Some Computable Structure Theory of Finitely Generated Structures

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Workshop on Computable Structures and Reverse Mathematics 2017
in celebration of the work of Rod Downey

Happy birthday Rod!

Outline

The main question:

Which classes of finitely generated structures contain complicated structures?

The particular focus will be on groups.

General outline:

1. Descriptions (Scott sentences) of finitely generated structures, and in particular groups, among countable structures.
2. A notion of universality using computable functors (or equivalently effective interpretations).
3. Descriptions (quasi Scott sentences) of finitely generated structures among finitely generated structures.

Scott Sentences of Finitely Generated Structures

Infinitary Logic

$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countably infinite conjunctions and disjunctions.

There is a hierarchy of $\mathcal{L}_{\omega_1\omega}$-formulas based on their quantifier complexity after putting them in normal form. Formulas are classified as either Σ^0_α or Π^0_α, for $\alpha < \omega_1$.

- A formula is Σ^0_0 and Π^0_0 is it is finitary quantifier-free.
- A formula is Σ^0_α if it is a disjunction of formulas $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is Π^0_β for $\beta < \alpha$.
- A formula is Π^0_α if it is a conjunction of formulas $(\forall \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is Σ^0_β for $\beta < \alpha$.

Examples of Infinitary Formulas

Example

There is a Π^0_2 sentence which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n \in \mathbb{N}} nx = 0.$$

Example

There is a Σ^0_1 formula which describes the dependence relation on triples x, y, z in a $\mathbb{Q}$-vector space:

$$\bigvee_{(a,b,c) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}} ax + by + cz = 0$$

Examples of Infinitary Formulas

Example

There is a Σ^0_3 sentence which says that a $\mathbb{Q}$-vector space has finite dimension:

$$\bigvee_{n\in\mathbb{N}} (\exists x_1,\dots,x_n)(\forall y)\; y \in \mathrm{span}(x_1,\dots,x_n).$$

Example

There is a Π^0_3 sentence which says that a $\mathbb{Q}$-vector space has infinite dimension:

$$\bigwedge_{n\in\mathbb{N}} (\exists x_1,\dots,x_n)\; \mathrm{Indep}(x_1,\dots,x_n).$$

Scott Sentences

Let $\mathcal{A}$ be a countable structure.

Theorem (Scott)

There is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that:

$$\mathcal{B} \text{ countable, } \mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

φ is a *Scott sentence* of $\mathcal{A}$.

Example

$(\omega, <)$ has a Π^0_3 Scott sentence consisting of the Π^0_2 axioms for infinite linear orders together with:

$$\forall y_0 \bigvee_{n\in\omega} \exists y_n < \cdots < y_1 < y_0 \left[\forall z \ (z > y_0) \vee (z = y_0) \vee (z = y_1) \vee \cdots \vee (z = y_n)\right].$$

Scott Rank

Let $\mathcal{A}$ be a countable structure.

Definition (Montalbán)

$SR(\mathcal{A})$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi^0_{\alpha+1}$ Scott sentence.

Theorem (Montalbán)

Let α a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Pi^0_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is Σ^0_{α}-definable without parameters.*
- *$\mathcal{A}$ is uniformly (boldface) $\mathbf{\Delta}^0_{\alpha}$-categorical without parameters.*

An Upper Bound on the Complexity of Finitely Generated Structures

Theorem (Knight-Saraph)

Every finitely generated structure has a Σ^0_3 Scott sentence.

Often there is a simpler Scott sentence.

A Scott Sentence for the Integers

Example

A Scott sentence for the group $\mathbb{Z}$ consists of:

- the axioms for torsion-free abelian groups,
- for any two elements, there is an element which generates both,
- there is a non-zero element with no proper divisors:

$$(\exists g \neq 0) \bigwedge_{n \geq 2} (\forall h)[nh \neq g].$$

d-Σ^0_2 Sentences

φ is d-Σ^0_2 if it is a conjunction of a Σ^0_2 formula and a Π^0_2 formula.

$$\begin{array}{ccccccccccc}
 & \Sigma_1 & & & \Sigma_2 & & & \Sigma_3 & & \\
 \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & & \searrow \\
\Sigma_1 \cap \Pi_1 & & \text{d-}\Sigma_1 \to \Sigma_2 \cap \Pi_2 & & & & \text{d-}\Sigma_2 \to \Sigma_3 \cap \Pi_3 & & & \text{d-}\Sigma_3 \to \cdots \\
 \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & & \nearrow \\
 & \Pi_1 & & & \Pi_2 & & & \Pi_3 & &
\end{array}$$

Theorem (Miller)

Let $\mathcal{A}$ be a countable structure. If $\mathcal{A}$ has a Σ^0_3 Scott sentence, and also has a Π^0_3 Scott sentence, then $\mathcal{A}$ has a d-Σ^0_2 Scott sentence.

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This is a d-Σ^0_2 Scott sentence.

A Scott Sentence for the Free Group

Example (CHKLMMMQW)

A Scott sentence for the free group $\mathbb{F}_2$ on two elements consists of:

- the group axioms,
- every finite set of elements is generated by a 2-tuple,
- there is a 2-tuple $\bar{x}$ with no non-trivial relations such that for every 2-tuple $\bar{y}$, $\bar{x}$ cannot be expressed as an "imprimitive" tuple of words in $\bar{y}$.

A pair u, v of words is primitive if whenever $\bar{x}$ is a basis for $\mathbb{F}_2$, $u(\bar{x}), v(\bar{x})$ is also a basis for $\mathbb{F}_2$.

This is a d-Σ^0_2 Scott sentence.

d-Σ_2^0 Scott Sentences for Many Groups

Theorem (Knight-Saraph, CHKLMMMQW, Ho)

The following groups all have d-Σ_2^0 Scott sentences:

- *abelian groups,*
- *free groups,*
- *nilpotent groups,*
- *polycyclic groups,*
- *lamplighter groups,*
- *Baumslag-Solitar groups $BS(1, n)$.*

Question

Does every finitely generated group have a d-Σ_2^0 Scott sentence?

Characterizing the Structures with d-Σ^0_2 Scott Sentences

The first step is to understand when a finitely generated structure has a d-Σ^0_2 Scott sentence.

Theorem (A. Miller, HT-Ho, Alvir-Knight-McCoy)

Let $\mathcal{A}$ be a finitely generated structure. The following are equivalent:

- *$\mathcal{A}$ has a Π^0_3 Scott sentence.*
- *$\mathcal{A}$ has a d-Σ^0_2 Scott sentence.*
- *$\mathcal{A}$ is the only model of its Σ^0_2 theory.*
- *some generating tuple of $\mathcal{A}$ is defined by a Π^0_1 formula.*
- *every generating tuple of $\mathcal{A}$ is defined by a Π^0_1 formula.*
- *$\mathcal{A}$ does not contain a copy of itself as a proper Σ^0_1-elementary substructure.*

Proof, First Direction

Suppose that $\mathcal{A}$ does not contain a copy of itself as a proper Σ_1^0-elementary substructure.

Let p be the $\forall$-type of a generating tuple for $\mathcal{A}$.

We can write down a d-Σ_2^0 Scott sentence for $\mathcal{A}$:

- there is a tuple $\bar{x}$ satisfying p, and
- for all tuples $\bar{x}$ satisfying p and for all y, y is in the substructure generated by $\bar{x}$.

Proof, Second Direction

Now suppose that $\mathcal{A}$ does contain a copy of itself as a proper Σ_1^0-elementary substructure.

Take the union of the chain

$$\mathcal{A} \prec_{\Sigma_1^0} \mathcal{A} \prec_{\Sigma_1^0} \mathcal{A} \prec_{\Sigma_1^0} \cdots \prec_{\Sigma_1^0} \mathcal{A}^*.$$

Then $\mathcal{A}^*$ has the same Σ_2^0 theory as $\mathcal{A}$, but is not finitely generated.

In particular, $\mathcal{A}$ does not have a d-Σ_2^0 Scott sentence.

A Complicated Group

Theorem (HT-Ho)

There is a computable finitely generated group G which does not have a d-Σ_2^0 Scott sentence.

The construction of G uses small cancellation theory and HNN extensions.

Theorem (HT-Ho)

There is a computable finitely generated ring $\mathbb{Z}[G]$ which does not have a d-Σ_2^0 Scott sentence.

This is just the group ring of the previous group.

No Complicated Fields

Theorem (HT-Ho)

Every finitely generated field has a d-Σ_2^0 Scott sentence.

Proof sketch:

Suppose that E is a proper Σ_1^0-elementary substructure of F, with E isomorphic to F.

Then E and F have the same transcendence degree.

So F/E is an algebraic extension.

Then the atomic type of the generators of F over E is isolated, and so cannot be realized in E.

Open Questions

Question

Does every finitely presented group have a d-Σ_2^0 Scott sentence?

Question

Does every commutative ring have a d-Σ_2^0 Scott sentence?

Question

Does every integral domain have a d-Σ_2^0 Scott sentence?

Computable Functors, Effective Interpretations, and Universality

Computable Structure Theory

Our structures are all countable structures with domain ω.

A structure is computable if its domain is computable, and its functions and relations are computable as functions $\omega^n \to \omega$ and as subsets of ω^n respectively.

Two computable copies of the same structure are isomorphic, but they are not necessarily computably isomorphic.

Computable Dimension

Definition

The computable dimension of a computable structure $\mathcal{A}$ is the number of computable copies up to computable isomorphism.

Theorem (Goncharov; Goncharov and Dzgoev; Metakides and Nerode; Nurtazin; LaRoche; Remmel)

All structures in each of the following classes have computable dimension 1 *or* ω*:*

- *algebraically closed fields,*
- *real closed fields,*
- *torsion-free abelian groups,*
- *linear orderings,*
- *Boolean algebras.*

Finite Computable Dimension > 1

Theorem (Goncharov)

For each $n > 0$ there is a computable structure with computable dimension n.

Theorem (Goncharov; Goncharov, Molokov, and Romanovskii; Kudinov)

For each $n > 0$ there are structures with computable dimension n in each of the following classes:

- *graphs,*
- *lattices,*
- *partial orderings,*
- *2-step nilpotent groups,*
- *integral domains.*

Hirschfeldt, Khoussainov, Shore, and Slinko in *Degree Spectra and Computable Dimensions in Algebraic Structures*, 2000:

> *Whenever a structure with a particularly interesting computability-theoretic property is found, it is natural to ask whether similar examples can be found within well-known classes of algebraic structures, such as groups, rings, lattices, and so forth.*

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Hirschfeldt, Khoussainov, Shore, and Slinko in *Degree Spectra and Computable Dimensions in Algebraic Structures*, 2000, continued:

> *The codings we present are general enough to be viewed as establishing that the theories mentioned above are computably complete in the sense that, for a wide range of computability-theoretic non-structure type properties, if there are any examples of structures with such properties then there are such examples that are models of each of these theories.*

Universal Classes

Theorem (Hirschfeldt, Khoussainov, Shore, Slinko)

Each of the classes

- *undirected graphs,*
- *partial orderings,*
- *lattices,*
- *integral domains,*
- *commutative semigroups, and*
- *2-step nilpotent groups.*

is complete with respect to

- *degree spectra of nontrivial structures,*
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- *Scott ranks,*
- *categoricity spectra, ...*

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Solution two (Montalbán): Use effective bi-interpretations.

Theorem (Montalbán)

If $\mathcal{A}$ and $\mathcal{B}$ are bi-interpretable, then they are essentially the same from the point of view of computable structure theory. In particular, the complexity of their optimal Scott sentences are the same.

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These two solutions are equivalent.

Effective Interpretations

Let $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \dots)$ where $P_i^{\mathcal{A}} \subseteq A^{a(i)}$.

Definition

$\mathcal{A}$ is *effectively interpretable* in $\mathcal{B}$ if there exist a uniformly computable Δ_1^0-definable relations $(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, R_1, \dots)$ such that

(1) $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$,

(2) $\sim$ is an equivalence relation on $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

(3) $R_i \subseteq (B^{<\omega})^{a(i)}$ is closed under $\sim$ within $\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}}$,

and a function $f_{\mathcal{A}}^{\mathcal{B}}: \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \to \mathcal{A}$ which induces an isomorphism:

$$(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} / \sim; R_0/\sim, R_1/\sim, \dots) \cong (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \dots).$$

Computable Functors

Definition

$\mathrm{Iso}(\mathcal{A})$ is the category of copies of $\mathcal{A}$ with domain ω. The morphisms are isomorphisms between copies of $\mathcal{A}$.

Recall: a functor F from $\mathrm{Iso}(\mathcal{A})$ to $\mathrm{Iso}(\mathcal{B})$

(1) assigns to each copy $\widehat{\mathcal{A}}$ in $\mathrm{Iso}(A)$ a structure $F(\widehat{\mathcal{A}})$ in $\mathrm{Iso}(\mathcal{B})$,

(2) assigns to each isomorphism $f\colon\widehat{\mathcal{A}}\to\widetilde{\mathcal{A}}$ in $\mathrm{Iso}(\mathcal{A})$ an isomorphism $F(f)\colon F(\widehat{\mathcal{A}})\to F(\widetilde{\mathcal{A}})$ in $\mathrm{Iso}(\mathcal{B})$.

Definition

F is *computable* if there are computable operators Φ and Φ_* such that

(1) for every $\widehat{\mathcal{A}}\in\mathrm{Iso}(\mathcal{A})$, $\Phi^{D(\widehat{\mathcal{A}})}$ is the atomic diagram of $F(\mathcal{A})$,

(2) for every isomorphism $f:\widehat{\mathcal{A}}\to\widetilde{\mathcal{A}}$, $F(f)=\Phi_*^{D(\widehat{\mathcal{A}})\oplus f\oplus D(\widetilde{\mathcal{A}})}$.

Equivalence

An effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ induces a computable functor from $\mathcal{B}$ to $\mathcal{A}$.

Theorem (HT-Melnikov-Miller-Montalbán)

Effective interpretations of $\mathcal{A}$ in $\mathcal{B}$ are in correspondence with computable functors from $\mathcal{B}$ to $\mathcal{A}$.

By "in correspondence" we mean that every computable functor is effectively isomorphic to one induced by an effective interpretation.

Effective Isomorphisms of Functors

Let $F, G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ be computable functors.

Definition

F is *effectively isomorphic* to G if there is a computable Turing functional Λ such that for any $\widetilde{\mathcal{B}} \in \mathrm{Iso}(\mathcal{B})$, $\Lambda^{\widetilde{\mathcal{B}}}$ is an isomorphism from $F(\widetilde{\mathcal{B}})$ to $G(\widetilde{\mathcal{B}})$, and the following diagram commutes:

$$\widetilde{\mathcal{A}}$$

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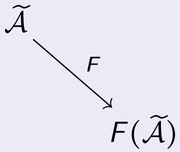

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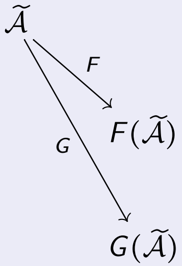

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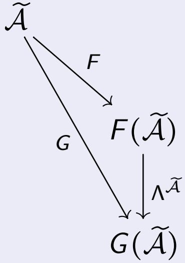

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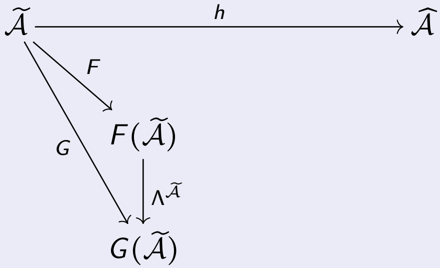

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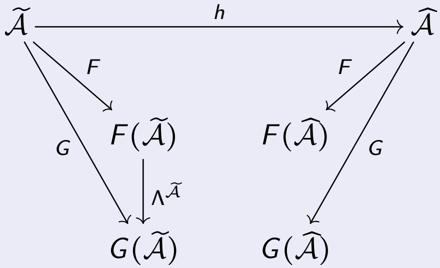

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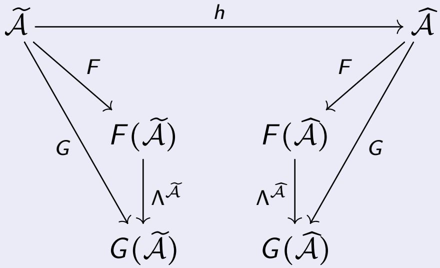

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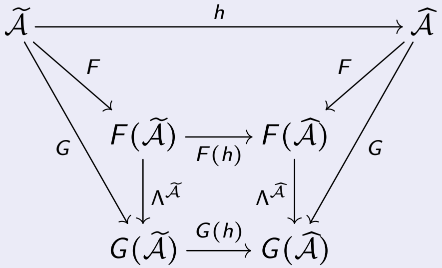

Effective Bi-Interpretations

Definition

$\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are effective interpretations of each in the other, and Δ^0_1-definable isomorphisms $\mathcal{D}om_{\mathcal{A}}^{(\mathcal{D}om_{\mathcal{B}}^{\mathcal{A}})} \to \mathcal{A}$ and $\mathcal{D}om_{\mathcal{B}}^{(\mathcal{D}om_{\mathcal{A}}^{\mathcal{B}})} \to \mathcal{B}$.

$$\begin{array}{ccc} & & \mathcal{B} \\ & & \rotatebox{90}{$\subseteq$} \\ \mathcal{A} & \longrightarrow & \mathcal{D}om_{\mathcal{A}}^{\mathcal{B}} \end{array}$$

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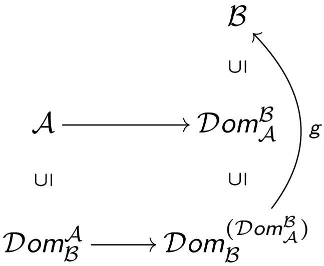

Computable Bi-transformations

Definition

A computable equivalence of categories between $\mathcal{A}$ and $\mathcal{B}$ consists of computable functors $F\colon \mathrm{Iso}(\mathcal{A}) \to \mathrm{Iso}(\mathcal{B})$ and $G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ such that both $F \circ G\colon \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{B})$ and $G \circ F\colon \mathrm{Iso}(\mathcal{A}) \to \mathrm{Iso}(\mathcal{A})$ are effectively isomorphic to the identity functor.

So if $\widehat{\mathcal{B}}$ is a copy of $\mathcal{B}$, then $F(G(\widehat{\mathcal{B}})) \cong \widehat{\mathcal{B}}$ and the isomorphism can be computed uniformly in $\widehat{\mathcal{B}}$.

Equivalence

Theorem (HT-Melnikov-Miller-Montalbán)

Effective bi-interpretations between $\mathcal{A}$ and $\mathcal{B}$ are in correspondence with effective equivalences of categories between $\mathcal{A}$ and $\mathcal{B}$.

Theorem (HT-Miller-Montalbán)

Non-effective bi-interpretations between $\mathcal{A}$ and $\mathcal{B}$ are in correspondence with Borel equivalences of categories between $\mathcal{A}$ and $\mathcal{B}$ and with continuous isomorphisms between the automorphism groups of $\mathcal{A}$ and $\mathcal{B}$.

A Definition of Universality

Definition

A class $\mathcal{C}$ of structures is universal if each structure $\mathcal{A}$ is uniformly effectively bi-interpretable with a structure in $\mathcal{C}$.

Theorem

Each of the following classes is universal:

- *undirected graphs,*
- *partial orderings,*
- *lattices, and*
- *fields,*

and, after naming finitely many constants,

- *integral domains,*
- *commutative semigroups, and*
- *2-step nilpotent groups.*

Classes Which Are Not Universal

Theorem

Each of the following classes is not universal:

- *algebraically closed fields,*
- *real closed fields,*
- *abelian groups,*
- *linear orderings,*
- *Boolean algebras.*

Proof: In these classes, the computable dimension can only be 1 or ω.

Universality for Finitely Generated Structures

What about for finitely generated structures? These are never going to be universal. So we have to restrict our attention to finitely generated structures.

Definition

Let $\mathcal{C}$ be a class of finitely-generated structures. $\mathcal{C}$ is universal among finitely generated structures if every finitely generated structure is uniformly effectively bi-interpretable with one in $\mathcal{C}$.

Example

Fields are not universal among finitely generated structures.

Finitely Generated Groups Are Universal

Theorem (HT-Ho)

Finitely generated groups are universal among finitely generated structures (after naming three constants).

Moreover, the orbits of the constants are Σ^0_1 definable.

So now, instead of constructing a finitely generated group with some property, we can construct a finitely generated structure in whatever language we like.

Scott Sentences and Constants

Recall that two structures which are bi-interpretable have Scott sentences of the same complexity.

Proposition

Let $\mathcal{A}$ be a countable structure and $\bar{c} \in \mathcal{A}$. If $\mathcal{A}$ has a Σ^0_α (respectively Π^0_α, d-Σ^0_α) Scott sentence, then so does $(\mathcal{A}, \bar{c})$.

Proposition

Let $\mathcal{A}$ be a countable structure and $\bar{c} \in \mathcal{A}$.

- *If $(\mathcal{A}, \bar{c})$ has a Σ^0_α Scott sentence, then so does $\mathcal{A}$.*
- *Suppose that the orbit of $\bar{c}$ is defined by a Σ^0_β formula for some $\beta < \alpha$. If $(\mathcal{A}, \bar{c})$ has a Π^0_α (respectively d-Σ^0_α) Scott sentence, then so does $\mathcal{A}$.*

Fields are not universal among finitely-generated structures, even adding finitely many constants, because every finitely generated field has a d-Σ^0_2 Scott sentence.

Quasi Scott Sentences

Quasi Scott Sentences

When we constructed a group with no d-Σ^0_2 Scott sentence before, we used an infinitely generated group with the same Σ^0_2 theory. What happens if we ask for a description of a finitely generated group within the class of finitely generated groups?

Definition

We say that a sentence φ is a quasi Scott sentence for a finitely generated structure $\mathcal{A}$ if $\mathcal{A}$ is the unique finitely generated structure satisfying φ.

Fact (HT-Ho)

Each finitely generated structure has a Π^0_3 quasi Scott sentence.

Let p be the atomic type of a generating tuple of $\mathcal{A}$. The Π^0_3 quasi Scott sentence for $\mathcal{A}$ says that every tuple is generated by a tuple of type p.

Quasi Scott Sentences, Constants, and Bi-interpretations

First, we show that we can consider arbitrary finitely generated structures instead just groups.

Proposition

Let $\mathcal{A}$ be a countable structure and $\bar{c} \in \mathcal{A}$. Suppose that the orbit of $\bar{c}$ is defined by a Σ^0_1 formula $\psi(\bar{x})$. Then $\mathcal{A}$ has a Σ^0_2 (respectively Π^0_2, d-Σ^0_2) quasi Scott sentence if and only if $(\mathcal{A}, \bar{c})$ does.

Proposition

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable (plus a little bit more). Then $\mathcal{A}$ has a d-Σ^0_2 quasi Scott sentence if and only if $\mathcal{B}$ does.

A Partial Characterization

When does a finitely generated structure have a simpler description?

It turns out that things are more complicated than they were before.

Theorem (HT-Ho)

Let $\mathcal{A}$ be a finitely generated structure. The following are equivalent:

- *The Σ^0_2 theory of $\mathcal{A}$ has more than one finitely generated model.*
- *There is a finitely generated structure $\mathcal{B}$ not isomorphic to $\mathcal{A}$ such that $\mathcal{A} \prec_{\Sigma^0_1} \mathcal{B}$ and $\mathcal{B} \prec_{\Sigma^0_1} \mathcal{A}$.*

The Σ^0_2 theory of $\mathcal{A}$ contains both the Σ^0_2 and the Π^0_2 sentences true of $\mathcal{A}$.

A Complicated Structure

Theorem (HT-Ho)

There is a finitely generated structure whose Σ^0_2 theory has more than one finitely generated model. This structure has no d-Σ^0_2 quasi Scott sentence.

In particular, it is possible to have both a Σ^0_3 and a Π^0_3 quasi Scott sentence without having a d-Σ^0_2 quasi Scott sentence.

Corollary (HT-Ho)

There is a finitely generated group whose Σ^0_2 theory has more than one finitely generated model. This group has no d-Σ^0_2 quasi Scott sentence.

d-Σ_2^0 Quasi Scott Sentences

We do not have a complete characterization of structures with d-Σ_2^0 quasi Scott sentences.

One can still often write one down by hand.

Simple Quasi Scott Sentence But No Simple Scott Sentence

Theorem (HT-Ho)

There is a finitely generated structure which has no d-Σ_2^0 Scott sentence, but has a d-Σ_2^0 quasi Scott sentence.

We build a finitely generated structure which is a Σ_1^0-elementary substructure of itself, but not of any other finitely generated structure.

Corollary (HT-Ho)

There is a finitely generated group which has no d-Σ_2^0 Scott sentence, but has a d-Σ_2^0 quasi Scott sentence.

Open Questions

The main question here that we are still working on is:

Question

Characterize the finitely generated structures which are the only finitely-generated models of their Σ^0_2 theory, but which do not have a d-Σ^0_2 quasi Scott sentence?

Question

Are there any such models?

Computability Theory and its Applications

University of Waterloo

June 4-8, 2018.

Arrival June 3, departure afternoon of June 8.

Organizers and Program Committee: Barbara Csima (chair); Matthew Harrison-Trainor, Laurent Bienvenu, Peter Cholak.

Public lecture by Antonio Montalbán.

Please contact Barbara Csima if interested.