T_0 = \{x_0, x_1, y_0\}$. This means that $B = (B' - y_0) \cup \{x_0, x_1\}$ is a basis of M. The fundamental circuit of y_0 relative to B is $\{y_0, x_0, x_1\}$, and the fundamental circuit of y_1 is $\{y_1, a, x_1\}$. Furthermore, the fundamental circuit of b, relative to B, is $\{b, c, x_0\}$.

We will consider a $\mathbb{P}$-matrix, A_1, such that $M = M[I|A_1]$. By performing pivots as necessary, we can assume that B labels the rows of A_1. The previous paragraph shows that the column labeled by b is non-zero only in the rows labeled by c and x_0. Also, $A_1[B, y_1]$ is non-zero only in the rows labeled by a and x_1, and $A_1[B, y_0]$ is non-zero only in the rows labeled by x_0 and x_1. By scaling these two rows, we will assume that $A_1[x_0, y_0] = A_1[x_1, y_0] = 1$. Any basis must intersect the cocircuit $\{b, x_0, x_1, y_1\}$. This implies that any 2×2 submatrix with rows labeled by x_0 and x_1 and columns labeled by y_0 and some other column that is not b or y_1 must have a zero determinant. As $A_1[x_0, y_0] = A_1[x_1, y_0]$, we deduce that the rows labeled by x_0 and x_1 are identical, except possibly in the columns labeled by b and y_1.

Next we scale columns b and y_1 so that $A_1[x_0, b]$ and $A_1[x_1, y_1]$ are equal to one. Finally, we scale row a so that $A_1[a, y_1]$ is equal to one. Thus we can assume that M is equal to $M[I|A_1]$, where A_1 is a $\mathbb{P}$-matrix with the following form.

$$
\begin{array}{c}
\\ \\ a \\ x_0 \\ x_1
\end{array}
\begin{array}{c|ccc}
 & b & y_0 & y_1 \\
D & \mathbf{b} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{a}^T & 0 & 0 & 1 \\
\mathbf{x}^T & 1 & 1 & 0 \\
\mathbf{x}^T & 0 & 1 & 1
\end{array}
$$

Here $\mathbf{b}$ and $\mathbf{0}$ are $(r(M)-3)\times 1$ vectors where all the entries of $\mathbf{0}$ are zero, while $\mathbf{a}$ and $\mathbf{x}$ are $(|E(M)|-r(M)-3)\times 1$ vectors.

We define A_0 to be the following matrix.

$$
\begin{array}{c}
\\ \\ a \\ x_0
\end{array}
\begin{array}{c|cc}
 & b & y_0 \\
D & \mathbf{b} & \mathbf{0} \\
\hline
\mathbf{x}^T - \mathbf{a}^T & 0 & 1 \\
\mathbf{x}^T & 1 & 1
\end{array}
$$

Note that A_0 is the matrix we obtain from A_1 if we pivot on (x_1, y_1), then multiply the row labeled by a with the scalar -1, and finally, delete the row labeled y_1 and the column labeled x_1. Therefore A_0 is a $\mathbb{P}$-matrix.

For each positive integer $i > 1$, we define a derived matrix A_i. If i is odd, then A_i is the following matrix.

$$
\begin{array}{c}
\\ \\ a \\ x_0 \\ x_1 \\ \vdots \\ x_{i-2} \\ x_{i-1} \\ x_i
\end{array}
\begin{array}{c|ccccccc}
 & b & y_0 & y_1 & \cdots & y_{i-2} & y_{i-1} & y_i \\
D & \mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{a}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\mathbf{x}^T & 1 & 1 & 0 & & 0 & 0 & 0 \\
\mathbf{x}^T & 0 & 1 & 1 & & 0 & 0 & 0 \\
\vdots & \vdots & & & \ddots & & & \vdots \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 0 & 0 \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 1 & 0 \\
\mathbf{x}^T & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{array}
$$

If i is even, we define A_i as follows.

$$
\begin{array}{c}
\\ \\ a \\ x_0 \\ x_1 \\ \vdots \\ x_{i-2} \\ x_{i-1} \\ x_i
\end{array}
\begin{array}{c|cccccccc}
 & b & y_0 & y_1 & \cdots & y_{i-2} & y_{i-1} & y_i \\
D & \mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{x}^T - \mathbf{a}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\mathbf{x}^T & 1 & 1 & 0 & & 0 & 0 & 0 \\
\mathbf{x}^T & 0 & 1 & 1 & & 0 & 0 & 0 \\
\vdots & \vdots & & & \ddots & & & \vdots \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 0 & 0 \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 1 & 0 \\
\mathbf{x}^T & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{array}
$$

Lemma 3.1. *The matrix A_i is a $\mathbb{P}$-matrix, for every non-negative integer i.*

Proof. Let us assume that the lemma fails, and that i is the least non-negative integer such that A_i is not a $\mathbb{P}$-matrix. Certainly A_1 is a $\mathbb{P}$-matrix, and we have already argued that A_0 is a $\mathbb{P}$-matrix, so $i \geq 2$.

If a matrix with entries from R contains a square submatrix with a determinant not in $G \cup \{0\}$, then we will say that this is a *bad* submatrix. Therefore i is the least non-negative integer such that A_i contains a bad submatrix.

Let A'_i be the matrix obtained from A_i by pivoting on (x_i, y_i) and then multiplying the row labeled by a with the scalar -1. A straightforward calculation shows that A'_i is the following matrix.

$$
A'_i = \begin{array}{c} \\ \\ \\ \\ \\ y_i \end{array}
\left[\begin{array}{ccccc|c}
 & & & & & \mathbf{0} \\
 & & & & & 1 \\
 & & A_{i-1} & & & 0 \\
 & & & & & \vdots \\
 & & & & & 0 \\
\hline
\mathbf{x}^T & 0 & \cdots & 0 & 1 & 1
\end{array}\right]
\quad (\text{column } x_i)
$$

Since A_i can be obtained from A'_i by scaling and then pivoting on (y_i, x_i), it follows that A'_i cannot be a $\mathbb{P}$-matrix, or else A_i would be a $\mathbb{P}$-matrix also. Therefore A'_i contains a bad submatrix.

3.1.1. *Every bad submatrix of A'_i contains the row labeled y_i and the column labeled x_i.*

Proof. Assume for a contradiction that Z is a bad submatrix of A'_i and that Z avoids either row y_i or column x_i. It cannot be the case that Z avoids both, for that would imply that Z is a bad submatrix of A_{i-1}, and this contradicts our inductive assumption. First assume that Z contains column

x_i, but not row y_i. Then Z is a bad submatrix of $A'_i - y_i$. However, in this matrix, the column labeled by x_i contains only a single non-zero entry. From this we see that there is a bad submatrix that avoids both row y_i and column x_i, contradicting our earlier conclusion. Therefore Z contains row y_i, but not column x_i, so Z is a bad submatrix of the following matrix.

$$
A'_i - x_i = \begin{array}{c} \\ \\ a \\ x_0 \\ x_1 \\ \vdots \\ x_{i-3} \\ x_{i-2} \\ x_{i-1} \\ y_i \end{array}
\left[\begin{array}{c|ccccccc}
 & b & y_0 & y_1 & \cdots & y_{i-3} & y_{i-2} & y_{i-1} \\
D & \mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{c}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\mathbf{x}^T & 1 & 1 & 0 & & 0 & 0 & 0 \\
\mathbf{x}^T & 0 & 1 & 1 & & 0 & 0 & 0 \\
\vdots & \vdots & & & \ddots & & & \vdots \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 0 & 0 \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 1 & 0 \\
\mathbf{x}^T & 0 & 0 & 0 & & 0 & 1 & 1 \\
\mathbf{x}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right]
$$

Here $\mathbf{c}^T$ is either $\mathbf{a}^T$ or $\mathbf{x}^T - \mathbf{a}^T$ depending on the parity of i. Note that $A'_i - \{x_i, y_i\} = A_{i-1}$.

Assume that Z avoids row x_{i-1}. After deleting x_{i-1}, the column labeled y_{i-2} has only a single non-zero entry, so we can assume that Z is a bad submatrix of $A'_i - x_i$ that avoids both x_{i-1} and y_{i-2}. Deleting x_{i-1} and y_{i-2} from $A'_i - x_i$ produces a matrix that is identical to $A_{i-1} - y_{i-2}$. Therefore we have a contradiction to the minimality of i. We conclude that a bad submatrix in $A'_i - x_i$ must contain row x_{i-1}.

Let Z be a bad submatrix of $A'_i - x_i$, and assume that Z avoids column y_{i-2}. In $A'_i - \{x_i, y_{i-2}\}$, the rows labeled by x_{i-1} and y_i are identical, so we can choose Z so that it contains y_i, and not x_{i-1}. This contradicts the conclusion in the previous paragraph, so now any bad submatrix of $A'_i - x_i$ must contain x_{i-1} and y_{i-2}.

By Equation (1), there is a bad submatrix in the matrix, A'', obtained from $A'_i - x_i$ by pivoting on (x_{i-1}, y_{i-2}), and then deleting row y_{i-2} and column x_{i-1}. This matrix is shown below.

$$
A'' = \begin{array}{c} \\ \\ a \\ x_0 \\ x_1 \\ \vdots \\ x_{i-3} \\ x_{i-2} \\ y_i \end{array}
\begin{array}{c|ccccccc}
 & b & y_0 & y_1 & \cdots & y_{i-4} & y_{i-3} & y_{i-1} \\
D & \mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline
\mathbf{c}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\mathbf{x}^T & 1 & 1 & 0 & & 0 & 0 & 0 \\
\mathbf{x}^T & 0 & 1 & 1 & & 0 & 0 & 0 \\
\vdots & \vdots & & & \ddots & & & \vdots \\
\mathbf{x}^T & 0 & 0 & 0 & & 1 & 1 & 0 \\
\mathbf{0}^T & 0 & 0 & 0 & & 0 & 1 & -1 \\
\mathbf{x}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}
$$

Let Z be a bad submatrix of A'', and assume that Z avoids column y_{i-1}. After deleting y_{i-1} from A'', the row x_{i-2} contains only a single non-zero entry. Therefore we can assume that Z avoids y_{i-1} and x_{i-2}. After deleting y_{i-1} and x_{i-2} from A'', column y_{i-3} contains only a single non-zero entry, so we can assume that Z avoids y_{i-1}, x_{i-2}, and y_{i-3}. Therefore $A'' - \{y_{i-1}, x_{i-2}, y_{i-3}\}$ contains a bad submatrix. But this matrix is identical to $A_{i-1} - \{y_{i-3}, y_{i-2}, y_{i-1}, x_{i-1}\}$, so A_{i-1} contains a bad submatrix and we have a contradiction. Therefore any bad submatrix of A'' contains column y_{i-1}.

It is not difficult to see that $A'' - y_i$ is identical to the matrix obtained from A_{i-1} by pivoting on (x_{i-1}, y_{i-2}) and then deleting row y_{i-2} and column x_{i-1}. This matrix does not contain any bad submatrix, so $A'' - y_i$ does not contain any bad submatrix. Therefore any bad submatrix of A'' must contain column y_{i-1} and row y_i.

Equation (1) tells us that there is a bad submatrix in the matrix obtained from A'' by pivoting on (y_i, y_{i-1}), multiplying row a by -1, and then deleting row y_{i-1} and column y_i. But this matrix is identical to $A_{i-2} - y_{i-2}$, so A_{i-2} contains a bad submatrix, and we again have a contradiction to the minimality of i. Therefore every bad submatrix of A'_i contains row y_i and column x_i, as desired, and the proof of 3.1.1 is complete. □

By 3.1.1, any bad submatrix of A'_i contains row y_i and column x_i. Hence there is a bad submatrix in the matrix obtained from A'_i by pivoting on (y_i, x_i) and then deleting row x_i and column y_i. But this matrix is identical (up to scaling) to $A_i - \{x_i, y_i\}$. As the column labeled by y_{i-1} contains only a single non-zero entry in $A_i - \{x_i, y_i\}$, there is a bad submatrix in $A_i - \{x_i, y_i, y_{i-1}\}$.

$$A_i - \{x_i, y_i, y_{i-1}\} = \begin{array}{c} \\ \\ a \\ x_0 \\ x_1 \\ \vdots \\ x_{i-3} \\ x_{i-2} \\ x_{i-1} \end{array} \begin{array}{c|cccccccc} & b & y_0 & y_1 & \cdots & y_{i-4} & y_{i-3} & y_{i-2} \\ D & \mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{c}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{x}^T & 1 & 1 & 0 & & 0 & 0 & 0 \\ \mathbf{x}^T & 0 & 1 & 1 & & 0 & 0 & 0 \\ \vdots & \vdots & & & \ddots & & & \vdots \\ \mathbf{x}^T & 0 & 0 & 0 & & 1 & 1 & 0 \\ \mathbf{x}^T & 0 & 0 & 0 & & 0 & 1 & 1 \\ \mathbf{x}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}$$

Let Z be a bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$. Let us assume that Z avoids row x_{i-1}. After deleting x_{i-1}, the column y_{i-2} has a single non-zero entry. Therefore we can assume that Z avoids x_{i-1} and y_{i-2}. However, deleting x_{i-1} and y_{i-2} from $A_i - \{x_i, y_i, y_{i-1}\}$ produces a matrix that is identical to $A_{i-2} - y_{i-2}$. Therefore A_{i-2} contains a bad submatrix, and we have a contradiction to the minimality of i. Therefore any bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$ contains row x_{i-1}.

Let Z be a bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$, and assume that Z avoids row x_{i-2}. After deleting x_{i-2}, columns y_{i-2} and y_{i-3} contain single non-zero entries, so we can assume that Z avoids x_{i-2}, y_{i-2}, and y_{i-3}. This shows there is a bad submatrix that avoids y_{i-2} and y_{i-3}. However, after deleting y_{i-2}, and y_{i-3} from $A_i - \{x_i, y_i, y_{i-1}\}$, the rows x_{i-1} and x_{i-2} are identical, so there is a bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$ that does not contain x_{i-1}. This contradicts the conclusion of the previous paragraph. Now we know that any bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$ must contain x_{i-1} and x_{i-2}.

Let Z be a bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$, and assume that Z avoids column y_{i-3}. After deleting y_{i-3}, rows x_{i-1} and x_{i-2} are identical, and as Z must contain both these rows by the previous paragraph, it follows that the determinant of Z is zero. This contradiction shows that any bad submatrix of $A_i - \{x_i, y_i, y_{i-1}\}$ contains x_{i-1}, x_{i-2}, and y_{i-3}.

We pivot on (x_{i-2}, y_{i-3}), and then delete $\{y_{i-3}, x_{i-2}\}$. The previous paragraph shows that in the resulting matrix, A', there is a bad submatrix, Z, that contains row x_{i-1}.

$$A' = \begin{array}{c} \\ \\ a \\ x_0 \\ x_1 \\ \vdots \\ x_{i-4} \\ x_{i-3} \\ x_{i-1} \end{array} \left[\begin{array}{c|cccccccc} & b & y_0 & y_1 & \cdots & y_{i-5} & y_{i-4} & y_{i-2} \\ D & \mathbf{b} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{c}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{x}^T & 1 & 1 & 0 & & 0 & 0 & 0 \\ \mathbf{x}^T & 0 & 1 & 1 & & 0 & 0 & 0 \\ \vdots & \vdots & & & \ddots & & & \vdots \\ \mathbf{x}^T & 0 & 0 & 0 & & 1 & 1 & 0 \\ \mathbf{0}^T & 0 & 0 & 0 & & 0 & 1 & -1 \\ \mathbf{x}^T & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

Assume Z avoids y_{i-2}. After deleting y_{i-2}, the row x_{i-3} contains a single non-zero entry. Therefore we can assume that Z avoids y_{i-2} and x_{i-3}. After deleting y_{i-2} and x_{i-3} from A', the column y_{i-4} contains only a single non-zero entry. Therefore we can assume that Z contains x_{i-1}, but avoids y_{i-2}, x_{i-3}, and y_{i-4}. It is not too difficult to see that deleting y_{i-2}, x_{i-3}, and y_{i-4} from A' produces a matrix that is identical to $A_{i-2} - \{y_{i-2}, y_{i-3}, y_{i-4}, x_{i-2}\}$. Therefore A_{i-2} contains a bad submatrix, and we have a contradiction. It follows that Z, the bad submatrix of A' that contains x_{i-1}, also contains y_{i-2}.

Now we know there is a bad submatrix in the matrix obtained from A' by pivoting on (x_{i-1}, y_{i-2}), and then deleting $\{y_{i-2}, x_{i-1}\}$. But this matrix is identical to $A_{i-2} - \{y_{i-2}, y_{i-3}, x_{i-2}\}$, so we have a final contradiction to the minimality of i that completes the proof of Lemma 3.1. □

For any non-negative integer i, we let $M_i = M[I|A_i]$, so that M_1 is the matroid M from the statement of Theorem 1.3. By pivoting on (x_i, y_i), and deleting row y_i and column x_i from the resulting matrix, we discover the following relation.

Proposition 3.2. *If* $i > 1$, *then* $M_i/y_i \backslash x_i = M_{i-1}$.

A 4-*fan* of a matroid is a sequence, $(\alpha, \beta, \gamma, \delta)$, of distinct elements such that $\{\alpha, \beta, \gamma\}$ is a triangle and $\{\beta, \gamma, \delta\}$ is a triad.

Proposition 3.3. M_1 *contains no* 4-*fan.*

Proof. Assume that $(\alpha, \beta, \gamma, \delta)$ is a 4-fan. Note that $\{\alpha, \beta, \gamma, \delta\}$ has rank/corank at least two, and at most equal to three. If the rank or corank of $\{\alpha, \beta, \gamma, \delta\}$ is two, then $\lambda_{M_1}(\{\alpha, \beta, \gamma, \delta\}) \leq 1$, and the fact that M_1 is 3-connected means that M_1 contains at most five elements, contradicting statement (iii) in the hypotheses of Theorem 1.3. Therefore $r_{M_1}(\{\alpha, \beta, \gamma, \delta\}) = r^*_{M_1}(\{\alpha, \beta, \gamma, \delta\}) = 3$, and $\lambda_{M_1}(\{\alpha, \beta, \gamma, \delta\}) = 2$.

Let C be the complement of $\{\alpha, \beta, \gamma, \delta\}$ in M_1. Because M_1 is fused it follows that C has rank or corank at most two. As

$$r(M_1) = r_{M_1}(C) + |\{\alpha,\beta,\gamma,\delta\}| - r^*_{M_1}(\{\alpha,\beta,\gamma,\delta\}) \quad \text{and}$$
$$r^*(M_1) = r^*_{M_1}(C) + |\{\alpha,\beta,\gamma,\delta\}| - r_{M_1}(\{\alpha,\beta,\gamma,\delta\})$$

this means that M_1 has rank or corank at most three. In either case we have a contradiction to the hypotheses of Theorem 1.3. □

Corollary 3.4. *Both* $\mathbf{a}^T$ *and* $\mathbf{x}^T$ *contain non-zero entries.*

Proof. If $\mathbf{a}^T$ is everywhere zero, then $\{a, y_1\}$ is a series pair in M_1, contradicting 3-connectivity. If $\mathbf{x}^T$ is everywhere zero, then (a, y_1, x_1, y_0) is a 4-fan of M_1, contradicting Proposition 3.3. □

Let d be a column label of A_1 such that d is not equal to b, y_0, or y_1, and $A_1[x_0, d]$ is non-zero. Such a d exists by Corollary 3.4.

Lemma 3.5. *If* i *is a non-negative integer, then* M_i *is* 3*-connected.*

Proof. Assume the lemma fails, and that i is the least non-negative integer such that M_i is not 3-connected. The hypotheses of Theorem 1.3 imply that $i \geq 2$.

The choice of i means that $M_{i-1} = M_i/y_i\backslash x_i$ is 3-connected. Assume that $M_i\backslash x_i$ is not 3-connected. Then y_i is either a coloop in $M_i\backslash x_i$, or is contained in a series pair [6, Proposition 8.2.7]. Now $M_i\backslash x_i = M[I|A_i^{x_iy_i} - x_i]$, and $A_i^{x_iy_i} - x_i$ is obtained from A_{i-1} by scaling, and adding the row $[\mathbf{x}^T\ 0\ \cdots\ 0\ 1]$. The new row is labeled y_i, and as this row contains a non-zero entry, y_i is not a coloop in $M_i\backslash x_i$. Therefore y_i is contained in a series pair.

Corollary 3.4 implies that the row labeled by y_i contains at least two non-zero entries in $A_i^{x_iy_i} - x_i$. This means that y_i is not in a series pair of $M_i\backslash x_i$ with any element that labels a column of $A_i^{x_iy_i} - x_i$, so y_i is in a series pair with an element that labels a row. Let z be this element. By examining $A_i^{x_iy_i} - x_i$, we see that $\{a, x_{i-1}, y_{i-1}, y_i\}$ is a circuit of $M_i\backslash x_i$. Now orthogonality with the series pair $\{y_i, z\}$ implies that z is in $\{a, x_{i-1}, y_{i-1}\}$. As z labels a row, it is equal to either a or x_{i-1}. If $\{y_i, x_{i-1}\}$ is a series pair in $M_i\backslash x_i$, then we contradict orthogonality with the circuit $\{x_{i-2}, x_{i-1}, y_{i-2}\}$. Therefore $\{y_i, a\}$ is a series pair of $M_i\backslash x_i$. This means that we can scale so that the rows labeled by y_i and a are identical. If $i-1$ is odd, then we deduce that $\mathbf{a}^T = \mathbf{x}^T$, which implies that $\{a, y_0\}$ is a series pair in M_0. If $i-1$ is even, then $\mathbf{x}^T - \mathbf{a}^T = \mathbf{x}^T$, contradicting Corollary 3.4. In either case we have a contradiction, so we conclude that $M_i\backslash x_i$ is 3-connected.

Since M_i is not 3-connected, it follows that x_i is either a loop, or is in a parallel pair in M_i. The former is impossible, as x_i labels a row in A_i. Hence there is a column of A_i that contains a non-zero entry only in row x_i. This column cannot be in $\{b, y_0, \ldots, y_i\}$, and now we have a contradiction, since if the column is non-zero in row x_i, then it is also non-zero in row x_{i-1}. □

We complete the proof of Theorem 1.3 by establishing the next lemma.

Lemma 3.6. *If $i \geq 1$ is an odd integer, then M_i is fused.*

Before we begin the proof of Lemma 3.6, we note that M_i may not be fused when i is even, even if M_0 and M_1 are both fused. For example, if A_1 is the following matrix over GF(7):

$$\begin{array}{c} \\ c \\ a \\ x_0 \\ x_1 \end{array} \begin{array}{c} \begin{array}{cc|ccc} u & v & b & y_0 & y_1 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

then we can verify that M_0 and M_1 are fused, but $(\{a, c, u, x_2, y_2\}, \{x_0, x_1, v, b, y_0, y_1\})$ is a 3-separation certifying that M_2 is not fused.

Proof of Lemma 3.6. Assume the lemma fails. Let $i \geq 1$ be the least odd integer such that M_i is not fused. By hypothesis M_1 is fused, so $i \geq 3$.

Note that $\{a, x_i\}$ is a parallel pair in M_i/y_i. Since $M_i/y_i \backslash x_i = M_{i-1}$ is 3-connected by Lemma 3.5, it follows that the only 2-separation of M_i/y_i consists of $\{a, x_i\}$ and its complement.

3.6.1. *If (U, V) is a 3-separation of M_i/y_i, then either*

$$\min\{r_{M_i/y_i}(U), r_{M_i/y_i}(V)\} \leq 2 \quad \textit{or} \quad \min\{r^*_{M_i/y_i}(U), r^*_{M_i/y_i}(V)\} \leq 2.$$

Proof. We assume (U, V) is a 3-separation of M_i/y_i that does not satisfy the conditions of 3.6.1. First we claim that we can assume that either U or V contains $\{a, x_i\}$. Without loss of generality we assume that x_i is in U. If a is in U, then we are done, so assume $a \in V$.

Assume that $|V| = 3$. Then $|U| > 3$, so $(U - x_i, V)$ is a 3-separation of the 3-connected matroid $M_i/y_i \backslash x_i$. Therefore V is a triangle or a triad in $M_i/y_i \backslash x_i$. If V is a triangle, then $r_{M_i/y_i}(V) \leq 2$, contrary to hypothesis. Therefore V is a triad in $M_i/y_i \backslash x_i$. It is not a triad in M_i/y_i, or else $r^*_{M_i/y_i}(V) \leq 2$. Therefore $V \cup x_i$ is a cocircuit of M_i/y_i, so $x_i \in \mathrm{cl}^*_{M_i/y_i}(V)$. Then x_i is also in $\mathrm{cl}^*_{M_i/y_i}(U - x_i)$, or else $(U - x_i, V \cup x_i)$ is a 2-separation in M/y_i, and this is impossible as neither $U - x_i$ nor $V \cup x_i$ is equal to $\{a, x_i\}$. However, x_i cannot be in $\mathrm{cl}^*_{M_i/y_i}(U - x_i)$, or else we violate orthogonality with the circuit $\{a, x_i\}$. This contradiction implies that $|V| > 3$.

Now $|V - a| \geq 3$, so $(U \cup a, V - a)$ is a 3-separation of M/y_i. If both sides of this separation have rank and corank at least equal to three, then our claim is justified, so we assume this is not the case. Therefore $V - a$ has rank or corank at most equal to two. In fact, a cannot be contained in a cocircuit that is contained in V, by orthogonality with $\{a, x_i\}$. This means that $r^*_{M_i/y_i}(V - a) < r^*_{M_i/y_i}(V)$. It cannot be the case that $r_{M_i/y_i}(V - a) < r_{M_i/y_i}(V)$, for that would imply that $(U \cup a, V - a)$ is a 2-separation of M_i/y_i. Since neither side of $(U \cup a, V - a)$ is equal to $\{a, x_i\}$, this is impossible. From this we conclude that $r^*_{M_i/y_i}(V) = 3$, and $r^*_{M_i/y_i}(V-a) = 2$. However, we can apply symmetric arguments to U, and deduce that $r^*_{M_i/y_i}(U) = 3$. Because (U, V) is a 3-separation of M_i/y_i, this means that $4 = r^*(M_i/y_i) = r^*(M_i)$. As $r^*(M_1) > 3$ and $r^*(M_i) = r^*(M_{i-1}) + 1$ for every i, we see that $i = 1$, contradicting the earlier statement that $i \geq 3$.

Now we can assume, as we claimed, that (U, V) is a 3-separation of M_i/y_i that fails the conditions of 3.6.1, and that a and x_i are in U. Because $M_{i-1} = M_i/y_i\backslash x_i$ is fused, it follows that

$$\min\{r_{M_i/y_i\backslash x_i}(U - x_i), r_{M_i/y_i\backslash x_i}(V)\} \leq 2 \quad \text{or} \quad \min\{r^*_{M_i/y_i\backslash x_i}(U - x_i), r^*_{M_i/y_i\backslash x_i}(V)\} \leq 2.$$

Certainly the rank of V in $M_i/y_i\backslash x_i$ is identical to its rank in M_i/y_i. Moreover, there can be no cocircuit of M_i/y_i contained in $V \cup x_i$ that contains x_i, by orthogonality with $\{a, x_i\}$. This means that the corank of V in $M_i/y_i\backslash x_i$ is identical to its corank in M_i/y_i. The parallel pair $\{a, x_i\}$ tells us that $r_{M_i/y_i\backslash x_i}(U - x_i) = r_{M_i/y_i}(U)$. Therefore we conclude that $r^*_{M_i/y_i}(U) = 3$ and $r^*_{M_i/y_i\backslash x_i}(U - x_i) = 2$.

If $|U| = 3$, then $r_{M_i/y_i}(U) \leq 2$ because of the parallel pair $\{a, x_i\}$. This is a contradiction, so $|U - x_i| \geq 3$. As $M_i/y_i\backslash x_i = M_{i-1}$ is 3-connected, and $r^*_{M_i/y_i\backslash x_i}(U - x_i) = 2$, it follows that a is contained in a triad, T^*, in M_{i-1}. However, a is also contained in the triangle $T = \{a, x_{i-1}, y_{i-1}\}$. Now T^* and T must intersect in at least two elements, by orthogonality, and they must intersect in no more than two elements, or else $T = T^*$ is 2-separating in M_{i-1}. Thus M_{i-1} has a 4-fan. As M_{i-1} is fused, we can use exactly the same arguments as in Proposition 3.3 to show that M_{i-1} has rank or corank at most equal to three, and we have a contradiction. □

We let (U, V) be a 3-separation of M_i such that the rank/corank of U and V are at least equal to three. We assume that y_i is in U.

3.6.2. $y_i \notin \mathrm{cl}_{M_i}(V)$.

Proof. If y_i is in $\mathrm{cl}_{M_i}(V)$, then $(U - y_i, V)$ is a 2-separation in M_i/y_i. From Lemma 3.5 it follows that either $U - y_i$ or V is the parallel pair $\{a, x_i\}$. But $|V| \geq 3$, so $U = \{a, x_i, y_i\}$, and hence $r_{M_i}(U) = 2$, a contradiction. □

If $|U| = 3$, then U is a circuit or cocircuit in M_i, which means that it has rank or corank at most two. This contradiction implies $|U| \geq 4$, so $(U - y_i, V)$ is a 3-separation in M_i/y_i. We apply 3.6.1 to this separation.

Evidently $r^*_{M_i/y_i}(V) \geq 3$. From 3.6.2 we see that $r_{M_i/y_i}(V) \geq 3$. If

$$r^*_{M_i/y_i}(U - y_i) \leq 2 < r^*_{M_i}(U),$$

then y_i is not in $\mathrm{cl}^*_{M_i}(U - y_i)$. This is equivalent to y_i being in $\mathrm{cl}_{M_i}(V)$, contradicting 3.6.2. The only remaining possibility is that $r_{M_i/y_i}(U-y_i) \leq 2$. We conclude that $r_{M_i}(U) = 3$.

As (U, V) is an exact 3-separation of M_i, it follows that $r_{M_i}(V) = r(M_i) - 1$. Let $C^* \subseteq U$ be the cocircuit whose complement is $\mathrm{cl}_{M_i}(V)$. Then 3.6.2 implies y_i is in C^*.

3.6.3. $x_i \notin C^*$.

Proof. Assume that x_i is in C^*. Because $\{x_i, x_{i-1}, y_{i-1}\}$ is a triangle of M_i, orthogonality requires that x_{i-1} or y_{i-1} is in C^*, so C^*, and hence U, contains either $\{y_i, x_i, x_{i-1}\}$ or $\{y_i, x_i, y_{i-1}\}$. Both of these sets have rank three, and $r_{M_i}(U) = 3$. Therefore U is spanned by either $\{y_i, x_i, x_{i-1}\}$ or $\{y_i, x_i, y_{i-1}\}$ in M_i. Inspection of the matrix A_i shows that $\mathrm{cl}_{M_i}(\{a, x_i, x_{i-1}, y_{i-1}\})$ is $\{a, x_i, x_{i-1}, y_i, y_{i-1}\}$. From this it follows that $\{a, x_i, x_{i-1}, y_i, y_{i-1}\}$ contains U.

Because $\{x_{i-1}, x_{i-2}, y_{i-2}\}$ is a triangle, orthogonality requires that x_{i-1} is not in C^*. Therefore C^* is either $\{y_i, x_i, y_{i-1}\}$ or $\{y_i, x_i, y_{i-1}, a\}$. Assume that $\{y_i, x_i, y_{i-1}\}$ is a cocircuit. Inspection of A_i shows that this means $\mathbf{x}^T$ contains only zero entries, contradicting Corollary 3.4. Therefore $\{y_i, x_i, y_{i-1}, a\}$ is a cocircuit. Recall that i is odd. Corollary 3.4 asserts that $\mathbf{a}^T$ and $\mathbf{x}^T$ contain non-zero entries. Because every basis must intersect $\{y_i, x_i, y_{i-1}, a\}$, every 2×2 submatrix with rows labeled by a and x_i that contains neither column y_{i-1} nor column y_i has a zero determinant. It follows that $\mathbf{a}^T$ and $\mathbf{x}^T$ are identical (up to scaling by an element of G). This implies that $\{a, x_1, y_0\}$ is a triad in M_1, so (x_0, x_1, y_0, a) is a 4-fan, and we have a contradiction to Proposition 3.3. □

By applying 3.6.3 and orthogonality with the triangle $\{a, x_i, y_i\}$, we now see that C^* contains y_i and a.

3.6.4. $r_{M_i}(C^* - y_i) = 2$.

Proof. We start by showing that $C^* \cap \{x_i, x_{i-1}, y_{i-2}\} = \emptyset$. We know from 3.6.3 that $x_i \notin C^*$. If C^* contains x_{i-1}, then orthogonality with the triangle $\{x_{i-1}, x_{i-2}, y_{i-2}\}$ means that C^* contains either $\{a, y_i, x_{i-1}, x_{i-2}\}$ or $\{a, y_i, x_{i-1}, y_{i-2}\}$. Both of these sets are independent, so we contradict the fact that $r_{M_i}(C^*) \leq r_{M_i}(U) = 3$. Now assume that y_{i-2} is in C^*. Orthogonality with $\{x_{i-1}, x_{i-2}, y_{i-2}\}$, and the previous paragraph, tells us that x_{i-2} and y_{i-2} are in C^*. But this again leads to the contradiction that $r_{M_i}(C^*) \geq 4$. Therefore $C^* \cap \{x_i, x_{i-1}, y_{i-2}\} = \emptyset$, as claimed.

Certainly the rank of $C^* - y_i$ is at most three, since $C^* \subseteq U$. Because $C^* - y_i$ contains a, its rank is at least one. If $r_{M_i}(C^* - y_i) = 1$, then $C^* = \{y_i, a\}$. But this is a contradiction, as M_i is 3-connected, and therefore has no series pairs. Hence $r_{M_i}(C^* - y_i) \geq 2$. We assume for a contradiction that $r_{M_i}(C^* - y_i) = 3$. Let Z be a basis of $C^* - y_i$. Then Z spans U, so there is a circuit contained in $Z \cup y_i$ that contains y_i. This circuit violates orthogonality with the cocircuit $\{y_i, x_i, x_{i-1}, y_{i-2}\}$. □

Now we let z be an arbitrary element in $C^* - \{y_i, a\}$. By 3.6.4, $\{a, z\}$ is a basis of $C^* - y_i$.

First we assume that z is in $\{x_0, \dots, x_{i-1}\}$. By inspection of the matrix A_i, $\{a, z\}$ is not contained in a triangle that is contained in $C^* - y_i$. Thus $C^* - y_i$ contains no triangle, so 3.6.4 implies $C^* = \{y_i, a, z\}$. By Corollary 3.4, both $\mathbf{a}^T$ and $\mathbf{x}^T$ contain non-zero entries. As row z contains a non-zero entry in a column where row a has a zero entry, we can find a basis of M_i that does not intersect $\{y_i, a, z\}$. This contradiction, and 3.6.3, means that z is not in $\{x_0, \dots, x_i\}$.

Now we assume that z is in $\{b, y_0, \dots, y_{i-1}\}$. By the previous paragraph, $C^* - \{y_i, a\}$ does not contain any element in $\{x_0, \dots, x_i\}$. From this, and inspection of A_i, we can easily see that if there is a triangle of M_i containing $\{a, z\}$ that is contained in $C^* - y_i$, then the third element of the triangle labels a column of D. This third element labels a column that is constant in the rows $\{x_0, \dots, x_i\}$, while z labels a column that is not constant in these rows, unless $z = y_0$ and $i = 1$. But we have already noted that $i \geq 3$, so this is impossible. Therefore no such triangle exists, so 3.6.4 implies that $C^* = \{a, z, y_i\}$ is a cocircuit. This implies that the row labeled by a is zero everywhere except in the columns labeled by y_i and z. Thus $\mathbf{a}^T$ is everywhere zero, and we have a contradiction to Corollary 3.4. We have shown that the elements in $C^* - \{y_i, a\}$ all label rows and columns in D.

Assume that there is an element, z, in $C^* - \{a, y_i\}$ that labels a row of the matrix D. As $\{a, z\}$ is a basis of $C^* - y_i$, every element in $C^* - \{y_i, a, z\}$ labels a column of D. Any such column is non-zero only in the rows labeled by a and z. Assume that row a is non-zero only in columns labeled by elements of $C^* - \{a, z\}$. Then every basis of M_i intersects $C^* - z$, which contradicts the fact that this is a proper subset of a cocircuit. Therefore row a is non-zero in a column that is not labeled by an element of $C^* - \{a, z\}$. The same statement can be made about row z. As every basis intersects C^*, we see that any 2×2 submatrix with rows labeled by a and z has zero determinant, unless it has a column labeled by an element in $C^* - \{a, z\}$. This means that we can scale in such a way that row a and row z are identical in columns that are not labeled by elements in $C^* - \{a, z\}$. Now we can see that $(C^* - y_i) \cup y_1$ is a rank-three cocircuit in M_1.

Next assume that no element in $C^* - \{a, y_i\}$ labels a row of the matrix D, so that every such element labels a column of D. Then row a is zero everywhere except in columns labeled by elements in $C^* - a$. Therefore $(C^* - y_i) \cup y_1$ is a cocircuit in M_1. Any two elements in $C^* - \{a, y_i\}$ form a triangle of M_i with a. The columns they label must have non-zero elements other than in row a, or else M_i contains a parallel pair. By scaling, we can assume that any two such columns are identical, except in row a. This implies that $r_{M_1}(C^* - y_i) = 2$. Therefore $(C^* - y_i) \cup y_1$ is a rank-three cocircuit in M_1 in any case.

If $C^* = \{a, z, y_i\}$, then $\{a, z, y_1\}$ is a triad in M_1, and $\{a, y_1, x_1\}$ is a triangle. This gives us a contradiction to Proposition 3.3, so C^* contains at least four elements. Therefore $(C^* - y_i) \cup y_1$ is a cocircuit of M_1 with rank three and corank at least three, and $\lambda_{M_1}((C^* - y_i) \cup y_1) = 2$. Let

H be the complement of $(C^* - y_i) \cup y_1$ in M_1. As M_1 is fused, it follows that $r_{M_1}(H) \leq 2$ or $r^*_{M_1}(H) \leq 2$. In the former case, $r(M_1) \leq 3$, since H is a hyperplane in M_1. Therefore $r^*_{M_1}(H) \leq 2$. As $\{x_0, x_1, y_0\} \subseteq H$, this means that $\{x_0, x_1, y_0\}$ is a triangle and a triad in M_1, so it is 2-separating. This quickly leads to a contradiction, so we have completed the proof of Lemma 3.6, and hence the proof of Theorem 1.3. □

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