Statistical Modelling of Complex Emotions using Mixture of von Mises Distributions

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***Abstract*—The recognition of basic human emotions based on facial points has been studied extensively for many years. Since complex emotions are comprised of a number of the basic emotions, in order to identify them some way to interpolate between known basic emotions must be identified. In this paper, we introduce a finite mixture model to recognise complex emotions and represent them onto the activation-evaluation space, a popular model in psychology for emotion representation. Since the activation-evaluation space is circular, the popular probability distribution models for emotion recognition are inappropriate to characterise complex emotions. The model that we propose is based on a mixture of von Mises distributions, which is an approximation to the normal distribution when wrapped onto a circle and the most common model for describing directional data. This paper describes the process of estimating the parameters of the mixture model and tests the fit of an estimated model to a set of ground truth values of emotion direction and intensity.**

***Keywords*-complex emotions; emotion recognition; Mahalanobis distance; von Mises distribution.**

I. Introduction

There has been a lot of research into emotion recognition, analysis, and synthesis over the past three decades. Much of it has focussed on the basic emotions (happy, angry, sad, afraid, disgusted, surprised) and sometimes frustrated, excited, and neutral are often also included to make a total of nine [1], [2], [3]. However, most human emotions are not pure examples of one basic emotion, but a mixture of them [4], known as complex emotions. Although there are names for many complex emotions, they are often subjective. In addition, the number of them makes categorising them very hard. This is particularly true when the emotions are not posed, but 'real'. As a consequence of these difficulties, most studies have focused on identifying a limited set of complex emotions, e.g., [5], [6]. Also there has been an increasing interest in emotion synthesis to animate Embodied Conversational Agents (ECAs), most of which rely on generating new emotions by mixing two basic emotions (see Section I-A). In addition, there has been recent interest within the affective computing field to map emotions into some emotion space continuum rather that categorical labelling, in part to recognise the fact that emotions are a continuous phenomenon and in part to enable complex emotions to be identified without requiring labels.

In this paper, we introduce a flexible mixture model to recognise and represent complex emotions on the basis of known basic emotions. Following the psychological assumption that complex emotions can be conceived of as mixtures of basic emotions [7], we propose a hypothesis that it may be possible to develop a mixture model that combines each basic emotion in an appropriate amount to recognise and represent complex emotions. The proposed model is based on the activation-evaluation space, which is the most widely-used bipolar circumplex model for representing emotions in psychological studies [8]. We use data from the IEMOCAP dataset and identify first the basic emotions using shape models based on a training set of human labels. We demonstrate that the locations in activation-evaluation space correspond well to the positions for the emotions chosen by Whissell [9]. We then identify a statistical method to form appropriate mixtures of the basic emotions by identifying that we are dealing with directional data. This leads us to form a mixture of von Mises distributions, which is the circular analogue of the Gaussian distribution.

A. Related Work

The most common emotion space is the two dimensional valence-activation (or activation-evaluation) space. This is a model that represents emotions based on activation (how motivated a person is by that emotion) and valence (how positive or negative an emotion is). It forms a circular representation of emotion space with neutral at the origin [8]. The relative positions of emotions may be described by the specific angular locations on the circle based on their similarity to one another. For example, the angular location of anger is closer to frustration than to happiness. Neutral lies at the centre of the circle. The radial distance from the centre of the circle represents the intensity of emotion; the greater the distance, the stronger the emotion and vice-versa [8], [10].

This space has been used in emotion recognition work such as [11], which aimed to predict valence and activation based on video of the head and shoulders of the person, and an associated audio track. They used the output-associative relevance vector machine regression framework to predict valence and activation based on learning the input and output dependencies. The proposed framework is quite robust for continuous emotion classification in terms of valence and activation, but further analysis of spatio-temporal dynamics is needed in order to understand the correlation between these two dimensions. The blending of emotions was considered

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in [12], where a model consisting of the linear combination of distinct continuous spaces is produced, although no results of using it are shown. Some of the other computational methods using this space are [13], [14], [15]. Refer to [16] for a review of models and techniques related to the emotion continuum, synthesis and representation of emotions.

There has also been work to synthesise complex emotions by interpolating between two neighbouring basic emotions on the activation-evaluation space, such as [8], [17], which used Whissell's list of emotion words [9]. They used the angles of the emotions assigned by Whissell as a measure of similarity between emotions, while activation-values were used as a measure of intensity. [10] extended their technique by expressing the newly created emotion in a muscle-based talking head instead of 2D emotion space. Although there is an overlap between these techniques and that presented in this paper, our method is the other way round in that, given a set of facial points, we consider how to recognise the (basic or complex/intermediate) emotions as well as the corresponding intensities, and map them onto activation-evaluation space.

The activation-evaluation space describes a disk of potential emotions, and so a random variable describing emotional data is a circular random variable. Since circular data 'wraps' at 2π, the popular probability distribution models for emotion recognition such as (mixtures of) multi-variate Gaussian models are inadequate for characterising such data. The frequency distributions of circular or angular variables can be described by a large number of probability density functions (pdf) [18], [19], [20]. Based on observations of the datapoints mapped into activation-evaluation space (using the method described in Section II-C) we believe that the data is approximately Gaussian in a local neighbourhood. For this reason we have chosen to use the von Mises distribution, which is the most common model for symmetric uni-modal samples of circular data and is a close approximation to a 'wrapped normal distribution' and so a circular analogue of the normal distribution [19]. This distribution was introduced by Richard von Mises in 1918. In directional statistics its importance is almost the same as that of the normal distribution on the line. We have used the von Mises pdf to represent the distributions of six basic emotions, and their mixtures to interpolate complex emotions.

II. Methodology

A. The Dataset

In practical applications of emotion recognition the ideal would obviously be to deal with images of the face alone. However, for simplicity we have chosen to start with a simpler problem. We used the Interactive Emotional Dyadic Motion Capture (IEMOCAP) dataset, in which a pair of actors were recorded using a high-speed camera capturing 120 frames per second. One of the actors had a set of reflective markers on their face and the 3D positions of these markers was tracked with very high accuracy [1]. We based our analysis on the locations of these marker points. The videos of the actors were watched by three human experts who labelled the dataset, providing a ground truth labelling of the data. However, the experts did not always agree. Each conversation consists of almost 50 utterances with an average duration of 4.5 seconds each, It is these utterances that were annotated by the three human evaluators into categorical labels (neutral, happy, angry, sad, surprise, disgust, fear, frustration, and excitement) as well as psychological data about emotion intensity (valence, activation, and dominance). Although nine emotional labels were used by the humans, we chose to consider only six of them, as for the missing emotions (disgust, surprise, and fear) there was insufficient data. For further details on any part of the data capture and labelling, see [1].

We consider the two actors separately in everything that follows. We started by creating a training set of 4,000 frames of each of the six emotions (five basic emotions and neutral) to make a total set of 24,000 frames for each of the actors. These 24,000 frames were chosen from the set where all three human experts agreed on the annotation. We then created a test set based on seven continuous conversations, each lasting around 3 minutes, comprising almost $152,000$ frames in total.

B. The Shape Models

Based on the data from IEMOCAP we had sets of facial points with an associated emotion label. IEMOCAP has 61 marker points (2 on the head, 6 on the hands, and the rest on the face) in 3D. We restricted this to 28 points by ignoring the hands and head, and replacing correlated sets of points (such as the cheeks and forehead) by one point in each region. We then used principal component analysis (PCA) to develop three shape models of the dataset: the full face (28 markers), the upper face (17 markers) and lower face (11 markers) separately [21]. For the full face, the first 5 PCs covered 93% of the total variation of the training data, for the upper face, the first 5 PCs covered 93.4% of the total variance and for the lower face, the first 4 PCs covered almost 95% of the total variance of the data. We noticed that the first PC of full and lower face, which covers almost 50% of the total variation, was describing the upward and downward movement of the mouth points. This movement of lips was experimentally shown to be highly correlated with talking, which is not directly connected with emotion recognition, and not much else, and so we discarded the first PC.

Consequently, we chose to use four PCs (2-5) of full and lower face model and four PCs (1-4) of the upper face model for our analysis. We transformed the training data into the three different 4D spaces of these sets of four principal components. Each datapoint was then labelled with the majority vote of the three human experts, so that the training set consists of 24,000 points, each labelled with one of six emotions in three different 4D spaces.

For classification of a test frame, it was transformed into the 4D space of each model separately. We then computed the Mahalanobis distance between the test frame and each of the six emotion clusters. In this way, we get three sets of six distances; one for each emotion in each model space (total 18 distances). For each emotion we take the minimum distance across the three models and these distances are used to map the emotion into the activation-evaluation space.

C. Mapping Emotions into the Activation-Evaluation Space

There are two steps required to map the representation of the facial points of an image frame into activation-evaluation space: represent the basic emotions as points within that space, and then position each frame from there (using the six distances to the basic emotions). The first of these steps uses the training data, which are assumed to represent just one emotion (since all three experts agreed on their labels), while the second uses the test data.

As well as giving the emotion class label, the human experts annotated each utterance with valence and activation values (as a self-assessment manikins (SAMs) score between 1 and 5; we rescaled these values to [-1,+1]). For the 6,000 frames of each emotion this represented 10-15 utterances, so we had 10-15 values for three expert annotations for each emotion. We transformed these into polar coordinates (which corresponds to intensity of emotion in the radial direction and particular emotion in the angular direction) and then computed the mean average of the 30-45 values for each emotion.

For the radial component the linear statistical mean average is correct, but in order to average angles, it is not appropriate. For example, the arithmetic mean of the angles $1°$ and $359°$ is $180°$, which is different from the geometrical mean of $0°$ [19]. Therefore, we chose to calculate the geometric mean to get the angular position of each of the basic emotions in the activation-evaluation space. The compution of the mean direction and magnitude of the resultant vector for each of the six emotions separately is as follows:

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} val_i, \quad \bar{A} = \frac{1}{n}\sum_{i=1}^{n} act_i \tag{1}$$

where n is the number of utterances of each emotion.

$$\mu = \begin{cases} \tan^{-1}(\bar{A}/\bar{V}) & \bar{A} > 0, \bar{V} > 0 \\ \tan^{-1}(\bar{A}/\bar{V}) + \pi & \bar{V} < 0 \\ \tan^{-1}(\bar{A}/\bar{V}) + 2\pi & \bar{A} < 0, \bar{V} > 0 \end{cases} \tag{2}$$

$$\bar{R}^2 = \bar{V}^2 + \bar{A}^2 \tag{3}$$

μ is the mean direction and $\bar{R}$ corresponds to the mean emotion point (in terms of valence and activation) on the activation-evaluation space.

This gave us locations for the six basic emotions (including neutral). Each test frame is assumed to be a combination of the basic emotions, and so we needed to calculate the weighted average of basic emotions, where the weights correspond to the classification confidence of test frames for each basic emotion. We modelled the distribution of each basic emotion as a von Mises distribution and constructed a mixture model of them to calculate the weighted average of basic emotions for each test frame. This is described in section II-D. In the work described in this paper the emotions corresponding to each utterance are mapped into the activation-evaluation space frame by frame; however, we have also extended this work to the computational analysis of continuous emotion trajectories to understand emotion dynamics [22].

D. von Mises Mixture Model

A circular variable θ is said to have a von Mises distribution if the probability density function is given by:

$$m(\theta; K, \mu) = \frac{1}{\pi I_0(K)} e^{[K\cos(\theta-\mu)]}, \tag{4}$$

where $0 \leq \theta < 2\pi$, $K > 0$ and $0 \leq \mu < 2\pi$.

The parameter μ is the mean direction and K is the concentration parameter, which is analogous to the (inverse) variance: the density at the mode depends on e^{2K} and the larger the value of K, the greater is the clustering around the mode. The distribution is uni-modal and symmetric about μ. $I_0(K)$ is a normaliser to turn this into a probability density function and consists of a modified Bessel function of the first kind of order zero [23].

Although each emotion class is uni-modal, we cannot fit one von Mises distribution to the full data as it is the mixture of six different emotion classes. Such multi-modal distributions may be regarded as mixtures of uni-modal distributions. We used a finite mixture model of six uni-modal von Mises distributions, given by:

$$M = \sum_{j=1}^{6} \omega_j m_j(\theta) \tag{5}$$

where ω_j are non-negative weights that sum to one. We have already calculated the mean direction (μ_j) of each of the six reference emotions in the space using Eq. (2), and the method of estimating K_j and ω_j are described in the following section.

E. Estimating the Parameters of the Mixture Model

There are several ways to estimate the parameters on which the mixture model depends [19]. We have used the usual maximum likelihood estimate for K_j. However, the weights ω_j of each emotion model are estimated by using the distances to the six emotions calculated by the shape models.

1) Estimating the Concentration Parameter: The concentration parameter K_j is estimated by using the Fisher equation [18]:

$$\hat{K}_{ML} = \begin{cases} 2\bar{R} + \bar{R}^3 + 5\bar{R}^5/6 & \bar{R} < 0.53 \\ -0.4 + 1.39\bar{R} + 0.43/(1-\bar{R}) & 0.53 \leq \bar{R} < 0.85 \\ 1/(\bar{R}^3 - 4\bar{R}^2 + 3\bar{R}) & \bar{R} \geq 0.85 \end{cases} \tag{6}$$

$\hat{K}_{ML}$ may be biased if the sample size (n) and $\bar{R}$ are small (specially when $\bar{R} < 0.45$). For this reason, if $n \leq 15$, the following estimate is to be preferred:

$$\hat{K} = \begin{cases} \max(\hat{K}_{ML} - 2(n\hat{K}_{ML})^{-1}, 0) & \hat{K}_{ML} < 2 \\ (n-1)^3\hat{K}_{ML}/(n^3+n) & \hat{K}_{ML} \geq 2 \end{cases} \tag{7}$$

2) Estimating the Weights: We have calculated the Mahalanobis distance of each test frame to each of the six basic emotions using the shape models and retained the minimum distance of the three models for each emotion. We want to position each test frame in activation-evaluation space using the positions of the basic emotions. However, the Mahalanobis distance is an unsigned quantity and so we do not know the direction between the test frame and the mean of each of the clusters of basic emotions. Since we have assumed that each emotion lies along a radial line in the activation-evaluation space we want to compute the intensity of each of the basic emotions as a component of the complex emotion. We did this starting at the position of the basic emotion and then by applying a simple rule to move along that radial line: if the distance of the test frame from neutral is less than the mean of a particular emotion, then the distance of the test frame from that emotion must be towards neutral i.e., its intensity decreases and comes close to neutral and vice-versa. We convert these distances to weights (ω_j) by reciprocating their values.

III. Experimental Results

Fig. 1 plots the locations of the basic emotion in activation-evaluation space using the estimated mean positions of the training set. We observed that the estimated directions are quite close to those specified by Whissell in [9], except that of neutral which corresponds to a very passive emotional state with very low intensity (see Table I for numerical values).

Based on these positions for the basic emotions we were now able to compute the parameters of the mixture model and test it using initially single utterances with only one labelled emotion, and then full conversations with several emotion transitions. Fig. 2(a) shows the von Mises probability distributions of all emotions in the mixture model. The x-axis shows the emotion directions (angles in degrees) (see Table I for numerical values) and the y-axis shows the von Mises probability densities for each distribution in the mixture model. It is clear that there is a big overlap between the distributions of anger and frustration as well as of happiness and excitement. Sadness lies quite close to the anger/frustration distribution. This is to be expected based on the Whissell angles. Fig. 2(b) shows the von Mises probability distributions characterising

TABLE I: Angular values from Whissell's study and those estimated by the model.

Emotions	Whissell's Angles (in degrees)	Estimated Angles (in degrees)
Sadness	108.5	131.10
Frustration	200.6	188.48
Anger	212	194.84
Excitement	311	330.58
Happiness	323.7	341.45
Neutral	0	89.23 (inclined towards *very passive* state)

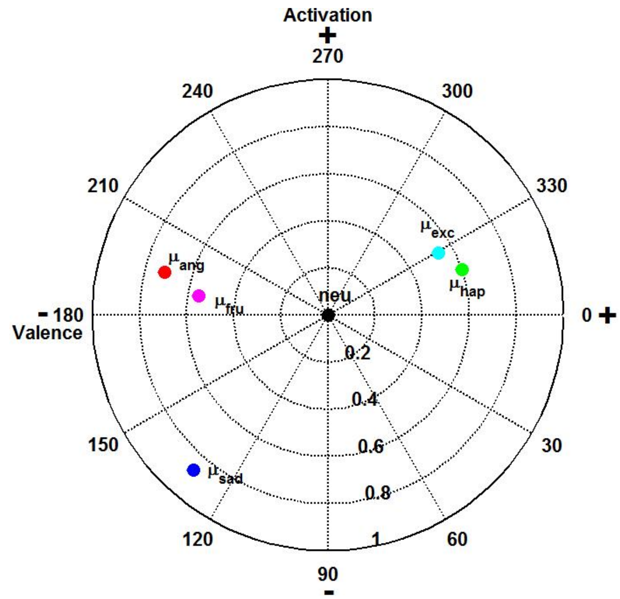


Fig. 1: The position of each basic emotion (based on the training set) in the activation-evaluation space.

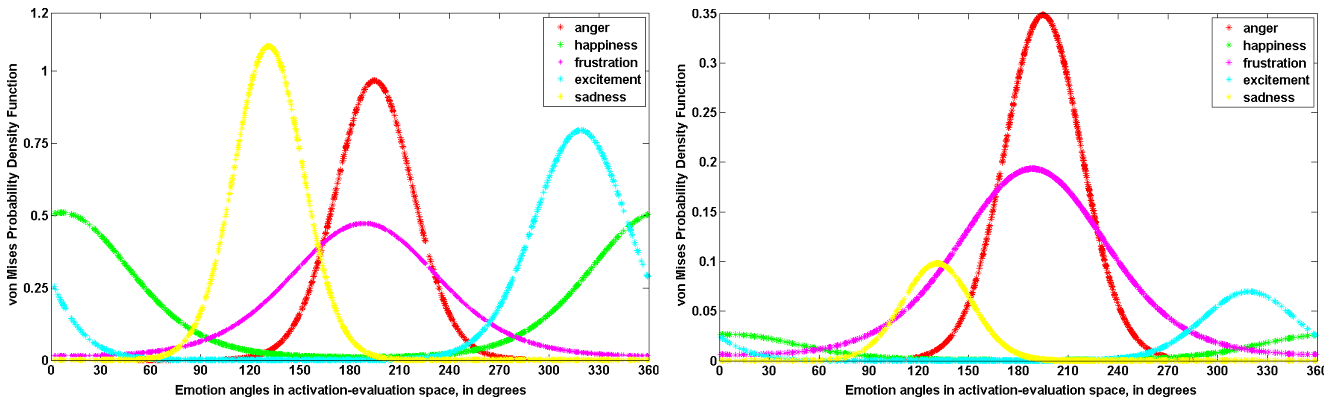
one frame of an utterance labelled as [angry, angry, frustrated] by three human experts.

We use the valence and activation values assigned to each utterance by three human experts to estimate the ground truth direction and intensity of emotion associated with that utterance, which we can compare to our results. In order to measure the association between two circular variables (the ground truth direction and that estimated by the model), we have used a measure of circular sample correlation coefficient $(\rho_{c,n})$ [20]. If $(\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)$ is a random sample, $\rho_{c,n}$ is defined as:

$$\rho_{c,n} = \frac{\sum_{i=1}^{n} \sin(\alpha_i - \mu) \sin(\beta_i - \nu)}{\sqrt{\sum_{i=1}^{n} \sin^2(\alpha_i - \mu)) \sin^2(\beta_i - \nu)}} \tag{8}$$

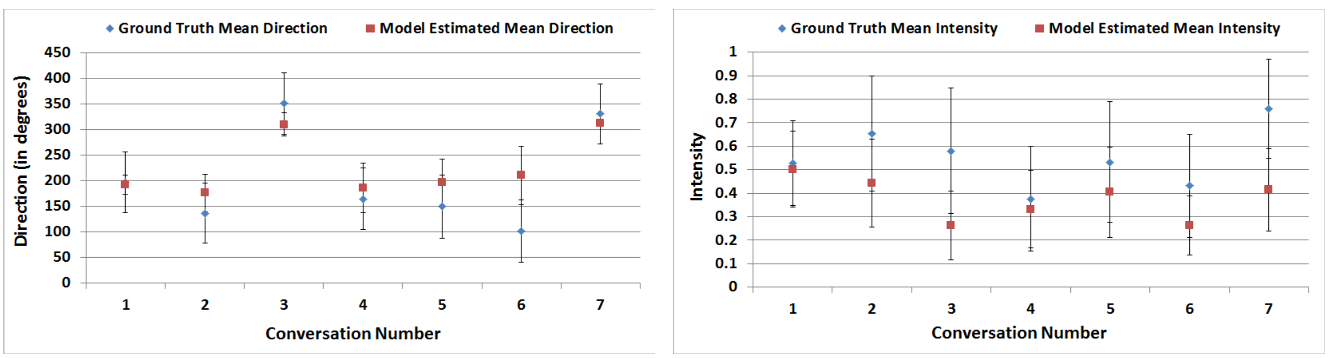
where μ and ν are the sample mean directions.

Only one ground truth measure of direction and intensity is available for each full utterance, while the model estimates the directions and intensity for each frame. In order to measure $\rho_{c,n}$, we generate a sample of (n=1000) random variables based on the von Mises probability density functions for both distributions (ground truth and model estimation). $\rho_{c,n}$ shows a significantly high correlation between samples of ground truth direction and those estimated by the model for each conversation. We also applied the pairwise *t*-test on the intensity values and found that the intensity calculated by ground truth values and those estimated by the mixture model are not statistically different $(p > 0.05)$. Fig. 3 presents the visual fit of ground truth mean directions and mean intensities



(a) (b)

Fig. 2: (a) von Mises Probability Distributions in the mixture model with unit weight, (b) von Mises Probability Distributions characterising one frame of an utterance labelled as [angry,angry,frustrated] by three human experts.



(a) (b)

Fig. 3: The test of fit for (a) the mean ground truth *directions* and those estimated by the mixture model (b) the mean ground truth *intensities* and those estimated by the mixture model, for each of the seven conversations in the test set. Lines mark one standard deviation.

with those estimated by the model for the test set consisting of seven different conversations.

Fig. 4 shows the mapping of continuous emotions through time corresponding to a single utterance (Ses01F_script02_1_F007) in the activation-evaluation space. The colour variation represents time, ranging from red (dark in grayscale) to yellow (light in grayscale). The utterance is labelled as angry/angry/frustrated by the three human observers, which matches the observation well. The figure also plots the ground truth values of the mean direction and intensity and those estimated by the model. The analysis of continuous emotions through time in activation-evaluation space is described in [22].

Figs. 3 and 4 show that the proposed mixture model fits the data well, despite the underlying problems with the ground truth labelling (that is, the fact that there is only one label associated with each utterance, which lasts for many frames while the model estimates the values for each frame). Furthermore, all 'silent' frames are unlabelled in the conversations while the model estimates the values for those frames as well. The intensity values do not fit as well as the directions because the small number of samples leads to high concentration around the mean as compared to the large number of frames in the test set.

IV. Conclusion

In this paper we have presented a statistical model for the recognition and representation of complex emotions in the activation-evaluation space. The proposed model is based on the psychological assumption that complex emotions are comprised of mixtures of basic emotions. There is still debate among psychologists on the number of basic emotions and which emotions should be considered as basic, and of the

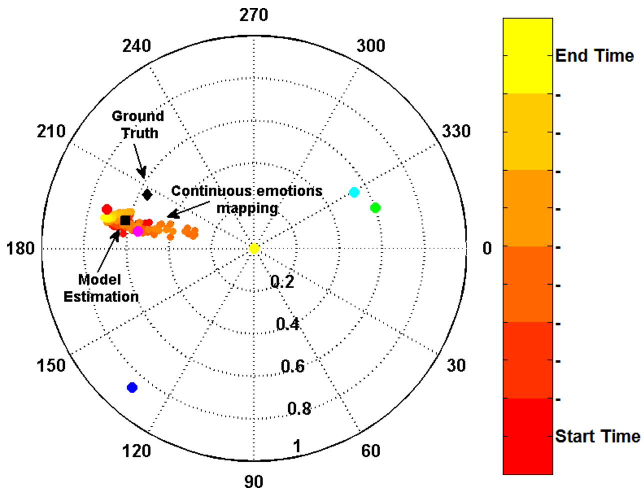


Fig. 4: The mapping of continuous emotions during an utterance on the activation-evaluation space, along with the mean ground truth direction and intensity and those estimated by the model. The movement of emotions through time is represented by changing colour spectrum from dark/red (start) to light/yellow (end).

six emotions that we have considered two (frustration and excitement) are candidate basic emotions [2], [3]. However, the proposed mixture model is quite flexible and can be applied to any set of basic emotions. We estimated the degree of similarity of each test frame to each of the basic emotions and project them into the activation-evaluation space using the von Mises mixture model, which takes into account the circular nature of the emotion space.

In this paper each continuous conversation is mapped into the activation-evaluation space frame by frame, but we have extended this work to the computational analysis of continuous emotion trajectories in the activation-evaluation space [22]. That paper focusses on the analysis of emotion dynamics, since emotions constitute several variations in the intensity, flow, persistence with time, and their relationships with other emotions. By analysing the emotion dynamics through time, we try to seek the answers about the 'common' paths between emotions, the smoothness of emotion trajectories, and how do we travel along emotion flows. The computational analysis of emotion dynamics may be helpful in better understanding of emotion trajectories as well as in the development of more flexible models for emotion recognition, representation, and synthesis.

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