Discrete Differential Geometry: The Non-Planar Quadrilateral Mesh.

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We consider the problem of constructing a discrete differential geometry defined on *non-planar* quadrilateral meshes. Physical models on discrete non-flat spaces are of inherent interest, as well as being used in applications such as computation for electromagnetism, fluid mechanics, and image analysis. However, the majority of analysis has focused on triangulated meshes. We consider two approaches: discretizing the tensor calculus, and a discrete, mesh version of differential forms. Whilst these two approaches are equivalent in the continuum, we show that this is not true in the discrete case. Nevertheless, we show that it is possible to construct mesh versions of the Levi-Civita connection (and hence the tensorial covariant derivative and the associated covariant exterior derivative), the torsion, and the curvature. We show how discrete analogs of the usual vector integral theorems are constructed, in such a way that the appropriate conservation laws hold exactly on the mesh, rather than only as approximations to the continuum limit. We demonstrate the success of our method by constructing a mesh version of classical electromagnetism, and discuss how our formalism could be used to deal with other physical models, such as fluids.

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I. INTRODUCTION

Physical theories on discrete spaces have a long history, and can be viewed in various ways: as simplified models, as computational devices, or as fundamental theories in their own right. For example, in solid state physics, the discrete and periodic nature of the crystal lattice is an obvious candidate for a discrete model, where the model can help elucidate the structure of the actual continuum theory, and help us focus on those aspects of the physics that depend on the underlying periodic structure (e.g., the Ising model as a model of ferromagnetism). In quantum field theory, many continuum theories are computationally intractable, leading to discretized versions of such theories on spacetime lattices (e.g., lattice gauge theories), which allow computer simulations to be performed, and physical quantities, such as particle masses[11], to be computed. Finally, we also have the possibility that spacetime itself may not be continuous, and that our fundamental physical theories need to be formulated in terms of some sort of discrete structure for spacetime at the Planck scale (e.g., causal set theory[3]).

If we consider the simplest case of classical physical theories on a discrete space or spacetime, then such theories are employed in various contexts: applications that consider the numerical simulation of physical theories per se (e.g., fluids[18], or electromagnetism[30]), and applications that utilize simplified, discrete models of physical theories as approaches to other problems. An example of the latter is the use of elasticity to allow for physically-realistic simulations of deformable objects when represented as meshes for computer graphics purposes, or the analysis of movies of deforming objects (e.g., see Grinspun *et al.*[12]).

However, there are also less obvious applications of discrete physical theories, where a discrete version of a physical model is used to accomplish some other computational task. A prime example of this is the task of image-matching via non-rigid registration. In this application, images are deformed diffeomorphically so that their appearance more closely matches a reference image, with the aim of assisting in tasks such as disease diagnosis from medical images. As part of the approach, an elastic[34] or a fluid[6] model is used to regularise the deformations. Similar, physically-based models can be used to solve the matching problem for object surfaces which, unlike images, are not flat spaces[7]. Other applications which use and manipulate discrete models of surfaces use methods from differential geometry directly. For example, in the field of medical image analysis, one important surface studied is the cortical surface of the human brain. Recent papers have applied various methods to such surfaces, for example, the Laplace-Beltrami operator[1], spherical conformal mapping[5], harmonic energy minimisation[13], Ricci flow[17], and harmonic and holomorphic one-forms[31].

When applying methods from differential geometry to such discrete surfaces, or when attempting to construct physical models on such surfaces, we have two main options. The first option is that we take the continuum mathematical tools that have the properties we desire for

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our algorithm, and try to construct a discrete approximation to the continuum formulation (e.g., see Meyer *et al.*[21] on averaged curvature operators for triangulated surfaces). This has the problem that the very properties we desire to retain, or the quantities we desire to compute, may be only approximated by our discretization procedure. The second option is to seek a discrete theory that admits parallels to that in the continuum, but that is inherently self-consistent, and where the desired properties are retained exactly. It is this second option – which has already been championed by Hirani et al. [8, 9, 14], and by Stern[28] – that we prefer, and that we follow in this paper.

Let us now briefly consider various ways to produce a discretized version of a 2D surface. If we take a set of sample points on the surface, and then perform bilinear interpolation of position between these points, this gives a triangulated mesh, with the associated piecewise-flat continuous approximation to the original surface. However, as noted by Desbrun *et al.*[8] (see their Section **15**), the intrinsic notion of what constitutes the tangent space at a vertex of such a non-flat mesh was then an open question, since the tangent spaces at the sample points have been excluded from the construction. We could instead proceed by taking the tangent planes to the sample points. The intersections of these tangent planes then define the links and vertices of a planar polyhedral mesh, which can then be triangulated without introducing any additional vertices. However, the original sample tangent planes now become associated with the cells of this mesh, hence the question as to the intrinsic tangent plane at a vertex still remains. For example, Desbrun *et al.*[8], using simplicial complexes, only defined discrete tangent vector fields for flat meshes (see their Remark 7.1), whereas Schwalm *et al.*[27] defined a vector-difference calculus on a general triangulated mesh, but were restricted to a topological approach, so that, as they noted (see end of their Section **V**), tangent spaces at adjacent points overlapped, but not completely.

However, consider the following alternative construction. Imagine a set of points, where at each point we have a pair of vectors (defined in terms of their lengths and the angle between them), which are a basis for the 2D tangent space at that point. If we now stitch together the set of points, using just the basis vectors, it can be seen that this naturally leads to the idea of a non-planar quadrilateral mesh, where the links of the mesh are now associated with the elements of the tangent spaces at the various vertices. It should be noted that for arbitrary length basis vectors, these cells will not in general close, and the condition that all cells do close, and the consequences of this, will be discussed in the sections where we construct the Levi-Civita connection.

Such quadrilateral meshes are of growing interest in several fields because they admit a coordinate space that is just the square grid, and interpolation on the square grid is much quicker than the barycentric interpolation required when working with triangulated meshes. Such ease of interpolation is important when we are considering applications such as surface matching, where we need to repeatedly compare quantities at corresponding points for two different surfaces, and this exact point was recently noted by Yeo *et al.*[36] for the case of cortical surface matching and coordinate charts on a sphere. As such, there exist various methods in the computer graphics literature for generating quadrangulations from a triangulated mesh (e.g., see Tong *et al.*[29] and the references therein).

Within the framework of seeking a discrete theory directly we consider two different approaches to the problem: by producing a discrete version of tensor calculus, and by producing a version of the discrete exterior calculus for quadrilateral meshes. While these would be equivalent in the continuum, we show that this is not the case here, and we show that, for example, the divergence of a vector field is different depending whether it is calculated using the covariant derivative of tensor calculus or the exterior derivative and Hodge star of differential forms. A description of the basic operators of discrete differential geometry are provided in an appendix.

In Section **II**, we introduce a discrete tensor calculus and work towards definitions of torsion and curvature. We show that on the quadrilateral mesh this leads to difficulties, since the construction of a torsion-free, Levi-Civita connection is hampered by the fact that the usual methods for constructing the torsion tensor led to an object is not a well-formed tensor on the mesh. We instead use a simple geometric argument to construct a suitable connection, and hence a tensorial covariant derivative.

In Section **III**, we then switch to using difference forms (the discrete equivalent of the differential form) and produce a version of the discrete exterior calculus, including the Hodge star and codifferential operators. This leads to a well-defined version of Stokes' theorem, and the recovery of Cartan's structure equations for torsion and curvature, as well as making clear the link between the connection we defined using a geometric argument, and the mesh equivalent of a torsion-free Levi-Civita connection.

In Section **IV** we use the discrete exterior calculus version of our theory and apply it to electrodynamics in 2+1 dimensions. We show that by requiring that we retain the identity of electromagnetism as a field theory with a $U(1)$ local gauge invariance, we obtain a mesh version of Maxwell's equations, whilst also retaining the usual links between electric currents and charges, and between these sources and electric and magnetic fluxes. We conclude with a brief discussion of how to apply the tools we have developed to the construction of mesh versions of other physical theories, such as fluids.

II. DISCRETE TENSOR CALCULUS (DTC)

We start by investigating how far an approach based on producing a tensor calculus on the mesh can take us.

Our approach is to seek mesh versions of the common objects of tensor calculus and investigate their equivalence in the naïve continuum limit to the continuous case. We work on a non-planar quadrilateral mesh $\mathcal{M} \subset \mathbb{R}^3$ with the same connectivity as a regular square grid in $\mathbb{R}^2$. Vertices on this mesh are labelled by i and edges by μ, ν so that from a site i we can travel along a (directed) edge in a direction μ to arrive at site $i+\mu$, and along a different edge in direction ν to arrive at site $i+\nu$. We will also define a coordinate chart on our mesh, which we take to be a regular square mesh of edge length 1 unit. We define a scalar function on the mesh at each vertex $\{f(i) : i \in \mathcal{M}\}$ and define difference operators in the usual way. The basic definitions of discrete differential geometry are given in the appendix

A. The Gradient Theorem

We begin by seeking a mesh version of the gradient theorem, which is the generalization of the fundamental theorem of calculus to curves, and says that if we integrate grad f along a path $\gamma(t)$, we find that:

$$\int_{\gamma(0)}^{\gamma(1)} \operatorname{grad} f \cdot \underline{dl} \;\equiv f(\gamma(1)) - f(\gamma(0)), \tag{1}$$

where $\underline{dl}$ represents an infinitesimal element of the path and the gradient operator is a first-order derivative operator, that when applied to a scalar field produces a vector. For a mesh, the continuum path is replaced by the set of individual directed links that form a path. However, we first have to define grad on the mesh, which we do as:

$$\operatorname{grad} f(i) \triangleq g^{\mu\nu}(i) d_\nu f(i) \underline{e}_{\;\mu}(i). \tag{2}$$

The gradient of a function hence lies in the tangent space V_i, and the scalar product of the gradient with an arbitrary vector field X is given by:

$$\langle \operatorname{grad} f, X \rangle \equiv X(f). \tag{3}$$

Now, by considering a path on the mesh as a set of individual connected links, we see that the path integral is a sum over contributions that each have the form (in this case, for the single link $i \rightarrow i + \underline{e}_{\;\alpha}(i)$):

$$\operatorname{grad} f(i) \cdot \underline{e}_{\;\alpha}(i) = g^{\mu\nu}(i) d_\nu f(i) \underline{e}_{\;\mu}(i) \cdot \underline{e}_{\;\alpha}(i)$$
$$= g^{\mu\nu}(i) d_\nu f(i) g_{\mu\alpha}(i) = \delta^\nu_\alpha d_\nu f(i) = f(i+\alpha) - f(i),$$

which gives the gradient theorem on the mesh.

The gradient theorem is a special case of Stokes' Theorem when applied to 0-forms (functions), and we will consider the full theorem in Section **III D** when we move onto forms. The importance of such theorems in the continuum is that they allows us to move between an integral form of the dynamics of a field to a differential equation form. To do this on the mesh we will need to consider a general version of a mesh derivative that is defined on vectors and general tensors (the covariant derivative). This is non-trivial, as tangent vectors at neighbouring vertices lie in different vector spaces ($V_{i+\mu}$ and V_i) and so we need a connection to transform between vectors defined in these two spaces.

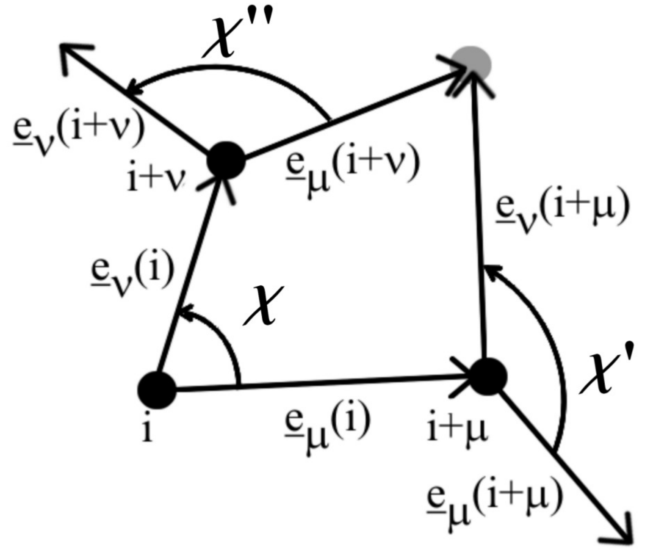

FIG. 1. A diagram showing how the quadrilateral formed by the four points shown can be flattened to a quadrilateral in the plane, without altering the metric (lengths and angles shown) at the 3 black points.

B. The Covariant Derivative

One approach to finding the covariant derivative on the mesh would be to proceed algebraically, and to derive a mesh version of the Levi-Civita connection by first defining torsion and requiring that it vanishes (since the fundamental theorem of Riemannian geometry states that there is a unique torsion-free connection that preserves the metric). However, the definition of torsion on the mesh seems to preclude this (see Section **II C**).

Instead, we adopt the approach of Leok *et al.*[19] (see very end of their Section **7**), where they considered constructing the curvature for the simplicial complexes used by Desbrun *et al.*[8]. This is to use local embeddings of the mesh into Euclidean space, and is a simple geometric construction (see Fig. 1), which involves a local flattening of the mesh. Unlike the case of triangulated meshes, there are two ways to flatten a non-planar quadrilateral cell, and we chose the one that corresponds to a convex quadrilateral when flattened, since otherwise we would have coincident vertices if we flattened a square cell. This flattening is metric preserving at the three vertices i, $i+\mu$ and $i+\nu$, since the lengths and angles between the basis vectors at each point are preserved. Once we have performed this flattening, the tangent spaces are now equivalent, and we can move vectors between tangent spaces by just translating in the plane. This now also elucidates the construction of the quadrilateral mesh that was discussed in the Introduction, since it is only for particular sets of basis vectors (i.e., those which can be formed into closed cells) that this construction defines a unique connection. Since identical sets of basis vectors always form closed cells, this closure constraint can be seen as a discrete version of the usual continuum notions of continuity and differentiability of the metric.

Consider a tangent vector X at $i+\mu$, parallel-transported to i. We define the ν^{th} component of the transported vector, in the basis defined at i, to be:

$$B^{(\mu)}_{\nu\alpha}(i)X^{\alpha}(i+\mu) \tag{4}$$
(no sum on μ, indicated by bracketed index),

where $\mathbf{B}^{(\mu)}(i)$ is a matrix corresponding to the change of basis vectors in the plane (The elements of this matrix can be computed from the lengths and angles given in the diagram using simple geometry). In terms of the flattened basis vectors and the metric, we have:

$$\begin{aligned} &B^{(\mu)}_{\nu\alpha}(i) \triangleq \underline{e}^{\ \nu}(i)\cdot\underline{e}_{\ \alpha}(i+\mu) \\ &\Rightarrow \mathbf{g}(i+\mu) = (\mathbf{B}^{(\mu)}(i))^T\mathbf{g}(i)\mathbf{B}^{(\mu)}(i). \end{aligned} \tag{5}$$

Since the matrix $\mathbf{B}^{(\mu)}(i)$ is based on parallel transport, it makes sense to rewrite it in terms of the Christoffel symbols $\{\Gamma^{\nu}_{\mu\alpha}(i)\}$:

$$\mathbf{B}^{(\mu)}(i) \triangleq \mathbb{I} + \mathbf{\Gamma}^{(\mu)}(i), \tag{6}$$

where $(\mathbf{\Gamma}^{(\mu)}(i))_{\alpha\beta} \triangleq \Gamma^{\alpha}_{\mu\beta}(i)$.

The covariant derivative D_μ is defined in terms of the difference between the vectors based at vertex i:

$$(D_\mu X)^{\nu}(i) \triangleq B^{(\mu)}_{\nu\alpha}(i)X^{\alpha}(i+\mu) - X^{\nu}(i). \tag{7}$$
$$\equiv d_\mu X^{\nu}(i) + \Gamma^{\nu}_{\mu\alpha}(i)X^{\alpha}(i+\mu). \tag{8}$$

In this final form, this mesh covariant derivative corresponds to the continuum covariant derivative.

Using the same geometric construction, it is possible to extend this covariant derivative to act on tensors of type $(m,0)$ by defining the parallel transport of a product of terms as the product of the parallel transport of the individual terms. To extend to general tensors, we have to define the action of parallel-transport on cotangent vectors. This can be done by requiring that scalar products between (tangent and cotangent) vectors are preserved under the transport. This is equivalent to metric compatibility of the connection, $D_\mu\mathbf{g}(i)\equiv 0$, since the components of the metric are defined using scalar products of basis vectors, $g_{\mu\nu}(i) \triangleq \langle \underline{e}_{\ \mu}(i), \underline{e}_{\ \nu}(i)\rangle$. Whereas the transport of tangent vectors is given by the action of the matrix $\mathbf{B}^{(\mu)}(i)$, the corresponding transport of cotangent vectors will involve the inverse of this matrix $(\mathbf{B}^{(\mu)}(i))^{-1}$. The covariant derivative of a cotangent vector ω is then given by:

$$\begin{aligned} \mathbb{I} - \widetilde{\mathbf{\Gamma}}^{(\mu)}(i) &\triangleq (\mathbf{B}^{(\mu)}(i))^{-1}, \\ \widetilde{\Gamma}^{\alpha}_{\mu\beta}(i) &\triangleq (\widetilde{\mathbf{\Gamma}}^{(\mu)}(i))_{\alpha\beta}, \\ (D_\mu\omega)_\nu(i) &\equiv d_\mu\omega_\nu(i) - \widetilde{\Gamma}^{\alpha}_{\mu\nu}(i)\omega_\alpha(i+\mu). \end{aligned} \tag{9}$$

Thus, on the mesh, we have two different sets of Christoffel symbols, $\Gamma^{\nu}_{\mu\alpha}(i)$ and $\widetilde{\Gamma}^{\alpha}_{\mu\beta}(i)$, corresponding to the connection on the tangent bundle, and the related (dual) connection on the cotangent bundle.

Both of these connections are derived from the matrix $\mathbf{B}^{(\mu)}(i)$. By considering the covariant derivatives of the tangent and cotangent vector fields defined by $X(i)=\underline{e}_{\ \alpha}(i)$, $X^{\mu}(i)=\delta^{\mu\alpha}$ and $\omega(i)=\underline{e}^{\ \alpha}(i)$, $\omega_\mu(i)=\delta_{\mu\alpha}$, where α is fixed, then we find that:

$$d_\mu\underline{e}_{\ \alpha}(i) \triangleq \underline{e}_{\ \alpha}(i+\mu) - \underline{e}_{\ \alpha}(i),$$
$$\Gamma^{\nu}_{\mu\alpha}(i) = \langle \underline{e}^{\ \nu}(i), d_\mu\underline{e}_{\ \alpha}(i)\rangle\,, \tag{10a}$$
$$\widetilde{\Gamma}^{\alpha}_{\mu\nu}(i) = -\langle \underline{e}_{\ \nu}(i), d_\mu\underline{e}^{\ \alpha}(i)\rangle\,, \tag{10b}$$

In this definition we have extended the usage of d_μ, previously defined as just the finite-difference operator on scalars, to a finite difference operator on basis vectors. For this to make sense, the difference has to be taken using the appropriate locally-flattened versions of the basis vectors, and it is the definition of local flattening in terms of preserving the metric that defines the connection. This point marks the point of divergence between our analysis, and that of other authors[15, 20, 35], in that they consider just the flat case, where $S_\mu\underline{e}_{\ \alpha}(i)=\underline{e}_{\ \alpha}(i+\mu)=\underline{e}_{\ \alpha}(i)$.

For the purposes of implementation it would be more convenient if we could define the connection in terms of the metric directly, rather than working with the basis vectors, since storing the metric for a mesh of N vertices/cells requires only $3N$ numbers, whereas storing the basis vectors and the dual basis vectors as vectors in $\mathbb{R}^3$ requires $12N$ numbers. Further, in order to compute the Christoffel symbols we also have to compute the local flattening operation for basis vectors at every vertex. Unfortunately, on the mesh:

$$d_\lambda g_{\alpha\beta} \equiv g_{\alpha\eta}\Gamma^{\eta}_{\lambda\beta} + g_{\beta\eta}\Gamma^{\eta}_{\lambda\alpha} + g_{\nu\eta}\Gamma^{\nu}_{\lambda\alpha}\Gamma^{\eta}_{\lambda\beta}$$

which is quadratic in the Christoffels, unlike the continuum result for $\partial_\lambda g_{\alpha\beta}$, which contains only the corresponding linear terms. Therefore, we cannot perform the usual trick of combining three such expressions to extract a simple useful formula for the Christoffels expressed purely in terms of the metric and its derivatives. Indeed, the quadratic nature of the above relation can be seen as the algebraic expression of the two possible choices for flattening of the cell shown in Fig. 1.

As regards the continuum limit, we note that because of our particular coordinate system, with unit edge length in the space of coordinates, the continuum limit has to be considered as the limit where the physical size of the mesh cells approaches 0, not the coordinate size. Hence in the naïve limit, all *differences* vanish in the limit, and in complicated expressions, the limiting form can be constructed by retaining only the leading-order terms, where order is defined in terms of the number of products of differences that occur. Hence in this case, we find that in the naïve continuum limit $\mathbf{\Gamma}^{(\mu)}(i) \approx \widetilde{\mathbf{\Gamma}}^{(\mu)}(i)$, and so we retain only a single set of Christoffel symbols, and a single connection, since:

$$(\mathbf{B}^{(\mu)}(i))^{-1} = \mathbb{I} - \widetilde{\mathbf{\Gamma}}^{(\mu)}(i) = \left(\mathbb{I} + \mathbf{\Gamma}^{(\mu)}(i)\right)^{-1} \approx \mathbb{I} - \mathbf{\Gamma}^{(\mu)}(i) \Rightarrow \ \mathbf{\Gamma}^{(\mu)}(i) \approx \widetilde{\mathbf{\Gamma}}^{(\mu)}(i), \tag{11}$$

We can now define the covariant derivative of a general tensor. The covariant derivative of a general product (which may involve contractions) of two tensors $\mathbf{F}$ and $\mathbf{G}$ is given by:

$$D_\mu(\mathbf{FG})(i) \equiv (D_\mu\mathbf{F}(i))(D_\mu\mathbf{G}(i)) + (D_\mu\mathbf{F}(i))\mathbf{G}(i) + \mathbf{F}(i)(D_\mu\mathbf{G}(i)), \tag{12}$$

and we see that, as with d_μ, the covariant derivative doesn't obey the Leibniz law.

We can also define a general directional covariant derivative (where $\mathbf{F}$ is a general tensor):

$$(D_X\mathbf{F})(i) \triangleq X^\mu(i)(D_\mu\mathbf{F})(i), \tag{13}$$

Given the covariant derivative, we might be tempted to define the covariant version of the divergence of a vector field as:

$$(D_\mu X)^\mu(i),$$

as is the case in the continuum. However, as we shall see later in Section **III D 2**, this is not an appropriate choice if we wish to retain the divergence theorem on the mesh.

C. The Torsion Tensor

We mentioned earlier that one way to compute the covariant derivative is to follow the case of Riemannian geometry and define a torsion-free connection. The action of the torsion in the continuum is defined as:

$$T(X,Y) \triangleq D_XY - D_YX - [X,Y]\,, \tag{14}$$

The Lie bracket $[\cdot,\cdot]$ has an intrinsic definition in terms of the action of two tangent vectors on a function:

$$[X,Y](f) \triangleq X(Y(f)) - Y(X(f)), \tag{15}$$

where the algebra closes, so that $[X,Y]$ is also a tangent vector.

However, on the mesh the fact that the difference is not a derivation means that the same property does not hold. One possible definition of a Lie bracket on the mesh is, however:

$$[X,Y]^\nu(i) \triangleq X^\alpha(i)d_\alpha Y^\nu(i) - Y^\alpha(i)d_\alpha X^\nu(i), \tag{16}$$

which would give us the torsion tensor on the mesh:

$$(T(X,Y))^\alpha(i) = \sum_{\mu,\nu}\Gamma^\alpha_{\mu\nu}(i)\left(X^\mu(i)Y^\nu(i+\mu) - Y^\mu(i)X^\nu(i+\mu)\right). \tag{17}$$

However, this is not a well-formed tensor on the mesh, since the expression does not contain the vector fields X and Y in the form $X^\mu(i)Y^\nu(i)$. In the naïve continuum limit it will vanish to first order ($X^\mu(i+\nu) \approx X^\mu(i)$, since $\Gamma^\alpha_{\mu\nu}(i)$ is of the same order as d_μ) provided that the Christoffel symbols are symmetric ($\Gamma^\alpha_{\mu\nu}(i) \equiv \Gamma^\alpha_{\nu\mu}(i)$). From this mesh definition, we find that:

$$\Gamma^\alpha_{\mu\nu}(i) - \Gamma^\alpha_{\nu\mu}(i) = \underline{e}_{\ \nu}(i+\mu) + \underline{e}_{\ \mu}(i) - (\underline{e}_{\ \mu}(i+\nu) + \underline{e}_{\ \nu}(i)), \tag{18}$$

which vanishes identically on the mesh, since if $\mu \neq \nu$, this is just the condition that the cell of the mesh is closed, and that $i + \underline{e}_{\ \mu}(i) + \underline{e}_{\ \nu}(i+\mu)$ is the same point as $i + \underline{e}_{\ \nu}(i) + \underline{e}_{\ \mu}(i+\nu)$. Hence, the torsion defined here becomes a well-formed tensor and vanishes in the continuum limit, but not on the mesh. If we had tried to proceed as in the continuum, constructing the Levi-Civita connection by requiring it to be torsion-free according to the above definition of torsion, we would have failed.

However, the above link between symmetry of the Christoffel symbols and the closure of the cells of the mesh is an interesting result. It should be noted that there is no similar, simple result for the other Christoffel symbols, hence we cannot assume that $\widetilde{\Gamma}^\alpha_{\mu\nu}(i) = \widetilde{\Gamma}^\alpha_{\nu\mu}(i)$ holds, except in the naïve continuum limit.

D. Curvature

In the continuum, the curvature tensor is often defined by considering the parallel transport of a vector around an infinitesimal closed loop, where the fact that the connection is not flat is indicated by the fact that the transported vector is not the same as the original vector, but has been rotated by some angle. We could consider a similar construction on the mesh, and parallel-transport a vector around a closed loop[21] (in our case, a cell of the mesh). However, if we did this, we would obtain expressions that would be of fourth-order in terms of the Christoffel symbols, whereas the curvature tensor in the continuum is of second-order. Because a closed path would have to contain links traversed in both positive and negative direction, we would also expect that such an expression would contain a mixture of the Christoffel symbols Γ and $\widetilde{\Gamma}$ for the connections on the tangent and cotangent bundles, which would introduce a further complication.

We instead return to our geometric construction of local flattening as shown in Fig. 1, but we now extend this diagram, and attempt to flatten the basis vectors at the i and the diagonally opposite point, $i+\mu+\nu$ as well (see Fig. 2). As can be seen from the diagram, the basis

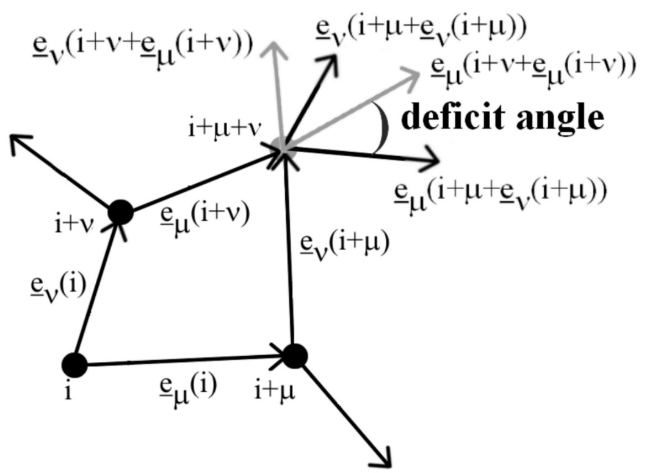

FIG. 2. A diagram showing how, when we start flattening at a point i, we obtain two different versions of the basis vectors at the diagonally opposite point $i+\mu+\nu$, depending on whether we respect the cell based at $i+\mu$, or the cell based at $i+\nu$. The difference between these two versions is the *deficit angle*, as shown in the diagram.

vectors (and any other vector) at $i+\mu+\nu$ cannot be uniquely defined in flattened form. The answer we get depends on whether we respect the flattening of the cell based at $i+\mu$, or whether we respect the flattening at the other cell based at $i+\nu$. The notation we will use for the different versions of the flattened vectors at $i+\mu+\nu$ is given in the diagram.

The two versions of the flattened vectors at $i+\mu+\nu$ differ by a rotation, since flattening preserves lengths. This rotation angle then represents the angle deficit (or angle excess) we find when we try to flatten the mesh about the vertex $i+\mu+\nu$, whilst still trying to preserve the metric. This construction is then in accord with the usual definition of curvature for the higher-dimensional equivalent of triangulated meshes in the Regge calculus[25].

In terms of the operations of DTC we have defined, we see that the curvature can be computed by considering a test vector at $i+\mu+\nu$, which we then transport to i via the two available paths. We will hence obtain an expression that is quadratic in the Christoffel symbols, and in particular, that contains only the Christoffel symbols of the connection on the tangent bundle. However, we will not do this here, but leave it to Section **III F**, where we will consider the mesh equivalent of the curvature tensor along with the mesh equivalent of the curvature 2-forms.

III. DISCRETE EXTERIOR CALCULUS (DEC)

The discrete exterior calculus[8, 14] aimed to introduce a discrete theory of differential calculus for simplicial complexes, thus enabling numerical integration schemes. It is based on differential forms, which have the benefits of being covariant and relatively simple to discretize. In this section we show how to construct a mesh version of this approach.

In the continuum, a k-form is a section of the k^{th} exterior power of the cotangent bundle. On the mesh, we will define a k-form $\alpha \in \Omega^k\mathcal{M}$ as:

$$\alpha(i) \triangleq \sum_{\mu_1<\mu_2\cdots<\mu_k} \alpha_{\mu_1\mu_2\ldots\mu_k}(i)\underline{e}^{\;\mu_1}(i)\wedge\underline{e}^{\;\mu_2}(i)\cdots\wedge\underline{e}^{\;\mu_k}(i), \tag{19}$$

where $\cdot\wedge\cdot$ is the antisymmetric, bilinear wedge product, which we need to define for the mesh.

Informally, for the discrete mesh a 0-form can be thought of as giving values to vertices, 1-forms giving values to edges, and so on. Since we are working on a two-dimensional mesh, we only need to consider 0-forms (that is, functions, with one degree of freedom at each point), 1-forms (two degrees of freedom at each point) , and 2-forms (one degree of freedom at each point). There is hence only one non-trivial case for which we need to define the wedge product. If ω and η are two 1-forms then the 2-form formed by the wedge product has components(s):

$$(\omega\wedge\eta)_{\mu\nu}(i) \triangleq \omega_\mu(i)\eta_\nu(i) - \omega_\nu(i)\eta_\mu(i). \tag{20}$$

A. The Exterior Derivative

As is reviewed in Section **A 3** we can construct an element of the cotangent space by applying a difference operator to a function:

$$df(i) \triangleq d_\mu f(i)\underline{e}^{\;\mu}(i). \tag{21}$$

In the language of forms, df is a 1-form constructed by applying the exterior derivative d to a 0-form. We can now generalize this in the usual way, so that d takes us from a k-form to a $k+1$-form, by including the definition:

$$\alpha\in\Omega^1\mathcal{M},\;\; (d\alpha)_{\mu\nu}(i) \triangleq d_\mu\alpha_\nu(i) - d_\nu\alpha_\mu(i). \tag{22}$$

From these definitions, it then follows that we retain the continuum property that $d^2\equiv 0$, but, as we might have expected, we do not retain the property that d is a derivation. The exterior derivative d as applied to exterior products of forms must be explicitly expanded in terms of components if required (Since we are in two dimensions, there is only one such expression which is non-trivial, which is $d(f\wedge\alpha)\equiv d(f\alpha)$, where f is a 0-form, and α is a 1-form).

The forms on our mesh involve the difference operator, hence we will use the terminology of Hydon and Mansfield[15, 20], and call such objects *difference forms*. So far, we have considered only the positive difference operator, hence could think of our forms as *positive-difference* forms. When we come to consider the codifferential operator (the formal adjoint to d, see Section **III C**), we will see that we also have to consider a negative difference operator, hence *negative-difference* forms. In what follows, forms should be taken to mean positive-difference forms unless further specified.

B. The Inner Product of Forms and the Hodge Star

We can also construct a (pointwise) inner product between two k-forms. This is trivial for 0-forms, and for 1-forms, we just use the previously-defined result for the inner product of cotangent basis vectors, to give:

$$g^{\mu\nu}(i) \triangleq \langle \underline{e}^{\,\mu}(i), \underline{e}^{\,\nu}(i)\rangle \;\Rightarrow \langle \omega,\eta\rangle(i) \triangleq g^{\mu\nu}(i)\omega_\mu(i)\eta_\nu(i), \tag{23}$$

which is just the usual tensorial convention for contraction of indices. The inner product of 2-forms is defined as:

$$\langle \alpha,\beta\rangle(i) \triangleq \sum_{\mu<\nu} \alpha_{\mu\nu}(i)\beta_{\eta\sigma}(i)g^{\mu\eta}(i)g^{\nu\sigma}(i) \tag{24}$$

$$\equiv \alpha_{12}(i)\beta_{12}(i)\left(\det \mathbf{g}(i)\right)^{-1}. \tag{25}$$

This form may seem a little odd on first encounter, since it does not accord with the usual tensorial convention of contraction over all indices. However, consider the physical vectors: $\underline{\alpha} \triangleq \alpha_{12}\,\underline{e}^{\,1} \times \underline{e}^{\,2}$, $\underline{\beta} \triangleq \beta_{12}\,\underline{e}^{\,1} \times \underline{e}^{\,2}$, where $\cdot \times \cdot$ is the usual vector cross-product in $\mathbb{R}^3$. If we then consider $\underline{\alpha} \cdot \underline{\beta}$, we see that we recover the inner product of 2-forms as given above.

Before we can define the Hodge star, we first need to define the area 2-form on our mesh da (the Levi-Civita tensor), with component:

$$\mathrm{da}(i)_{12} \triangleq |\underline{e}_{\,1}(i) \times \underline{e}_{\,2}(i)|, \tag{26}$$

where $\cdot \times \cdot$ refers to the usual Euclidean-space vector cross-product of the physical basis vectors. This can be related to the metric, since at every vertex i:

$$\begin{aligned}|\underline{e}_{\,1} \times \underline{e}_{\,2}|^2 &\equiv |\underline{e}_{\,1}|^2|\underline{e}_{\,2}|^2 - (\underline{e}_{\,1}\cdot\underline{e}_{\,2})^2 \\ &= g_{11}g_{22} - (g_{12})^2 = \det \mathbf{g},\end{aligned} \tag{27}$$

and so:

$$\mathrm{da}_{\mu\nu}(i) \equiv \varepsilon_{\mu\nu}\sqrt{\det \mathbf{g}(i)}, \tag{28}$$

where $\varepsilon_{\mu\nu}$ is the Levi-Civita symbol:

$$\varepsilon_{12} = -\varepsilon_{21} = 1,\;\; \varepsilon_{\mu\mu} = 0,\;\; \varepsilon^{\mu\nu} \triangleq \varepsilon_{\mu\nu}. \tag{29}$$

This is just the usual continuum definition of the volume form. However, for our mesh, it should be noted that $\mathrm{da}_{12}(i)$ is not the area of the physical cell based at i (which is not even defined, since in general the cell is non-planar), nor even the area of the flattened cell based at i, but the area of the parallelogram formed by the basis vectors at i. It is of course possible to compute the actual area of the flattened cell at i (which is: $\frac{1}{2}\left|(\underline{e}_{\,2}(i+1) - \underline{e}_{\,1}(i+2)) \times (\underline{e}_{\,1}(i) + \underline{e}_{\,2}(i+1))\right|$), but we can see that this expression is non-local in that it will involve the metric at points other than i. We hence retain the continuum definition, even if it is not the physical area of the cell.

Now that we have defined the inner product and wedge product for forms, we can also define the usual Hodge star $\star$ acting on forms thus:

$$\alpha \wedge \star\beta \triangleq \langle \alpha,\beta\rangle\,\mathrm{da}. \tag{30}$$

So we see that in n-dimensions, the Hodge star is a linear map from k-forms to $(n-k)$-forms. We then retain the usual continuum expressions:

$$\text{On functions: } (\star h)_{\mu\nu}(i) = \varepsilon_{\mu\nu}h(i)\sqrt{\det \mathbf{g}(i)}, \tag{31}$$

$$\text{On 1-forms: } (\star\omega)_\mu(i) = -\varepsilon_{\mu\nu}g^{\nu\lambda}(i)\omega_\lambda(i)\sqrt{\det \mathbf{g}(i)}, \tag{32}$$

$$\text{On 2-forms: } (\star\alpha)(i) = \frac{1}{2}\varepsilon_{\mu\nu}g^{\mu\lambda}(i)g^{\nu\eta}(i)\alpha_{\lambda\eta}(i)\sqrt{\det \mathbf{g}(i)} \equiv \frac{\alpha_{12}(i)}{\sqrt{\det \mathbf{g}(i)}}, \tag{33}$$

along with the result that on a k-form, $\star\star \equiv (-1)^{k^2}$. Using the flat operator we defined previously, we can also construct the Hodge dual of a vector X, where:

$$\begin{aligned}&\star X \triangleq \star(X^\flat) \\ \Rightarrow\; &(\star X)_\mu(i) \triangleq -\varepsilon_{\mu\nu}X^\nu(i)\sqrt{\det \mathbf{g}(i)}, \\ \Rightarrow\; &\star\star X \equiv -X^\flat.\end{aligned} \tag{34}$$

C. The Codifferential Operator

So far, we have managed to construct versions of many of the operators used in exterior calculus, in a form identical to that in the continuum. However, as we try to construct the codifferential operator, the discrete version starts to diverge, as we will now show.

In the continuum, the codifferential operator is defined as the formal adjoint of the exterior derivative:

$$\int \langle d\alpha, \beta\rangle \, \mathrm{da} \triangleq \int \langle \alpha, \delta\beta\rangle \, \mathrm{da}. \tag{35}$$

The importance of the codifferential is that it allows us to "pass derivatives across" within integrals, and hence obtain Euler-Lagrange equations by considering the functional variation of action integrals. It is also used, along with d, to construct the Laplace-de Rham operator.

It is clear that we cannot construct the adjoint of d on the mesh by applying this relation at a vertex, since $d\alpha(i)$ involves the difference of components of the k-form α evaluated at neighbouring points. We hence have to consider the mesh version of this relation:

$$\int_{\mathcal{M}} \langle d\alpha, \beta\rangle \, \mathrm{da} \triangleq \int_{\mathcal{M}} \langle \alpha, \delta\beta\rangle \, \mathrm{da}, \tag{36}$$

where by $\int_{\mathcal{M}}$ we mean the sum over vertices $\sum_i$. Starting from this definition for the two cases where α is either a 0-form or a 1-form, it is straightforward to expand this expression in terms of components, which shows that it is possible to construct a codifferential operator dual to d on the mesh, but only at the cost of also using the *negative-difference* operator $\overline{d}$ see (A3):

$$\delta \equiv - \star \overline{d} \star, \tag{37}$$
$$\overline{d}_\mu G_A(i + \underline{e}_{\,\mu}(i)) \triangleq G_A(i + \underline{e}_{\,\mu}(i)) - G_A(i) \equiv d_\mu G_A(i),$$

where $G_A(i)$ are the components of some object on the mesh.

We hence see that, as we might have predicted, the codifferential δ involves $\overline{d}$, which is just a difference operator acting in the reverse direction to d. By requiring that the *positive* difference operator has an adjoint, we are forced to consider the *negative* difference operator as well. We hence now briefly consider the new mesh structures generated by this operator.

1. *Negative-Difference Forms*

We define the components of a negative-difference 1-form as:

$$\overline{d}_\mu f(i) \triangleq f(i) - f(i-\mu) \;\Rightarrow\; \overline{d} f \triangleq \overline{d}_\mu f(i) \underline{e}^{\,\mu}(i-\mu), \tag{38}$$

The general negative-difference 1-form is then the object:

$$\bar{\omega}(i) \triangleq \bar{\omega}_\mu(i) \underline{e}^{\,\mu}(i-\mu). \tag{39}$$

In terms of the meaning of the flattened basis vectors we established in Fig. 1, this combination of vectors is well-defined and unique, since flattening from the site $j = i - \mu - \nu$ places the vectors $\{\underline{e}_{\,\mu}(i-\mu-\nu)$, $\underline{e}_{\,\nu}(i-\mu-\nu)$, $\underline{e}_{\,\mu}(i-\mu)$, $\underline{e}_{\,\nu}(i-\nu)\}$ in a common space.

We now need to clarify our earlier definition of $\overline{d}$ (37), since this earlier usage seemed to show it acting on a positive 0- or 1-form to produce a positive 1-form or 2-form. If we consider the basic negative difference operator acting on a function (A3):

$$(\overline{d}_\mu f)(i) \triangleq f(i) - f(i-\mu),$$

we can construct either a positive or negative 1-form using these values:

$$(\overline{d}_\mu f)(i) \underline{e}^{\,\mu}(i) \;\text{ or }\; (\overline{d}_\mu f)(i) \underline{e}^{\,\mu}(i-\mu), \tag{40}$$

and it is the former that we used in the definition of δ. Similarly, for $\overline{d}$ acting on a positive or negative 1-form, we produce either a positive or negative 2-form (where $(1 \leftrightarrow 2)$ is the usual notational shorthand for the first expression with 1 and 2 switched over):

$$(\overline{d}\omega)_{12}(i) \triangleq \overline{d}_1 \omega_2(i) - (1 \leftrightarrow 2), \tag{41a}$$
$$(\overline{d}\bar{\omega})_{12}(i) \triangleq \overline{d}_1 \bar{\omega}_2(i) - (1 \leftrightarrow 2), \tag{41b}$$
$$(\overline{d}\omega)(i) \triangleq (\overline{d}\omega)_{12}(i) \underline{e}^{\,1}(i) \wedge \underline{e}^{\,2}(i), \tag{41c}$$
$$(\overline{d}\bar{\omega})(i) \triangleq (\overline{d}\bar{\omega})_{12}(i) \underline{e}^{\,1}(i-1) \wedge \underline{e}^{\,2}(i-2). \tag{41d}$$

We hence see that we have actually used is a mapping between positive and negative forms in a coordinate basis, by just passing across the component values. Similarly, we can construct the operator dual to $\overline{d}$ acting on positive forms, which is then found to be:

$$\bar{\delta} \equiv - \star d \star. \tag{42}$$

We could construct a pointwise inner product of negative-difference forms in a similar way to the procedure for positive-difference forms. We just need the corresponding metric:

$$< \bar{\omega}, \bar{\eta} > (i) \triangleq \bar{\omega}_\mu(i) \bar{\eta}_\nu(i) \underline{e}^{\,\mu}(i-\mu) \cdot \underline{e}^{\,\nu}(i-\nu) \tag{43a}$$
$$\Rightarrow \;\; \bar{g}^{\mu\nu}(i) \triangleq \underline{e}^{\,\mu}(i-\mu) \cdot \underline{e}^{\,\nu}(i-\nu), \tag{43b}$$

which is uniquely defined by the geometric construction of flattening and so can be derived from the metric given previously. It should be noted, however, that in general this is not the same as the value we would obtain if we mapped the negative 1-forms to positive 1-forms.

In what follows, we will be using mainly positive forms, and will hence only require the operators d, $\overline{d}$, δ and $\bar{\delta}$, and the Hodge star $\star$ defined as acting on such forms.

D. Stokes' Theorem

Stokes' theorem says that for a manifold M with boundary ∂M:

$$\int_M d\omega = \int_{\partial M} \omega. \tag{44}$$

This generalizes the gradient, curl, and divergence theorems. In this section we demonstrate that Stokes' theorem holds on the mesh (where the manifold M will be replaced by some part of the mesh), and give the mesh formulations of the curl and divergence theorems.

We begin with the definition of the codifferential given in (37), first for the case where β is a specific 1-form:

$$\beta_\nu(j) = \pm\frac{1}{\sqrt{\det \mathbf{g}(j)}} g_{\nu\mu}(j)\delta_{j-i}, \tag{45}$$

where here δ_{j-i} is the Dirac delta function, and the vertex i and direction μ are fixed, defining a single link of the mesh. The sign defines whether the link will be traversed in the positive or negative direction. We can then construct $\delta\beta$ (37) as:

$$\begin{aligned} (\star\beta)_\nu(j) &= \pm\varepsilon_{\mu\nu}\delta_{i-j} \\ \Rightarrow\ (\overline{d}\star\beta)_{12}(j) &= \pm\left(\delta_{i-j} - \delta_{i+\mu-j}\right), \\ \Rightarrow \delta\beta(j) &= -\star\overline{d}\star\beta(j) \\ &= \frac{\pm 1}{\sqrt{\det \mathbf{g}(j)}}\left(\delta_{i+\mu-j} - \delta_{i-j}\right). \end{aligned}$$

Inserting this into the right-hand side of (36), we obtain the result:

$$\int\limits_{\mathcal{M}} \langle \alpha, \delta\beta\rangle\,\mathrm{da} = \pm\left(\alpha(i+\mu) - \alpha(i)\right), \tag{46}$$

which is just the difference of the values of the 0-form α at the beginning and end of the link defined by the form of β given above. For this value of β the left-hand side of (36) becomes:

$$\int\limits_{\mathcal{M}} \langle d\alpha, \beta\rangle\,\mathrm{da} \equiv \int\limits_{\mathcal{M}} d\alpha\wedge\star\beta = \pm\, d_\mu\alpha(i), \tag{47}$$

and by comparing these two equations we see that our definition of the codifferential operator (37) is consistent in this very simple case.

Consider now the link $i \to i+\mu$ (although note that it could also be traversed as $i+\mu \to i$) as forming part of a path γ on the mesh, with start point $\gamma(0)$ and end point $\gamma(1)$. By considering the set of links forming a continuous path we see that the above result can be generalized, and written in the form:

$$\int\limits_{\gamma} d\alpha \equiv \oint\limits_{\partial\gamma} \alpha \triangleq \alpha(\gamma(1)) - \alpha(\gamma(0)), \tag{48}$$

where $\partial\gamma$ is the boundary operator ∂ applied to the path. This is just Stokes' theorem for 0-forms. If we consider the grad operator as defined in (2), we then see that this result is just the gradient theorem written for differential forms.

To obtain Stokes' theorem for 1-forms we consider the specific 2-form:

$$\beta_{12}(j) = \sqrt{\det \mathbf{g}(j)}\delta_{i-j}. \tag{49}$$

By substituting into the left-hand side of (36) we obtain:

$$\sum_j \frac{(d\alpha)_{12}(j)\beta_{12}(j)}{\det \mathbf{g}(j)}\,\mathrm{da}(j) = (d\alpha)_{12}(i). \tag{50}$$

This can be considered as the integral of $d\alpha$ over a region $\mathcal{N}$, $\int_{\mathcal{N}} d\alpha$, where the form of β given above defines this region $\mathcal{N}$ as just the cell based at i, which we will denote by $\Diamond_i$.

We now consider the right-hand side of (36):

$$\int\limits_{\mathcal{M}} \langle \alpha, \delta\beta\rangle\,\mathrm{da} \equiv \int\limits_{\mathcal{M}} \alpha\wedge\star\delta\beta = -\int\limits_{\mathcal{M}} \alpha\wedge\star\star\overline{d}\star\beta = \int\limits_{\mathcal{M}} \alpha\wedge\overline{d}\star\beta, \tag{51}$$

where we have used the result that $\star\star = -1$ when acting on 1-forms. We hence only need to compute $\overline{d}\star\beta$:

$$\begin{aligned} \beta_{12}(j) &= \sqrt{\det \mathbf{g}(j)}\delta_{i-j} \\ \Rightarrow\ (\star\beta)(j) &= \delta_{i-j} \\ \Rightarrow\ \overline{d}_\mu(\star\beta)(j) &= \delta_{i-j} - \delta_{i+\mu-j}. \end{aligned}$$

This gives us that:

$$\int\limits_{\mathcal{M}} \alpha\wedge\overline{d}\star\beta = \alpha_1(i) + \alpha_2(i+1) - \alpha_1(i+2) - \alpha_2(i), \tag{52}$$

which is the result we obtained previously, confirming again that our construction of the codifferential operator is consistent.

We now consider the cell $\Diamond_i$, and define the boundary $\partial\Diamond_i$ (traced in the positive, anticlockwise direction) as the sequence of moves:

$$\underline{e}_{\,1}(i),\ \ \underline{e}_{\,2}(i+1),\ \ -\underline{e}_{\,1}(i+2),\ \ -\underline{e}_{\,2}(i). \tag{53}$$

We can then consider the α terms in (52) as representing the integral of α around this boundary. By considering an arbitrary collection of cells, forming some region $\mathcal{N}$, we then find:

$$\int_{\mathcal{N}} d\alpha \equiv \oint_{\partial\mathcal{N}} \alpha, \tag{54}$$

which is Stokes' theorem for 1-forms, and completes our derivation of Stokes' theorem on the mesh.

To derive the mesh equivalent of the curl and divergence theorems we will consider cases where the 1-form α is generated from a vector field. We have two ways of constructing a linear mapping from the space of tangent vectors to the space of 1-forms: the flat operator (A15) $X^\flat$, and the Hodge dual (34) $\star X$. We start with the former.

1. The Curl Theorem

We take Stokes' Theorem for 1-forms (54), and define $\alpha = X^\flat$. In components:

$$\begin{aligned} \alpha_\nu(i) &= g_{\nu\mu}(i)X^\mu(i) \\ &\equiv (\underline{e}_{\,\nu}(i)\cdot\underline{e}_{\,\mu}(i))X^\mu(i) \equiv \underline{e}_{\,\nu}(i)\cdot\underline{X}\,(i). \end{aligned}$$

The 2-form $d\alpha$ is then given by:

$$(d\alpha)_{12}(i) = d_1(\underline{e}_{\,2}(i) \cdot \underline{X}\,(i)) - (1 \leftrightarrow 2).$$

The curl operator acting on the vector $\underline{X}\,(i)$ can then be defined as:

$$\operatorname{curl} \underline{X}\,(i) \triangleq \frac{1}{\sqrt{\det \mathbf{g}(i)}} \left(d_1(\underline{e}_{\,2}(i) \cdot \underline{X}\,(i)) - (1 \leftrightarrow 2)\right). \tag{55}$$

Note that in terms of the 1-form α and the exterior derivative, this can also be written as:

$$\operatorname{curl} \alpha^\sharp \triangleq \star\, d\alpha, \;\; \operatorname{curl} X \triangleq \star(dX^\flat), \tag{56}$$

which agrees with the usual continuum expression in two dimensions.

Expanding out, we find that:

$$\begin{aligned} &\sqrt{\det \mathbf{g}(i)}\operatorname{curl} \underline{X}\,(i) = \\ &\underline{e}_{\,1}(i) \cdot \underline{X}\,(i) + \underline{e}_{\,2}(i+1) \cdot \underline{X}\,(i+1) \\ &-\underline{e}_{\,1}(i+2) \cdot \underline{X}\,(i+2) - \underline{e}_{\,2}(i) \cdot \underline{X}\,(i), \end{aligned} \tag{57}$$

which can be written as the path integral of $\underline{X}$ around the cell boundary $\partial\Diamond_i$. By considering a collection of such cells, we hence obtain the mesh version of the curl theorem:

$$\int_{\mathcal{N}} \operatorname{curl} \underline{X}\,\mathrm{da} \equiv \oint_{\partial\mathcal{N}} \underline{X}\,\cdot \underline{dl}\,,$$

where $\underline{dl}$ is an element of the path, and here da is just the scalar area measure $\mathrm{da}(i) \triangleq \sqrt{\det \mathbf{g}(i)}$ rather than the related 2-form.

Note that if we take the special case $\underline{X} = \operatorname{grad} f$ (2), we can also then show that $\operatorname{curl}(\operatorname{grad} f) \equiv 0$ by using (56) with $\alpha = df$, as a consequence of the fact that $d^2 \equiv 0$ on all forms.

2. The Divergence Theorem

As before, we start from Stokes' Theorem for 1-forms (54), but we now take $\alpha = \star X$. From (34):

$$\alpha_\nu(i) = -\varepsilon_{\nu\mu} X^\mu(i)\sqrt{\det \mathbf{g}(i)},$$

and so:

$$(d\alpha)_{12}(i) = d_\mu\left(\sqrt{\det \mathbf{g}(i)} X^\mu(i)\right).$$

Therefore, for this case the left-hand side of (54) can be written as:

$$\int_{\mathcal{N}} \frac{1}{\sqrt{\det \mathbf{g}}}\, d_\mu\left(\sqrt{\det \mathbf{g}} X^\mu\right) \mathrm{da},$$

where as before, da is the *scalar* area measure $\sqrt{\det \mathbf{g}}$, rather than the area 2-form. We hence define the divergence of a vector field as:

$$\operatorname{div} \underline{X} \;\triangleq \frac{1}{\sqrt{\det \mathbf{g}}}\, d_\mu\left(\sqrt{\det \mathbf{g}} X^\mu\right) \equiv \star\, d \star X, \tag{58}$$

which then gives $\star\, d\star$ as the divergence operator on either vectors or 1-forms. (If we define the divergence of a 1-form as $\operatorname{div} \omega^\sharp$, then it is straightforward to show that $\operatorname{div} \omega^\sharp \equiv \star d \star \omega$.) In the continuum this is the usual definition of the divergence using the exterior derivative, and there is an identity relating it to a version using the covariant derivative. However, as we might have expected given the issues with the Lorentz law (and as we noted earlier when we constructed the covariant derivative) the same identity does not hold on the mesh. That is:

$$(D_\mu X)^\mu \neq \frac{1}{\sqrt{\det \mathbf{g}}}\, d_\mu\left(\sqrt{\det \mathbf{g}} X^\mu\right). \tag{59}$$

To see this, consider calculating $d_\mu\left(\sqrt{\det \mathbf{g}} X^\mu\right)$. From (5), we can see that this will involve a term containing (in matrix notation):

$$\det \mathbf{B}^{(\mu)}(i) \equiv 1 + \operatorname{Tr} \mathbf{\Gamma}^{(\mu)}(i) + \det \mathbf{\Gamma}^{(\mu)}(i),$$

which is quadratic in the Christoffel symbols, unlike the covariant derivative, hence we see the inequality (59) must hold in general on the mesh, with equality only recovered in the naïve continuum limit.

Now consider the righthand side of (54) for the case of the integral around a single cell $\partial\Diamond_i$. As in (52), we have:

$$\oint_{\partial\Diamond_i} \alpha = \alpha_1(i) + \alpha_2(i+1) - \alpha_1(i+2) - \alpha_2(i).$$

Taking the first term and using the result in (A14):

$$\alpha_1(i) = -X^2(i)\sqrt{\det \mathbf{g}(i)} = \underline{X}\,(i) \cdot \left(-\frac{\underline{e}^{\,2}(i)}{|\underline{e}^{\,2}(i)|}\right) |\underline{e}_{\,1}(i)|,$$

This expression has a simple interpretation. The vector $(-\underline{e}^{\,2}(i)/|\underline{e}^{\,2}(i)|)$ is a unit vector perpendicular to $\underline{e}_{\,1}(i)$, which is the unit normal to the link, and the sign means that it is the outwardly directed normal to the cell $\Diamond_i$ at this link. It is scaled by the related path length $|\underline{e}_{\,1}(i)|$. The same conclusions apply if we consider the other terms. By considering an arbitrary collection of such cells, we can hence derive the divergence theorem on the mesh:

$$\int_{\mathcal{N}} \operatorname{div} \underline{X}\,\mathrm{da} \equiv \oint_{\partial\mathcal{N}} \underline{\hat{n}}\,\cdot \underline{X}\,dl, \tag{60}$$

where $\underline{\hat{n}}$ is the unit outward normal for any portion of the path, which has path length dl.

To summarize our progress so far, we have defined forms on the mesh, and we have defined the exterior derivative d, constructed using the difference operator d_μ on components of forms. This exterior derivative then maps k-forms to $k+1$-forms. We have shown that we can retain the curl and divergence theorems on the mesh, but at the cost of breaking the usual continuum link between the divergence of a vector field defined using the covariant derivative of tensor calculus, and the divergence defined using the exterior derivative and the Hodge star.

In the continuum the tensor calculus and exterior calculus are equivalent, but this is not true on the mesh. We next proceed to look for an object in the discrete exterior calculus that corresponds to the covariant derivative of discrete tensor calculus.

E. The Covariant Exterior Derivative

Consider a 1-form $\omega(i) \triangleq \omega_\mu(i)\underline{e}^{\ \mu}(i)$. This can also be viewed as a cotangent-bundle valued 0-form (function). Considering it in this light, if we apply the difference operator to it, then we have to take the difference of cotangent-values at two different points on the mesh. This will involve the connection $\widetilde{\Gamma}^\alpha_{\mu\nu}(i)$ on the cotangent bundle (10):

$$\begin{aligned} d_\mu(\omega(i)) &\triangleq \omega(i+\mu) - \omega(i) \\ &\equiv \omega_\nu(i+\mu)\underline{e}^{\ \nu}(i+\mu) - \omega_\nu(i)\underline{e}^{\ \nu}(i), \\ &= \left[d_\mu\omega_\eta(i) - \omega_\nu(i+\mu)\widetilde{\Gamma}^\nu_{\mu\eta}(i)\right]\underline{e}^{\ \eta}(i). \end{aligned}$$

Collecting components, we then obtain the usual tensorial covariant derivative of a cotangent vector (9):

$$D\omega(i) = \sum_{\mu,\eta}\left[d_\mu\omega_\eta(i) - \omega_\nu(i+\mu)\widetilde{\Gamma}^\nu_{\mu\eta}(i)\right]\underline{e}^{\ \mu}(i)\otimes\underline{e}^{\ \eta}(i).$$

We then antisymmetrize, to obtain the mesh formulation of covariant exterior derivative:

$$d^{\widetilde{\Gamma}}\omega(i) \triangleq \sum_{\mu<\eta}\left[\,(d_\mu\omega_\eta(i) - \omega_\nu(i+\mu)\widetilde{\Gamma}^\nu_{\mu\eta}(i)) - (\mu\leftrightarrow\eta)\right]\underline{e}^{\ \mu}(i)\wedge\underline{e}^{\ \eta}(i). \tag{61}$$

This acts on forms in a manner analogous to the exterior derivative d, in that it maps k-forms to $k+1$ forms.

In the naïve continuum limit:

$$(d^{\widetilde{\Gamma}}\omega)_{\mu\eta}(i) \approx (d\omega)_{\mu\eta}(i) - \omega_\nu(i)\left(\widetilde{\Gamma}^\nu_{\mu\eta}(i) - \widetilde{\Gamma}^\nu_{\eta\mu}(i)\right).$$

Hence, in the continuum, $d^{\widetilde{\Gamma}}$ acting on 1-forms is equivalent to d provided that:

$$\tilde{T}^\nu_{\mu\eta}(i) \triangleq \widetilde{\Gamma}^\nu_{\mu\eta}(i) - \widetilde{\Gamma}^\nu_{\eta\mu}(i) = 0,$$

where $\tilde{T}^\nu_{\mu\eta}$ is the continuum torsion tensor. This is true in the continuum for a Levi-Civita connection, by definition. However, we have already seen (in Section **II C**) that, on the mesh, the Christoffel symbols for the Levi-Civita connection on the cotangent bundle do not have this property. In the next section, we will look at the torsion in the context of the discrete exterior calculus, and in particular, the torsion 2-forms.

Note that we can also define a covariant exterior derivative d^Γ using the connection on the tangent bundle, by considering a tangent vector $X(i)$ as a tangent-bundle valued 0-form, and hence obtain the tangent-bundle valued 1-form:

$$\begin{aligned} d^\Gamma X(i) &\triangleq d_\mu\left(X^\nu(i)\underline{e}_{\ \nu}(i)\right)\underline{e}^{\ \mu}(i) \\ &\equiv (D_\mu X)^\nu(i)\underline{e}_{\ \nu}(i)\otimes\underline{e}^{\ \mu}(i), \end{aligned} \tag{62}$$

which is just the usual tensorial covariant derivative of a tangent vector (8). We can also write this in the useful form:

$$(d^\Gamma\underline{X}\ )_\mu(i) = \underline{X}\ (i+\mu) - \underline{X}\ (i), \tag{63}$$

where we now use $\underline{X}\ (i+\mu)$ to denote the flattened physical vector at $i+\mu$, where we have started flattening at i.

F. Torsion, Curvature, and the Cartan Structure Equations

Given the mesh version of the covariant exterior derivative, we can now use this derivative to construct the mesh formulation of the torsion and curvature forms.

From the definition of the Christoffel symbols (10):

$$\begin{aligned} d_\mu\underline{e}_{\ \alpha}(i) &= \Gamma^\nu_{\mu\alpha}(i)\underline{e}_{\ \nu}(i), \\ d_\mu\underline{e}^{\ \alpha}(i) &= -\widetilde{\Gamma}^\alpha_{\mu\nu}(i)\underline{e}^{\ \nu}(i). \end{aligned}$$

If we consider $\underline{e}_{\ \alpha}(i)$ as a tangent-bundle valued 0-form, then $d_\mu\underline{e}_{\ \alpha}(i)$ are the components of a tangent-bundle valued 1-form. We hence define the matrix of connection 1-forms:

$$\varpi^\nu_\alpha(i) \triangleq \Gamma^\nu_{\mu\alpha}(i)\underline{e}^{\ \mu}(i). \tag{64}$$

Similarly, we define the connection 1-forms for the connection on the cotangent bundle:

$$\widetilde{\varpi}^\alpha_\mu(i) \triangleq \widetilde{\Gamma}^\alpha_{\mu\nu}(i)\underline{e}^{\ \nu}(i) \tag{65}$$

For a fixed α, $\underline{e}^{\ \alpha}(i)$ lies in the cotangent space, and so can be considered as a 1-form $\omega(i)$ with constant components $\omega_\mu(i) = \delta^\alpha_\mu$. From (61), the covariant exterior derivative of this is:

$$(d^{\widetilde{\Gamma}}\underline{e}^{\ \alpha}(i))_{\mu\eta} = \widetilde{\Gamma}^\alpha_{\eta\mu}(i) - \widetilde{\Gamma}^\alpha_{\mu\eta}(i).$$

We hence define the torsion 2-form of the connection on the cotangent bundle as:

$$(\widetilde{T}^\alpha(i))_{\mu\eta} \triangleq (d^{\widetilde{\Gamma}}\underline{e}^{\ \alpha}(i))_{\mu\eta} = \widetilde{\Gamma}^\alpha_{\eta\mu}(i) - \widetilde{\Gamma}^\alpha_{\mu\eta}(i). \tag{66}$$

This then gives us the first Cartan structure equation on the mesh, for the torsion on the cotangent bundle:

$$\widetilde{T}^{\alpha} \triangleq d^{\widetilde{\Gamma}} \underline{e}^{\,\alpha} = d\underline{e}^{\,\alpha} + \widetilde{\varpi}^{\alpha}_{\lambda} \wedge \underline{e}^{\,\lambda}. \tag{67}$$

Since we are using a coordinate frame, the first term vanishes identically, but we include it here for the sake of completeness, since this is the form of the equation in an arbitrary frame.

We will now compute the torsion for the connection on the tangent bundle. Consider the quantity:

$$E(i) \triangleq \underline{e}_{\,\alpha}(i) \otimes \underline{e}^{\,\alpha}(i), \;\; E_{\alpha}(i) = \underline{e}_{\,\alpha}(i). \tag{68}$$

This can be described as a tangent-bundle valued 1-form, which is the identity on any tangent vector. We now differentiate this as a 1-form, but using the connection on the tangent bundle. This is hence equivalent to computing the covariant exterior derivative d^{Γ}. In components:

$$(d^{\Gamma}E)_{\mu\nu}(i) = d_{\mu}E_{\nu}(i) - (\mu \leftrightarrow \nu) \tag{69a}$$
$$= (\Gamma^{\eta}_{\mu\nu}(i) - \Gamma^{\eta}_{\nu\mu}(i))\underline{e}_{\,\eta}(i), \tag{69b}$$

and we see that $d^{\Gamma}E$ is a tangent-bundle valued 2-form. The torsion of the connection on the tangent bundle is then defined by:

$$T^{\alpha}(i)\underline{e}_{\,\alpha}(i) \triangleq d^{\Gamma}E(i) \tag{70a}$$
$$\Rightarrow \;\; T^{\alpha}_{\mu\nu}(i) \triangleq \Gamma^{\alpha}_{\mu\nu}(i) - \Gamma^{\alpha}_{\nu\mu}(i). \tag{70b}$$

In terms of the connection 1-forms (64), we can then write:

$$T^{\alpha}_{\mu\nu}(i) = \varpi^{\alpha}_{\nu\mu}(i) - \varpi^{\alpha}_{\mu\nu}(i) \;\; \Rightarrow T^{\alpha} = d\underline{e}^{\,\alpha} + \varpi^{\alpha}_{\lambda} \wedge \underline{e}^{\,\lambda}, \tag{71}$$

which is the first Cartan structure equation for the connection on the tangent bundle.

From the closure condition on the mesh, the torsion of the connection on the tangent bundle vanishes identically. We hence see that we have indeed managed to construct the equivalent of the torsion-free, Levi-Civita connection on the mesh, although we have had to use the DEC version of the torsion, rather than the DTC version we attempted to define in Section **II C**.

In the continuum, the curvature 2-forms are defined by considering the action of two covariant exterior derivatives on a tangent vector field. As before, we consider a tangent vector field $X(i)$ as a tangent-bundle valued 0-form. Applying d^{Γ} once generates a tangent-bundle valued 1-form, and applying it again generates a tangent-bundle valued 2-form. Using (63), we can write:

$$\begin{aligned}(d^{\Gamma}(d^{\Gamma}\underline{X}\,))_{\mu\nu}(i) &= \big((d^{\Gamma}\underline{X}\,)_{\nu}(i+\mu) - (d^{\Gamma}\underline{X}\,)_{\nu}(i)\big) - (\mu \leftrightarrow \nu)\\ &= \underline{X}\;\,(i + \underline{e}_{\,\mu}(i) + \underline{e}_{\,\nu}(i + \underline{e}_{\,\mu}(i))\;) - \underline{X}\;\,(i + \underline{e}_{\,\nu}(i) + \underline{e}_{\,\mu}(i + \underline{e}_{\,\nu}(i))\;)\,,\end{aligned}$$

where, as before, we have the used the long form of $i + \mu + \nu$ to make clear the two different flattened versions of any vector at $i + \mu + \nu$ (for the case where $\mu \neq \nu$), when we have started flattening at i. As we saw in Section **II D**, these two vectors are not equivalent, but are related by a rotation in the plane, the angle of rotation being the angle deficit at the vertex $i + \mu + \nu$. However, note that to maintain the consistency of our notation, this difference is used to define the curvature 2-forms at i, rather than at $i + \mu + \nu$.

We now restrict ourselves to the case $X(i) = \underline{e}_{\,\sigma}(i)$, with σ fixed (that is, the tangent vector field has constant components $X^{\lambda}(i) = \delta^{\lambda}_{\sigma}$). Our curvature is now the matrix of 2-forms R^{η}_{σ} defined by:

$$(d^{\Gamma}(d^{\Gamma}\underline{e}_{\,\sigma}))(i) \triangleq R^{\eta}_{\sigma}(i)\underline{e}_{\,\eta}(i), \tag{72}$$

which can then be written as:

$$(d^{\Gamma}(d^{\Gamma}\underline{e}_{\,\sigma}))_{\mu\nu}(i) = \underline{e}_{\,\sigma}(i + \underline{e}_{\,\mu}(i) + \underline{e}_{\,\nu}(i + \underline{e}_{\,\mu}(i))) - \underline{e}_{\,\sigma}(i + \underline{e}_{\,\nu}(i) + \underline{e}_{\,\mu}(i + \underline{e}_{\,\nu}(i))).$$

Using the definitions of the Christoffel symbols in terms of the basis vectors (10), we find (after some algebra) that:

$$(R^{\eta}_{\sigma})_{\mu\nu}(i) \equiv d_{\mu}\Gamma^{\eta}_{\nu\sigma}(i) - d_{\nu}\Gamma^{\eta}_{\mu\sigma}(i) + \Gamma^{\lambda}_{\nu\sigma}(i+\mu)\Gamma^{\eta}_{\mu\lambda}(i) - \Gamma^{\lambda}_{\mu\sigma}(i+\nu)\Gamma^{\eta}_{\nu\lambda}(i). \tag{73}$$

This cannot be written in a totally straightforward manner in terms of the connection 1-forms ϖ^{α}_{β} (64), since it includes the Christoffel symbols at the three vertices i, $i + \mu$, and $i + \nu$. However:

$$R^{\eta}_{\sigma} = d\varpi^{\eta}_{\sigma} + \varpi^{\eta}_{\lambda} \wedge \varpi^{\lambda}_{\sigma} + \text{third-order terms,}$$

so that we recover Cartan's second structure equation in the naïve continuum limit.

We can also construct the tensor form of the curvature, the Riemann curvature tensor, defined by:

$$\mathcal{R}^{\eta}_{\sigma\mu\nu} \triangleq (R^{\eta}_{\sigma})_{\mu\nu}. \tag{74}$$

In terms of covariant derivatives of a tangent vector field X, we then find that:

$$(D_\mu(D_\nu X) - D_\nu(D_\mu X))^\eta (i) = \mathcal{R}^\eta_{\lambda\mu\nu}(i) X^\lambda(i+\mu+\nu).$$

This formalizes the link between the curvature form, and the non-equivalence of parallel transport two ways across a cell that we described in Section **II D**. It also gives us the usual link between curvature and the non-commutation of second-order covariant derivatives (in a coordinate basis).

Thus, by using a discrete exterior calculus approach we are able to construct a representation of the geometry that is sufficient to include Cartan's structure equations. We now demonstrate the use of our discrete differential geometry on a sample dynamical system to show how it works in practice.

IV. AN EXAMPLE DYNAMICAL SYSTEM ON THE MESH: ELECTROMAGNETISM AND U(1) LOCAL GAUGE INVARIANCE

In this section we will construct a mesh equivalent of classical electromagnetism as a demonstration of our discrete differential geometry on quadrilateral meshes. We will take spacetime to be the simple $(2+1)$ dimensional product $\mathcal{M} \otimes \mathbb{R}$, where the spacelike sections (the mesh) are fixed and discrete, and time is continuous and flat.

Even in the classical case[24] electromagnetism in $(2+1)$ dimensions is far from straightforward, and has been of interest for several decades[10, 26]. Aside from the field-theoretic studies, such theories are also of interest in condensed-matter physics, because of their relevance to phenomena such as the quantum Hall effect[22], and high-T^c superconductivity[2]. It provides a non-trivial test case for demonstrating the applicability of our current formalism. Previous formulations of discrete electromagnetism include those of Bossavit[4] and Stern[28]. The latter also used a discrete exterior calculus approach, but based upon triangular meshes.

The naïve approach to electromagnetism on the mesh would be to take the continuum version of the Maxwell equations, in either tensor calculus form or exterior calculus form, and try to construct a mesh version of these. We will dismiss the DTC approach, since we have already seen that the DTC form of the divergence theorem does not hold on the mesh (59), and the divergence theorem for electric and magnetic flux is a far from trivial part of the theory. We hence choose to apply the tools of DEC.

As regards the field content of our theory, we note that in $(2+1)$ dimensions, the Maxwell-Faraday equation $\mathrm{curl}\,\underline{E} = -\partial \underline{B}/\partial t$, and the absence of magnetic monopoles, mean that the magnetic field only has a single orthogonal component. The electric field possesses two degrees of freedom, and is therefore purely planar. The physical field content is therefore composed of one 2-form field (the orthogonal magnetic field), and the 1-form electric field. We could now take these fields, and try to re-create the exterior calculus form of the continuum Maxwell equations on the mesh. However, there is a more theoretically satisfying way to proceed.

Electromagnetism (either classical or quantum) can be considered as a field theory that possesses a U(1) local gauge symmetry[32] (see Jackson and Okun[16] for a historical review of the development of this idea for electromagnetism). In classical continuum electromagnetism, the physical electric and magnetic fields can be described in terms of a scalar potential and a vector potential. These potential fields are sufficient to define the physical fields, but are not themselves totally determined by the values of the physical fields. So, for example, in electrostatics, the electric field is given by the spatial gradient of the scalar potential, but the zero of the potential is arbitrary. It is these symmetries of the physical fields under suitable arbitrary transformations of the unphysical potential fields that is the local gauge symmetry, as will be explained further below. Our approach to constructing the mesh theory is to construct it so that it possesses the same fundamental symmetries as the continuum theory, and is locally gauge invariant.

We take the standard approach to classical U(1) gauge theory, and consider a complex scalar field φ defined on the mesh – for the moment, we will consider just the spatial dependance of this field $\varphi(j) \in \mathbb{C}$. Since this field is a scalar, there is no geometric contribution to the notion of a derivative of the field along a link. However, we can introduce an extra structure, and specify a gauge connection: when the field value $\varphi(j+\mu)$ is transported from $j+\mu$ to j, and we define the corresponding value at j to be given by:

$$\varphi(j+\mu) \longrightarrow U_\mu(j)\varphi(j+\mu).$$

(Note that we have here used j to denote the vertex index, in order to avoid confusion with the complex number $i = \sqrt{-1}$.) In particular, we take the connection $U_\mu(j)$ to be of the form:

$$\begin{aligned} U_\mu(j) &\triangleq \exp(iA_\mu(j)) \in \mathrm{U}(1), \\ \Rightarrow\ D^A_\mu \varphi(j) &\triangleq \exp(iA_\mu(j))\varphi(j+\mu) - \varphi(j), \end{aligned} \tag{75}$$

where $A_\mu(j) \in \mathbb{R}$ is a real 1-form field (a gauge field). We hence see that the action of transport on a complex number is defined to consist of a phase rotation that leaves real-valued magnitudes such as $\varphi^\dagger(j)\varphi(j)$ unchanged (an obvious analogue of our earlier geometrical construction (see Section **II B**) that parallel transport of vectors changed their direction, but did not change their lengths).

Real quantities of the form $\varphi^\dagger(j)\varphi(j)$ are invariant under either a *global* phase rotation $\varphi(j) \to e^{i\theta}\varphi(j)$ of the field, or a *local* phase rotation $\varphi(j) \to e^{i\theta(j)}\varphi(j)$. We can extend this local symmetry to the derivatives defined above, provided that the fields $A_\mu(j)$ transform as well.

It can be seen that as $\varphi(j) \rightarrow e^{i\theta(j)}\varphi(j)$, so:

$$A_\mu(j) \rightarrow A_\mu(j) - \theta(j+\mu) + \theta(j) = A_\mu(j) - d_\mu\theta(j) \tag{76a}$$

$$\Rightarrow \; D^A_\mu \varphi(j) \rightarrow e^{i\theta(j)} D^A_\mu \varphi(j), \tag{76b}$$

which is the local gauge transformation for our mesh fields. The defining characteristic of such classical or quantum gauge field theories is that the requirement of local gauge invariance places a restriction on the interaction between the various fields in the theory. For example, from (75) we see that a possible gauge-invariant interaction term between the gauge field and the complex scalar field is:

$$\varphi^\dagger(j)\exp(iA_\mu(j))\varphi(j+\mu). \tag{77}$$

If we consider gauge-invariant fields constructed solely from the connection 1-form $A_\mu(j)$:

$$U_\Diamond(j) \triangleq e^{iA_\mu(j)} e^{iA_\nu(j+\mu)} e^{-iA_\mu(j+\nu)} e^{-iA_\nu(j)}, \;\; \mu \neq \nu \tag{78}$$

$$(dA)_{\mu\nu}(j) \triangleq d_\mu A_\nu(j) - (\mu \leftrightarrow \nu). \tag{79}$$

The variable $U_\Diamond(j)$ is associated with a cell of the mesh, (referred to as a *plaquette* in the lattice gauge theory literature), and is the Wilson loop variable[33]. The associated variable dA can be seen as the gauge-invariant spatial curvature 2-form of the spatial part of the gauge connection, $A_\mu(i)$. In order to preserve the gauge-invariance, we consider transport around a closed loop, rather than comparing two paths across a cell as in the geometrical case (72). We can now interpret the gauge-invariant quantity dA as a physical field. The single degree of freedom it represents corresponds to the single degree of freedom of the purely orthogonal magnetic field in $(2+1)$ dimensions, hence we take:

$$B \triangleq dA \;\; \Rightarrow \;\; dB \equiv 0. \tag{80}$$

We then have the usual relation between the magnetic field and the magnetic vector potential, with the curl of the magnetic field vanishing identically in $(2+1)$ dimensions.

To obtain the electric field, we need to consider non-static forms of local gauge transformations, and the addition of a gauge connection term to the temporal partial derivative:

$$\frac{\partial}{\partial t} \;\; \rightarrow \frac{\partial}{\partial t} + i\psi(j,t), \;\; \varphi(j,t) \;\; \rightarrow \;\; e^{i\theta(j,t)}\varphi(j,t).$$

It is then straightforward to show that the connection/scalar potential $\psi(i,t)$ must transform under a local, non-static gauge transformation thus:

$$\psi(j,t) \;\; \rightarrow \;\; \psi(j,t) - \frac{\partial\theta(j,t)}{\partial t} \triangleq \psi(j,t) - \dot{\theta}(j,t). \tag{81}$$

The corresponding spatio-temporal curvature is found by considering a closed path formed by traversing a single link forwards and backwards at times t and $t+\delta t$. The gauge-invariant physical variable is then given by:

$$E_\mu(j,t) \triangleq -\frac{\partial A_\mu(j,t)}{\partial t} + d_\mu\psi(j,t), \tag{82a}$$

$$E \triangleq d\psi - \dot{A} \;\; \Rightarrow \;\; dE \equiv -\dot{B}, \tag{82b}$$

which is the electric field as a 1-form, represented in terms of the magnetic potential 1-form A and the electric scalar potential ψ in the usual way. We hence, as in the continuum, obtain the first two of Maxwell's equations, based on the gauge fields alone.

We now consider the Lagrangian density $\mathcal{L}$ for the gauge fields A and ψ without the scalar field. We couple the field to external, non-dynamic sources (charges and currents), rather than to the mobile charges that the scalar field represents. The source term in the Lagrangian we take as the simplest coupling to the gauge fields:

$$L_{\text{source}} = \int dt \sum_i \left(J^\mu(i,t)A_\mu(i,t) + Q(i,t)\psi(i,t)\right) \tag{83}$$

$$\Rightarrow \; \mathcal{L}_{\text{source}}(i,t) = \langle \mathfrak{J}(i,t), A(i,t)\rangle + \langle \rho(i,t), \psi(i,t)\rangle, \tag{84}$$

$$\mathfrak{J}(i,t) \triangleq \frac{1}{\sqrt{\det \mathbf{g}(i)}} \; J^{\flat}(i,t), \;\; \rho(i,t) \triangleq \frac{Q(i,t)}{\sqrt{\det \mathbf{g}(i)}}, \tag{85}$$

where $J(i,t) \triangleq J^\mu(i,t)\underline{e}_{\;\mu}(i,t)$ is the current vector, $Q(i,t)$ is the charge at vertex i at time t, and $\mathfrak{J}(i,t)$ and $\rho(i,t)$ are the corresponding 1-form and 0-form densities. This corresponds to an intuitive physical model of the sources, where the charges live at the sites of the mesh, and the currents flow along the links.

The complete Maxwell Lagrangian density can then be written as:

$$\mathcal{L} \triangleq \frac{1}{2}\langle E, E\rangle - \frac{1}{2}\langle B, B\rangle + \langle \mathfrak{J}, A\rangle + \langle \rho, \psi\rangle. \tag{86}$$

Note that we could also consider the addition of various Chern-Simons (CS)[10, 26] terms to the Lagrangian density. For example, following Guo and Fang (see their equation (2.4)), for the case where time is continuous, we would then have the analogue of the continuum Maxwell-Chern-Simons theory[24], by the addition of the lattice point-split CS term:

$$L_{CS} = \int dt \sum_i \left[\psi(i,t)B_{12}(i,t) + \epsilon^{\mu\nu}A_\mu(i-\mu,t)E_\nu(i,t)\right].$$

This is invariant under a general gauge transformation (76) & (81), provided that:

$$\sum_i \theta(i,t)f(i+y,t) = \sum_i \theta(i-y,t)f(i,t),$$

where y is some lattice displacement. That is, we either do not have a spatial boundary, or we have periodic spatial boundary conditions.

Returning to the Maxwell case, as in the case of the scalar field (77), the requirement that the Lagrangian is gauge-invariant places a condition on the source terms. By considering such a transformation $\theta(i,t)$, and computing the functional derivative of the Maxwell Lagrangian, we see that:

$$\frac{\delta L}{\delta\theta(i,t)} = 0 \;\Rightarrow\; \delta\mathfrak{J} - \frac{\partial\rho}{\partial t} = 0$$
$$\Rightarrow \overline{d}_\mu J^\mu(i,t) + \frac{\partial Q(i,t)}{\partial t} = 0, \qquad (87)$$

where:

$$\overline{d}_\mu J^\mu(i,t) \triangleq \sum_\mu \left(J^\mu(i,t) - J^\mu(i-\mu,t)\right) \qquad (88)$$
$$\equiv J^1(i,t) - J^1(i-1,t) + J^2(i,t) - J^2(i-2,t).$$

We can therefore think of $\overline{d}_\mu J^\mu(i,t)$ as a measure of the divergence of the current vector field at a point; $J^\mu(i,t)$ represents currents flowing away from the point i along the links $i \to i+\mu$, whereas $-J^\mu(i-\mu,t)$ represents the currents flowing away from i to the points $i-\mu$. Hence, the constraint on the sources required by local gauge invariance is just the condition that charge is conserved (an example of Noether's first theorem on our mesh), and that a net current flow out of the point i is reflected by an decrease in the charge $Q(i,t)$ at i.

The divergence operator div (defined in Section **III D 2**) acts on vectors to produce a scalar density, defined on a cell of the mesh, which is then integrated over a cell or cells to find the related total flux out of the area. In contrast, the divergence measure defined here defines the total flow out of a point, is a scalar rather than a scalar density, and requires only summation to compute the total outward flux. From (58), we have:

$$\operatorname{div} \underline{X} \triangleq \frac{1}{\sqrt{\det \mathbf{g}}}\; d_\mu\left(\sqrt{\det\mathbf{g}}X^\mu\right) \equiv \star\, d \star X,$$

whereas from (88) & (85):

$$\begin{aligned}\overline{d}_\mu J^\mu(i,t) &= \overline{d}_\mu \left(g^{\mu\nu}(i)\mathfrak{J}_\nu(i,t)\sqrt{\det\mathbf{g}(i)}\right)\\ &= \left(\overline{d}(\star\mathfrak{J})\right)_{12}(i,t)\\ &= \sqrt{\det\mathbf{g}(i)}\star\,\overline{d}\star\mathfrak{J}(i,t),\\ \Rightarrow\;\; \frac{1}{\sqrt{\det\mathbf{g}}}\,\overline{d}_\mu J^\mu(i,t) &= \star\,\overline{d}\star\mathfrak{J}(i,t).\end{aligned}$$

We hence see that the operators for the cell-wise and point-wise divergences are related, by interchanging the positive and negative difference operators d and $\overline{d}$.

The Euler-Lagrange equations of the fields can now be obtained by calculating the functional derivative of the Lagrangian with respect to the gauge fields. We find that:

$$\frac{\delta L}{\delta A} = 0 \;\Rightarrow\; \delta B = \mathfrak{J} + \frac{\partial E}{\partial t} \quad \text{Ampère's circuital Law} \qquad (89)$$

$$\frac{\delta L}{\delta \psi} = 0 \;\Rightarrow\; \delta E + \rho = 0, \qquad \text{Gauss' Law} \qquad (90)$$

Together with the relations:

$$dE = -\frac{\partial B}{\partial t} \qquad \text{Faraday's Law of induction} \qquad (91)$$
$$dB = 0 \qquad \text{Gauss' Law for magnetism} \qquad (92)$$

these form the mesh version of Maxwell's equations. The form of Maxwell's equations on our non-planar quadrilateral mesh in $(2+1)$ dimensions is very similar in form to the discrete model constructed by Schwalm et al.[27], who used simplex methods on triangulated meshes[23]. However, note that their treatment was primarily topological, in that it only used adjacency rather than a metric and/or a connection. As we will now see, utilizing the extra structure provided by the metric allows a deeper understanding of the mesh model.

We now show how these equations for forms correspond with the usual continuum concepts of magnetic and electric flux and flux densities. Starting with Ampère's Law, we apply the Hodge star to get:

$$\overline{d}\star B = \star\,\mathfrak{J} + \frac{\partial}{\partial t}(\star\, E). \qquad (93)$$

The field B is the magnetic 2-form, corresponding to purely orthogonal magnetic fields. A general 2-form can be considered as containing the area of the cell within it, hence we will consider $B(i,t)$ as representing the *total* magnetic flux piercing the cell $\Diamond_i$ at time t. We hence introduce the *scalar* flux density for the cell at i thus:

$$\mathfrak{B}(i,t) \triangleq \star\, B(i,t) = \frac{B_{12}(i,t)}{\sqrt{\det\mathbf{g}(i)}}. \qquad (94)$$

As regards the electric flux, we take the 1-form E as a flux *density*, in the sense of being a 1-form density. We hence define the total flux vector:

$$\Phi(i,t) \triangleq E^\sharp(i,t)\sqrt{\det\mathbf{g}(i)} \;\Rightarrow\; (\star E)_\mu(i,t) \equiv -\epsilon_{\mu\nu}\Phi^\nu(i,t), \qquad (95)$$

where $\Phi^\mu(i,t)$ now represents the total electric flux passing along the link $i \to i+\mu$. We hence see that the electric flux lies wholly in the surface defined by the mesh, and is confined to flux tubes lying along the links. Ampère's Law (89) can now be written in flux form as:

$$\overline{d}_\mu\mathfrak{B} = -\epsilon_{\mu\nu}\left[J^\nu + \frac{\partial}{\partial t}\Phi^\nu\right]. \qquad (96)$$

The difference in magnetic flux density $\mathfrak{B}$ between the two cells lying on either side of a link depends on the current flowing along that link, and the rate of change of electric flux along the link. This accords with the usual continuum picture in $(3+1)$ dimensions, where the loop integral of the magnetic field varies according to the current passing through the loop, and the rate of change of electric flux through the loop. Similarly, Gauss' Law (90) can now be written in the flux form:

$$\overline{d}_\mu \Phi^\mu(i,t) = Q(i,t). \tag{97}$$

As in the interpretation of the law for charge conservation, this says that the total electric flux flowing away from a site, along all the links connected to the site, is equal to the charge at that site.

We can also obtain the mesh form of the wave equations for the field variables E and B, which are:

$$\frac{\partial^2 B}{\partial t^2} + (d\delta + \delta d)B = d\mathfrak{J}, \tag{98}$$
$$\frac{\partial^2 E}{\partial t^2} + (d\delta + \delta d)E = -d\rho - \frac{\partial \mathfrak{J}}{\partial t}. \tag{99}$$

Terms such as δd acting on B (which is of course identically zero since B is a 2-form), have been included to make it clear that both wave equations contain the mesh version of the Laplace-de Rham operator:

$$\Delta \triangleq d\delta + \delta d. \tag{100}$$

Now let us consider the field energy, which is given by the Hamiltonian $\mathcal{H}$. At time t, the total energy is:

$$\mathcal{H}_{\mathcal{M}}(t) \triangleq \sum_{i\in\mathcal{M}} \left[\frac{1}{2}\langle E,E\rangle + \frac{1}{2}\langle B,B\rangle\right] \mathrm{da}.$$

It is then straightforward to show that, if there are no currents, the field energy is conserved and $\dot{\mathcal{H}}_{\mathcal{M}} = 0$. If we now restrict ourselves to a single point i, then the rate of change of field energy at that point is given by:

$$\dot{\mathcal{H}}_i = -(E \wedge \star\mathfrak{J})_{12} + (\delta B \wedge \star E)_{12} + \mathfrak{B}\dot{B}_{12}.$$

The first term is the usual exchange of energy from the currents to the fields. If we define the point-split vector:

$$S^1(i,t) \triangleq -\mathfrak{B}(i,t)E_2(i+1,t), \tag{101a}$$
$$S^2(i,t) \triangleq \mathfrak{B}(i,t)E_1(i+2,t), \tag{101b}$$

then the rate of change of energy can be written in the compact form:

$$\dot{\mathcal{H}}_i = -E_\mu(i,t)J^\mu(i,t) + \overline{d}_\mu S^\mu(i,t). \tag{102}$$

The vector $S^\mu(i,t)$ (101) is the mesh point-split version of the Poynting vector, which represents electromagnetic energy flux. As in the case of electric currents and electromagnetic flux (88), $\overline{d}_\mu S^\mu(i,t)$ can be considered as the total energy flux along all the links connected to the site i, hence (102) is the mesh version of Poynting's Theorem.

V. DISCUSSION: OTHER PHYSICAL MODELS

We have seen that it is possible to construct a mesh version of electromagnetism, by requiring that the fundamental symmetries of the continuum theory are retained exactly. We hence now try to apply the same principle to a different physical theory, that of a viscous, compressible fluid in $(2+1)$ dimensions. We use the space $\mathcal{M}\otimes\mathbb{R}$ as before.

The first obvious physical variables are the density of the fluid $\rho(i,t)$, and the flow velocity of the fluid $\underline{v}\,(i,t) = v^\mu(i,t)\underline{e}\,_\mu(i)$. Now let us assume we have an *intensive* physical quantity $\mathbf{G}(i,t)$ (which may be scalar, vector etc.). Given the derivation of the divergence in (58), we then see that the continuity equation for this physical quantity $\mathbf{G}(i,t)$ can be written as:

$$\frac{\partial \mathbf{G}(i,t)}{\partial t} = -\operatorname{div}\left(\mathbf{G}(i,t)\underline{v}\,(i,t)\right) + \mathbf{q}(i,t).$$

Note that the subscript that appears in the d_μ term in the expansion of the divergence links to the superscript in the components of $\underline{v}\,(i,t)$, *not* to any components that the quantity $\mathbf{G}(i,t)$ may possess.

The physical meaning of this equation is that the rate of change of the relevant physical quantity for cell $\lozenge_i$ equals the rate at which this flows out of the cell, plus the extra term $\underline{q}\,(i,t)$, which is the source or sink of the corresponding quantity.

To be specific, for fluids we consider first the scalar intensive quantity which is density, and the statement that mass is conserved for the flow, with no sources or sinks. Hence we have $\mathbf{G}(i,t) = \rho(i,t)$ in this case, and the mass-conservation condition can be written as:

$$\frac{\partial \rho(i,t)}{\partial t} + \operatorname{div}\left(\rho(i,t)\underline{v}\,(i,t)\right) = 0.$$

We cannot simplify this by expanding it out, since differences are not derivations. However, for the case of an incompressible fluid, we do obtain the usual divergence-free condition on the flow: $\operatorname{div}\underline{v}\, = 0$.

Momentum conservation is then defined by taking $\mathbf{G}(i,t) = \rho(i,t)\underline{v}\,(i,t)$, with forces being a source or sink of momentum. In order to make clear the action of the divergence on this dyadic product, we now expand out the divergence, to obtain the final form:

$$\mathrm{da}(i)\frac{\partial}{\partial t}\left[\rho(i,t)\underline{v}\,(i,t)\right] = -d_\mu\left[\mathrm{da}(i)\rho(i,t)\underline{v}\,(i,t)v^\mu(i,t)\right] + \mathrm{da}(i)\underline{b}\,(i,t),$$

where $\underline{b}\,(i,t)$ are the relevant body forces (e.g., pressure and viscous forces, as well as external applied forces), and da is the scalar area measure $\mathrm{da}(i) \triangleq \sqrt{\det \mathbf{g}(i)}$. In the naïve continuum limit, d_μ would become the usual partial derivative, and this would then give the usual velocity-dependent terms that appear in the continuum Navier-Stokes equations in a curved space (note that the partial derivatives of the basis vectors would give the Christoffel symbols (10), hence covariant derivatives in this limit).

We hence see that for fluids, we have been able to select the important physical laws that we wish to maintain exactly on the mesh (conservation of mass and conservation of momentum), and shown how these conservation laws can be written in the usual form suitable for numerical simulation. A complete treatment would entail specifying the nature of the viscous and pressure forces on the mesh, but for reasons of space, we do not pursue this further here.

VI. CONCLUSIONS

In this paper we have introduced a discrete differential geometry for quadrilateral meshes. This has applications wherever interpolation is an important part of the proposed system, such as in image analysis and some field theories, since triangular meshes require considerably more computational effort for this. Our approach has been to construct a discrete theory that admits parallels to that in the continuum, but that is inherently self-consistent. We have considered two ways to do this: by developing a tensor calculus, and by developing difference forms on the mesh and producing a discrete exterior calculus, which has some similarities to that of Hirani and co-workers[8, 9, 14, 28], but based on quadrilaterals, rather than simplicial complexes.

One striking difference between the continuum case and our meshes is that these two approaches do not match, since the 'divergence' of a vector field is different depending whether it is calculated using the covariant derivative of tensor calculus or the exterior derivative and Hodge star of differential forms. This has led to us favoring the discrete exterior calculus version of our formulation, which admits Stokes' theorem.

We have therefore used this version for the construction of the mesh version of electromagnetism, which illustrates several points. Most importantly, it shows that it is possible to construct mesh versions of continuum theories that respect exactly the relevant symmetries and conservation laws of the continuum theory. We have shown how this can be extended to other physical theories, such as fluids, where the relevant conservation laws are for mass and momentum.

Finally, we note that this paper considered just the case of a curved 2D surface, and in our future work we will consider the extension of these methods to curved 3D spaces and to spacetimes.

Appendix A: Basic Definitions

In this appendix we introduce the basic definitions of discrete differential geometry for the reader who is not familiar with them. The fundamental space that we consider is a non-planar, quadrilateral mesh $\mathcal{M} \subset \mathbb{R}^3$ with the same connectivity as a regular square grid in $\mathbb{R}^2$. We label the vertices of this mesh by i and assign direction labels to links in the mesh in a consistent manner so that from a site i we can travel along a (directed) edge in a direction μ to arrive at site $i+\mu$, and along a different edge in direction ν to arrive at site $i+\nu$. We will also define a coordinate chart on our mesh, which we take to be a regular square mesh of edge length 1 unit.

1. Difference Operators

A scalar function on the mesh is defined by specifying the value of the function f at each vertex: $\{f(i) : i \in \mathcal{M}\}$. We can then take the difference of values along a link in the obvious way[15, 20, 35], by defining a shift operator, and hence the basic *forward* (or *positive*) difference operator:

$$S_\mu f(i) \triangleq f(i+\mu) \tag{A1}$$

$$\Rightarrow \quad d_\mu f(i) \triangleq (S_\mu - 1) f(i) \equiv f(i+\mu) - f(i) \tag{A2}$$

We can define a *backward* (or *negative*) difference operator similarly:

$$\overline{d}_\mu f(i) \triangleq f(i) - f(i-\mu) \tag{A3}$$

Based on our coordinate choice, in the naïve continuum limit the difference operator d_μ (and also the difference operator $\overline{d}_\mu$) become the coordinate partial derivative ∂_μ. However, since they are difference operators, they are not derivations, and do not obey the Leibniz (or product) law. That is, we have the deformed Leibniz rule[35]:

$$d_\mu(fh)(i) \equiv f(i+\mu)h(i+\mu) - f(i)h(i) \tag{A4}$$

$$\equiv h(i+\mu) d_\mu f(i) + f(i) d_\mu h(i) \tag{A5}$$

$$\neq f(i) d_\mu h(i) + h(i) d_\mu f(i). \tag{A6}$$

2. Directional Derivatives

We have defined a scalar-valued function in terms of values defined at the vertices of our mesh. The complete mesh consists of vertices *and* links, hence it is natural to consider also a mesh field defined in terms of values assigned to links. Let $X^\mu(i)$ denote the value of the field X on the *directed* link $i \to i+\mu$. The directional derivative of a function can then be defined as:

$$X(f)(i) \triangleq X^\mu(i) d_\mu f(i) \equiv \sum_\mu X^\mu(i) \left[f(i+\mu) - f(i) \right]. \tag{A7}$$

Note that unless otherwise stated, we will employ the Einstein summation convention, so that paired subscripts and superscripts $A^{\cdots\mu\cdots}_{\cdots} B^{\cdots}_{\cdots\mu\cdots}$ are summed over. We hence see that X can be considered as a *tangent vector* to the mesh since it defines a linear mapping from the space of functions to the reals. The value of X at a vertex i is thus an element of $T_i\mathcal{M}$(which, for convenience, we denote as V_i) and is given by the components $\{X^\mu(i) : \mu = 1, 2\}$. This is in accord with the definition of tangent spaces that we used in the Introduction.

3. Basis Frame

Now let us suppose we have some physical realization of our mesh in $\mathbb{R}^3$. The edge from i to $i+\mu$ then corresponds to a physical vector $\underline{e}_{\ \mu}(i)$. The set of vectors $\{\underline{e}_{\ \mu}(i) : \mu = 1, 2\}$ then form a basis for the tangent space V_i at i and an abstract tangent vector field $X = \{X^\mu(i)\}$ can be represented by a physical vector $\underline{X}\ (i) \triangleq X^\mu(i)\underline{e}_{\ \mu}(i)$ at each vertex of the mesh. We can also define the dual basis at i, $\{\underline{e}^{\ \alpha}(i)\}$, where:

$$\langle \underline{e}^{\ \alpha}(i), \underline{e}_{\ \beta}(i)\rangle \triangleq \delta^\alpha_\beta, \tag{A8}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product, and δ^α_β is the Kronecker delta. In what follows, for reasons of brevity, we will sometimes write the Euclidean scalar product as $\underline{e}^{\ \alpha}(i)\cdot\underline{e}_{\ \beta}(i)$. This dual basis spans the cotangent space denoted by $T^*_i\mathcal{M}$ or V^*_i, dual to V_i. So the cotangent vector (one form) defined by df with components $\{d_\mu f(i)\}$ at i can then also be written as a physical vector:

$$\underline{df}\ \triangleq d_\mu f(i)\underline{e}^{\ \mu}(i), \tag{A9}$$

and the action of the vector X on the function f can be written in the form:

$$X(f)(i) \triangleq X^\mu(i)d_\mu f(i) \equiv \left\langle \underline{df}\ (i), \underline{X}\ (i)\right\rangle = (df, X)\,(i), \tag{A10}$$

where $(\cdot,\cdot)$ is the contraction of a cotangent vector with a tangent vector.

Rather than using the notation $\underline{df}$ and $\underline{X}$, it is common to refer to the scalar product $\overline{\langle\cdot,\cdot\rangle}$ between abstract objects, where this scalar product is to be understood as that generated by the Euclidean scalar product when we move from the abstract basis vectors $\{\underline{e}_{\ \alpha}(i)\}, \{\underline{e}^{\ \alpha}(i)\}$ to some appropriate physical realization of these basis vectors. What exactly we mean by "appropriate" should be clear in the paper, based on the relation between basis vectors at neighbouring vertices.

4. The Metric

With the definition of the frames $\{\underline{e}_{\ \alpha}(i)\}, \{\underline{e}^{\ \alpha}(i)\}$ given in Section **A 3**, we can now define a general well-formed tensor field of type (m, n), $\mathbf{A} \in T^m_n\mathcal{M}$, which we will take to be of the form:

$$\begin{aligned}\mathbf{A}(i) \triangleq & A^{\mu_1\mu_2\cdots\mu_m}_{\nu_1\nu_2\ldots\nu_n}(i)\left(\underline{e}_{\ \mu_1}(i)\otimes\underline{e}_{\ \mu_2}(i)\cdots\otimes\underline{e}_{\ \mu_m}(i)\right)\\ &\otimes\left(\underline{e}^{\ \nu_1}(i)\otimes\underline{e}^{\ \nu_2}(i)\cdots\otimes\underline{e}^{\ \nu_n}(i)\right).\end{aligned}$$

Note that $\mathbf{A}(i)$ lies in $T^m_n\mathcal{M}_i$, formed by taking products of V_i and V^*_i. We could also define quantities on the mesh with components, and products of m tangent basis vectors and n cotangent basis vectors, but these quantities are only well-formed if the tangent and cotangent spaces all lie at the same base point i.

A metric can now be defined as the positive-definite, symmetric bilinear mapping from $V_i \otimes V_i$ to the reals defined by:

$$g(X,Y)(i) \triangleq \langle X, Y\rangle\,(i) = \langle \underline{e}_{\ \alpha}(i), \underline{e}_{\ \beta}(i)\rangle\, X^\alpha(i)Y^\beta(i), \tag{A11}$$

which is a symmetric tensor of type $(0, 2)$ with components:

$$g_{\alpha\beta}(i) \triangleq \langle \underline{e}_{\ \alpha}(i), \underline{e}_{\ \beta}(i)\rangle\,. \tag{A12}$$

From this definition of the metric it is clear that for a physical mesh the 3 independent degrees of freedom of the metric at a point i are the lengths of the two links $i \to i+\mu$ and $i \to i+\nu$, and the angle between them.

Considering $\mathbf{g}(i)$ as a matrix, we also have the inverse matrix $\mathbf{g}^{-1}(i)$, which is a tensor of type $(2, 0)$, with components:

$$g^{\alpha\beta}(i) \triangleq \left\langle \underline{e}^{\ \alpha}(i), \underline{e}^{\ \beta}(i)\right\rangle. \tag{A13}$$

We note here a relation that is useful in the paper: by the definition of $g^{\mu\nu}(i)$ as the elements of the inverse of $\mathbf{g}(i)$, we see that $g^{11}(i)\det\mathbf{g}(i) = g_{22}(i)$ and $g^{22}(i)\det\mathbf{g}(i) = g_{11}(i)$. Since $g_{11}(i) \equiv |\underline{e}_{\ 1}(i)|^2$ and $g^{22}(i) \equiv |\underline{e}^{\ 2}(i)|^2$, we find that:

$$\frac{|\underline{e}_{\ 1}(i)|}{|\underline{e}^{\ 2}(i)|} = \sqrt{\det\mathbf{g}(i)} = \frac{|\underline{e}_{\ 2}(i)|}{|\underline{e}^{\ 1}(i)|}. \tag{A14}$$

The metric $\mathbf{g}$ and its inverse $\mathbf{g}^{-1}$ can be used to raise and lower indices in the usual way, defining the sharp $\sharp$ and flat $\flat$ operators:

$$(\omega^\sharp)^\mu(i) \triangleq g^{\mu\nu}(i)\omega_\nu(i),\ \ (X^\flat)_\mu(i) \triangleq g_{\mu\nu}(i)X^\nu(i). \tag{A15}$$

Note that this definition means that for a given physical mesh, there are two different possible choices, corresponding to which angle of the quadrilateral we chose to be the one in the metric. The other angle is not included in the metric, corresponding to a freedom in bending a single quadrilateral about the corresponding diagonal. However, in general this folding will not be possible for a physical mesh, since each cell is constrained by others. In effect, the choice is equivalent to choosing a particular triangulation of the quadrilateral mesh, where the triangulated mesh metric is defined by giving the lengths for all links. And note that we cannot include both angles in the quadrilateral in the metric, since then we have too many degrees of freedom to make the link with the continuum case. Also, fixing both angles does not allow us to locally flatten the mesh (see Fig. 1), which is the mesh version of the locally-Euclidean property of a Riemannian manifold.

[1] Angenent S., Haker S., Tannenbaum A., and Kikinis R., "Laplace-Beltrami operator and brain flattening," IEEE Transactions on Medical Imaging, **18**, 700–711 (1999)

[2] Bednorz J. and Müller K. A., "Possible high-T^c superconductivity in the Ba-La-Cu-O system," Z. Phys. B, **64**, 189–193 (1986)

[3] Bombelli L., Lee J., Meyer D., and Sorkin R. D., "Spacetime as a causal set," Phys. Rev. Lett., **59**, 521–524 (1987)

[4] Bossavit A., *Computational Electromagnetism* (Academic Press, Boston, Massachusetts, 1998)

[5] Brechbuhler C., Gerig G., and Kubler O., "Parametrization of closed surfaces for 3-D shape description," Computer Vision and Image Understanding, **61**, 154–170 (1995)

[6] Christensen G. E., Rabbitt R. D., and Miller M. I., "Deformable templates using large deformation kinematics," IEEE Transactions on Image Processing, **5**, 1435–1447 (1996)

[7] Davies R. H., Twining C. J., and Taylor C., "Groupwise surface correspondence by optimization: Representation and regularization," Medical Image Analysis, **12**, 787–796 (2008)

[8] Desbrun M., Hirani A., Leok M., and Marsden J. E., "Discrete exterior calculus," (2005), arXiv:math/0508341v2 [math.DG]

[9] Desbrun M., Hirani A., and Marsden J. E., "Discrete exterior calculus for variational problems in computer graphics and vision," in *Proceedings of the* 42^{nd} *IEEE Conference on Decision and Control*, Vol. 5 (2003) pp. 4902–4907

[10] Deser S., Jackiw R., and Templeton S., "Topologically massive gauge theories," Ann. Phys. (NY), **140**, 372–411 (1982)

[11] Durr S., Fodor Z., Frison J., Hoelbling C., Hoffmann R., Katz S. D., Krieg S., Kurth T., Lellouch L., Lippert T., Szabo K. K., and Vulvert G., "Ab initio determination of light hadron masses," Science, **322**, 1224–1227 (2008)

[12] Grinspun E., Desbrun M., Hirani A., and Schröder P., "Discrete shells," in *Proceedings of ACM SIGGRAPH/Eurographics Symposium on Computer Animation*, edited by D. Breen and M. Lin (2003)

[13] Gu X. and Yau S., "Computing conformal structures of surfaces," Communications in Information and Systems, **2**, 121–146 (2002)

[14] Hirani A., *Discrete Exterior Calculus*, Ph.D. thesis, California Institute of Technology (2003)

[15] Hydon P. E. and Mansfield E. L., "A variational complex for difference equations," Foundations of Computational Mathematics, **4**, 187–217 (2004)

[16] Jackson J. D. and Okun L. B., "Historical roots of gauge invariance," Rev. Mod. Phys., **73**, 663–680 (2001)

[17] Jin M., Kim J., Luo F., and Gu X. F., "Discrete surface Ricci flow," IEEE Transactions on Visualization and Computer Graphics, **14**, 1030–1043 (2008)

[18] Langtangen H.-P., Mardal K.-A., and Winter R., "Numerical methods for incompressible viscous flow," Advances in water resources, **25**, 1125–1146 (2002)

[19] Leok M., Marsden J. E., and Weinstein A. D., "A discrete theory of connections on principal bundles," (2005), arXiv:math/0508338v2 [math.DG]

[20] Mansfield E. L. and Hydon P. E., "Difference forms," Foundations of Computational Mathematics, **8**, 427–467 (2008)

[21] Meyer M., Desbrun M., Schröder P., and Barr A. H., "Discrete differential-geometry operators for triangulated 2-manifolds," in *Visualization and Mathematics III*, edited by H.-C. Hege and K. Polthier (Springer-Verlag, 2003) pp. 35–57

[22] Morandi G., *Quantum Hall Effect* (Bibliopolis, Naples, 1988)

[23] Moritz B. and Schwalm W., "Triangle lattice Green functions for vector fields," J. Phys. A, **34**, 589–602 (2001)

[24] Moura-Melo W. A. and Helayël-Neto J. A., "Remarks on Dirac-like monopoles, Maxwell and Maxwell-Chern-Simons electrodynamics in $d = 2 + 1$ dimensions," Phys. Rev. D, **63**, 065013 (2001)

[25] Regge T., "General relativity without coordinates," Nuovo Cimento (10), **19**, 558–571 (1961)

[26] Schonfeld J. F., "A mass term for three-dimensional gauge fields," Nucl. Phys., **185**, 157–171 (1981)

[27] Schwalm W., Moritz B., Giona M., and Schwalm M., "Vector difference calculus for physical lattice models," Phys. Rev. E, **59**, 1217–1233 (1999)

[28] Stern A., *Geometric discretization of Lagrangian mechanics and field theories*, Ph.D. thesis, California Institute of Technology (2009)

[29] Tong Y., Alliez P., Cohen-Steiner D., and Desbrun M., "Designing quadrangulations with discrete harmonic forms," *SGP '06: Proceedings of the fourth Eurographics symposium on Geometry processing*, ACM International Conference Proceeding Series, **256**, 201–210 (2006)

[30] Tonti E., "Finite formulation of the electromagnetic field," Progress In Electromagnetics Research (PIER), **32**, 1–44 (2001)

[31] Wang Y., Gu X. F., Chan T. F., Thompson P. M., and Yau S.-T., "Conformal slit mapping and its applications to brain surface parameterization," in *Proceedings of Medical Image Computing and Computer-Assisted Intervention (MICCAI)*, Lecture Notes in Computer Science (LNCS), Vol. 5241 (Springer-Verlag, 2008) pp. 585–593

[32] Weyl H., "Elektron und gravitation. I," Z. Phys., **56**, 330–352 (1929), in german. English translation reproduced in *The dawning of gauge theory*, Lochlainn O'Raifeartaigh (Princeton, 1997).

[33] Wilson K. G., "Confinement of quarks," Phys. Rev. D, **10**, 2445–2459 (1974)

[34] Wörz S. and Rohr K., "Physics-based elastic image registration using splines and including landmark localization uncertainties," in *Proceedings of Medical Image Computing and Computer-Assisted Intervention (MICCAI)*, Lecture Notes in Computer Science (LNCS), Vol. 4191 (Springer, 2006) pp. 678–685

[35] Wu Ke, Zhao Weizhong, and Guo Hanying, "Difference discrete connection and curvature on cubic lattice," Science in China Series A-Mathematics, **49**, 1458–1476 (2006)

[36] Yeo B., Sabuncu M. R., Vercauteren T., Ayache N., Fischl B., and Golland P., "Spherical demons: Fast diffeomorphic landmark-free surface registration," IEEE Transactions on Medical Imaging, **29**, 650–668 (2010)