Computable structures on a cone

Matthew Harrison-Trainor

University of California, Berkeley

Sets and Computations, Singapore, April 2015

Overview

Setting: $\mathcal{A}$ a computable structure.

Suppose that $\mathcal{A}$ is a "natural structure".

OR

Consider behaviour on a cone.

What are the possible:

- computable dimensions of $\mathcal{A}$? (McCoy)
- degrees of categoricity of $\mathcal{A}$? (Csima, H-T)
- degree spectra of relations on $\mathcal{A}$? (H-T)

Conventions

All of our languages will be computable.

All of our structures will be countable with domain ω.

A structure is computable if its atomic diagram is computable.

Natural structures

What is a "natural structure"?

A "natural structure" is a structure that one would expect to encounter in normal mathematical practice, such as $(\omega, <)$, a vector space, or an algebraically closed field.

A "natural structure" is *not* a structure that has been constructed by a method such as diagonalization to have some computability-theoretic property.

Key observation: Arguments involving natural structures tend to relativize.

Cones and Martin measure

Definition

The cone of Turing degrees above $\mathbf{c}$ is the set

$$C_{\mathbf{c}} = \{\mathbf{d} : \mathbf{d} \geq \mathbf{c}\}.$$

Theorem (Martin 1968, assuming AD)

Every set of Turing degrees either contains a cone, or is disjoint from a cone.

Think of sets containing a cone as "large" or "measure one" and sets not containing a cone as "small" or "measure zero."

Note that the intersection of countably many cones contains another cone.

Relativizing to a cone

Suppose that P is a property that relativizes. We say that property P holds on a cone if it holds relative to all degrees $\mathbf{d}$ on a cone.

Definition

$\mathcal{A}$ is **$\mathbf{d}$-computably categorical** if every two $\mathbf{d}$-computable copies of $\mathcal{A}$ are $\mathbf{d}$-computably isomorphic.

Definition

$\mathcal{A}$ is **computably categorical on a cone** if there is a cone $C_{\mathbf{c}}$ such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical for all $\mathbf{d} \in C_{\mathbf{c}}$.

Theorem (Goncharov 1975, Montalbán 2015)

The following are equivalent:

(1) *$\mathcal{A}$ is computably categorical on a cone,*

(2) *$\mathcal{A}$ has a Scott family of $\Sigma_1^{\mathtt{in}}$ formulas,*

(3) *$\mathcal{A}$ has a $\Sigma_3^{\mathtt{in}}$ Scott family.*

Proving results about natural structures

Recall that arguments involving natural structures tend to relativize. So a natural structure has some property P if and only if it has property P on a cone.

We can study natural structures by studying all structure relative to a cone. If we prove that all structures have property P on a cone, then natural structures should have property P relative to $\mathbf{0}$.

Computable Dimension

Computable dimension

Definition

$\mathcal{A}$ has computable dimension $n \in \{1, 2, 3, \ldots\} \cup \{\omega\}$ if $\mathcal{A}$ has n computable copies up to computable isomorphism.

Theorem (Goncharov 1980)

For each $n \in \{1, 2, 3, \ldots\} \cup \{\omega\}$ there is a computable structure of computable dimension n.

Computable dimension 1 or ω

Theorem

The following structures have computable dimension 1 or ω:

1. *computable linear orders,* *[Remmel 81, Dzgoev and Goncharov 80]*
2. *Boolean algebras,* *[Goncharov 73, Laroche 77, Dzgoev and Goncharov 80]*
3. *abelian groups,* *[Goncharov 80]*
4. *algebraically closed fields,* *[Nurtazin 74, Metakides and Nerode 79]*
5. *vector spaces,* *[ibid.]*
6. *real closed fields,* *[ibid.]*
7. *Archimedean ordered abelian groups* *[Goncharov, Lempp, Solomon 2000]*
8. *differentially closed fields,* *[H-T, Melnikov, Montalbán 2014]*
9. *difference closed fields.* *[ibid.]*

Computable dimension relative to a cone

Definition

The computable dimension of $\mathcal{A}$ relative to $\mathbf{d}$ is the number $\mathbf{d}$-computable copies of $\mathcal{A}$ up to $\mathbf{d}$-computable isomorphism.

Definition

The computable dimension of $\mathcal{A}$ on a cone is the n such that the computable dimension of $\mathcal{A}$ is n for all $\mathbf{d}$ on a cone.

The computable dimension of $\mathcal{A}$ on a cone is well-defined.

Theorem on computable dimension

Let $\mathcal{A}$ be a computable structure.

Theorem (McCoy 2002)

If for all $\mathbf{d}$, $\mathcal{A}$ *has computable dimension* $\leq n \in \omega$, *then*
for all $\mathbf{d}$, $\mathcal{A}$ *has computable dimension one.*

Let $\mathcal{A}$ be a countable structure.

Corollary

Relative to a cone:
$\mathcal{A}$ *has computable dimension* 1 *or* ω.

Degrees of Categoricity

Degrees of categoricity

Definition

$\mathcal{A}$ is **d**-computably categorical if

$\mathbf{d}$ computes an isomorphism between $\mathcal{A}$ and any computable copy of $\mathcal{A}$.

Definition

$\mathcal{A}$ has degree of categoricity $\mathbf{d}$ if:

(1) $\mathcal{A}$ is $\mathbf{d}$-computably categorical and

(2) if $\mathcal{A}$ is $\mathbf{e}$-computably categorical, then $\mathbf{e} \geq \mathbf{d}$.

Equivalently: $\mathbf{d}$ is the least degree such that $\mathcal{A}$ is $\mathbf{d}$-computably categorical.

Example

$(\mathbb{N}, <)$ has degree of categoricity $0'$.

Which degrees are degrees of categoricity?

Theorem (Fokina, Kalimullin, Miller 2010; Csima, Franklin, Shore 2013)

If α is a computable ordinal then $0^{(\alpha)}$ is a degree of categoricity.

If α is a computable successor ordinal and $\mathbf{d}$ is d.c.e. in and above $0^{(\alpha)}$, then $\mathbf{d}$ is a degree of categoricity.

Theorem (Anderson, Csima 2014)

(1) *There is a Σ^0_2 degree $\mathbf{d}$ which is not a degree of categoricity.*

(2) *Every non-computable hyperimmune-free degree is not a degree of categoricity.*

Question (Fokina, Kalimullin, Miller 2010)

Which degrees are a degree of categoricity?

Strong degrees of categoricity

Definition

d is a strong degree of categoricity for $\mathcal{A}$ if

(1) $\mathcal{A}$ is **d**-computably categorical and

(2) there are computable copies $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ such every isomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$ computes **d**.

Every known example of a degree of categoricity is a strong degree of categoricity.

Question (Fokina, Kalimullin, Miller 2010)

Is every degree of categoricity a strong degree of categoricity?

Relative notions of categoricity

Definition

$\mathcal{A}$ is **d**-computably categorical relative to **c** if **d** computes an isomorphism between $\mathcal{A}$ and any **c**-computable copy of $\mathcal{A}$.

Definition

$\mathcal{A}$ has degree of categoricity **d** relative to **c** if:

1. $\mathbf{d} \geq \mathbf{c}$,
2. $\mathcal{A}$ is **d**-computably categorical relative to **c** and
3. if $\mathcal{A}$ is **e**-computably categorical relative to **c**, then $\mathbf{e} \geq \mathbf{d}$.

Equivalently: **d** is the least degree above **c** such that $\mathcal{A}$ is **d**-computably categorical relative to **c**.

Theorem on degrees of categoricity

Let $\mathcal{A}$ be a countable structure.

Theorem (Csima, H-T 2015)

Relative to a cone:

$\mathcal{A}$ has strong degree of categoricity $0^{(\alpha)}$ for some ordinal α.

More precisely:

Theorem (precisely stated)

There is an ordinal α such that for all degrees $\mathbf{c}$ on a cone, $\mathcal{A}$ has strong degree of categoricity $\mathbf{c}^{(\alpha)}$ relative to $\mathbf{c}$.

α is the Scott rank of $\mathcal{A}$:

it is the least α such that $\mathcal{A}$ has a $\Sigma^{\mathtt{in}}_{\alpha+2}$ Scott sentence.

Degree Spectra of Relations

Degree spectra

Let $\mathcal{A}$ be a (computable) structure and R an automorphism-invariant relation on $\mathcal{A}$.

Definition (Harizanov 1987)

The degree spectrum of R is

$$\mathrm{dgSp}(R) = \{d(R^{\mathcal{B}}) : \mathcal{B} \text{ is a computable copy of } \mathcal{A}\}$$

Many pathological examples have been constructed:

- $\{0, \mathbf{d}\}$, $\mathbf{d}$ is Δ^0_3 but not Δ^0_2 degree. [Harizanov 1991]
- the degrees below a given c.e. degree. [Hirschfeldt 2001]
- $\{0, \mathbf{d}\}$, $\mathbf{d}$ is a c.e. degree. [Hirschfeldt 2001]

Degree spectra of linear orders

For particular relations and structures, degree spectra are often nicely behaved.

Theorem (Mal'cev 1962)

Let R be the relation of linear dependence of n-tuples in an infinite-dimensional $\mathbb{Q}$-vector space. Then

$$\operatorname{dgSp}(R) = \textit{c.e. degrees.}$$

Theorem (Knoll 2009; Wright 2013)

Let R be a unary relation on $(\omega, <)$. Then

$$\operatorname{dgSp}(\omega, R) = \Delta^0_1 \text{ or } \operatorname{dgSp}(\omega, R) = \Delta^0_2.$$

Degree spectra of c.e. relations

Theorem (Harizanov 1991)

Suppose that R is computable. Suppose moreover that the property $(*)$ holds of $\mathcal{A}$ and R. Then

$$\mathrm{dgSp}(R) \neq \{\mathbf{0}\} \Rightarrow \mathrm{dgSp}(R) \supseteq \text{c.e. degrees}.$$

$(*)$ *For every $\bar{a}$, we can computably find $a \in R$ such that for all $\bar{b}$ and quantifier-free formulas $\theta(\bar{z}, x, \bar{y})$ such that $\mathcal{A} \models \theta(\bar{a}, a, \bar{b})$, there are $a' \notin R$ and $\bar{b}'$ such that $\mathcal{A} \models \theta(\bar{a}, a', \bar{b}')$.*

On a cone, the effectiveness condition holds.

Degree spectra relative to a cone

Definition

The degree spectrum of R below the degree $\mathbf{d}$ is

$$\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{d(R^{\mathcal{B}}) \oplus \mathbf{d} : \mathcal{B} \cong \mathcal{A} \text{ and } \mathcal{B} \leq_T \mathbf{d}\}$$

Corollary (Harizanov)

One of the following is true for all degrees $\mathbf{d}$ *on a cone:*

1. $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \{\mathbf{d}\}$, *or*
2. $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} \supseteq$ *degrees c.e. in and above* $\mathbf{d}$.

Relativised degree spectra

For any degree $\mathbf{d}$, either:

(1) $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} = \mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$,

(2) $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} \subsetneq \mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$,

(3) $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}} \supsetneq \mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$, or

(4) none of the above.

By Borel determinacy, exactly one of these four options happens on a cone.

Definition (Montalbán)

The degree spectrum of $(\mathcal{A}, R)$ on a cone is equal to that of $(\mathcal{B}, S)$ if we have equality on a cone, and similarly for containment and incomparability.

Two classes of degrees

Definition

A set A is d.c.e. if it is of the form $B - C$ for some c.e. sets B and C.

A set is n-c.e. if it has a computable approximation which is allowed n alternations.

We omit the definition of α-c.e.

Definition

A set A is CEA in B if A is c.e. in B and $A \geq_T B$.

A is n-CEA if there are sets $A_1, A_2, \dots, A_n = A$ such that A_1 is c.e., A_2 is CEA in A_1, and so on.

We omit the definition of α-CEA.

Natural classes of degrees

Let Γ be a natural class of degrees which relativises. For example the Δ^0_α, Σ^0_α, Π^0_α, α-c.e., or α-CEA degrees.

For any of these classes Γ of degrees, there is a structure $\mathcal{A}$ and a relation R such that, for each degree $\mathbf{d}$,

$$\mathrm{dgSp}_{\leq \mathbf{d}}(\mathcal{A}, R) = \Gamma(\mathbf{d}) \oplus \mathbf{d}.$$

So we may talk, for example, about a degree spectrum being equal to the Σ_α degrees on a cone.

Main question about degree spectra

Harizanov's result earlier showed that degree spectra on a cone behave nicely with respect to c.e. degrees.

Corollary (Harizanov)

Any degree spectrum on a cone is either equal to Δ^0_1 or contains Σ^0_1.

Question

What are the possible degree spectra on a cone?

D.c.e. relations

Theorem (H-T 2014)

There is are computable structures $\mathcal{A}$ and $\mathcal{B}$ with relatively intrinsically d.c.e. relations R and S on $\mathcal{A}$ and $\mathcal{B}$ respectively with the following property:

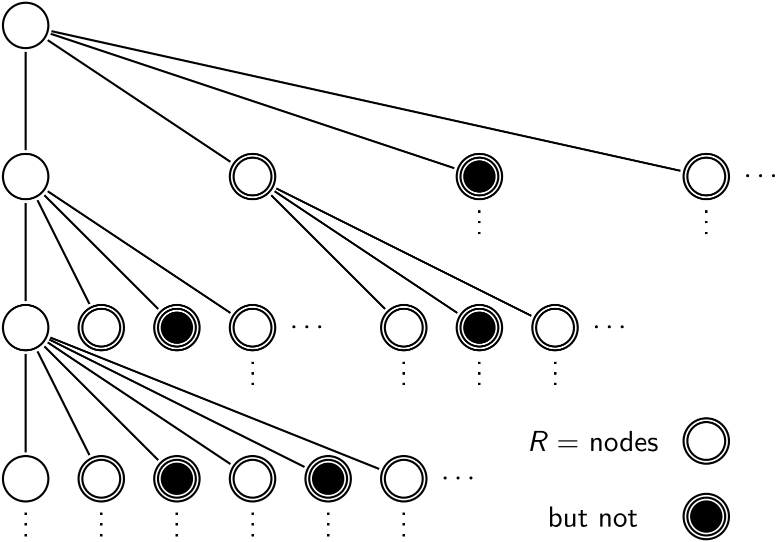
for any degree $\mathbf{d}$, $\mathrm{dgSp}(\mathcal{A}, R)_{\leq \mathbf{d}}$ and $\mathrm{dgSp}(\mathcal{B}, S)_{\leq \mathbf{d}}$ are incomparable.

Corollary (H-T 2014)

There are two degree spectra on a cone which are incomparable, each contained within the d.c.e. degrees and containing the c.e. degrees.

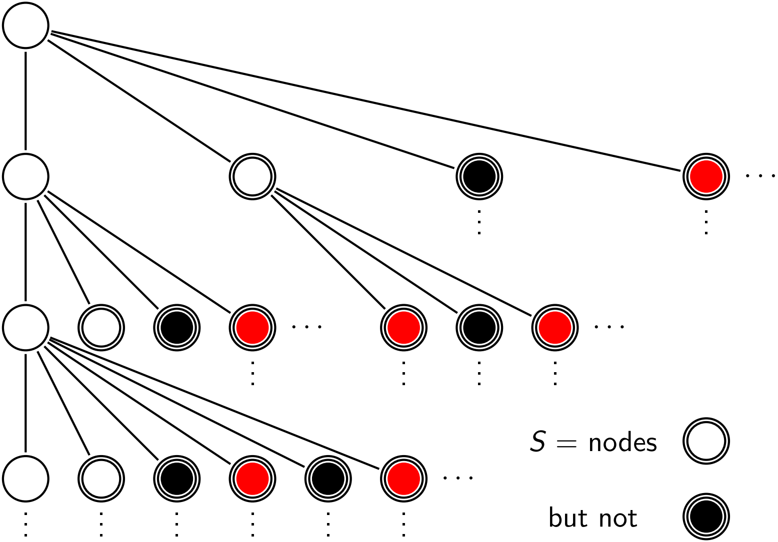
The structure $\mathcal{A}$

$\mathcal{A}$ is a tree with a successor relation.



The structure $\mathcal{B}$

$\mathcal{B}$ is a tree with a tree-order.



A question of Ash and Knight

Question (Ash, Knight 1997)

(Assuming some effectiveness condition):

Is any relation which is not intrinsically Δ^0_α realizes every α-CEA degree?

Stated in terms of degree spectra on a cone:

Does any degree spectrum on a cone which is not contained in Δ^0_α contain α-CEA?

Ash and Knight [1995] showed that we cannot replace α-CEA with Σ^0_α.

A question of Ash and Knight

Ash and Knight gave a result which goes towards answering this question.

Theorem (Ash, Knight 1997)

Let $\mathcal{A}$ be a computable structure with an additional computable relation R. Suppose that R is not relatively intrinsically Δ^0_α.

Moreover, suppose that $\mathcal{A}$ is α-friendly and that for all $\bar{c}$, we can find $a \notin R$ which is α-free over $\bar{c}$.

Then for any Σ^0_α set C, there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that

$$R^{\mathcal{B}} \oplus \Delta^0_\alpha \equiv_T C \oplus \Delta^0_\alpha$$

where Δ^0_α is a Δ^0_α-complete set.

The class 2-CEA

For the case of 2-CEA, we can answer this question:

Theorem (H-T 2014)

Let $\mathcal{A}$ be a structure and R a relation on $\mathcal{A}$. Then one of the following is true relative to all degrees on a cone:

1. $\text{dgSp}(\mathcal{A}, R) \subseteq \Delta^0_2$, *or*
2. $2\text{-}CEA \subseteq \text{dgSp}(\mathcal{A}, R)$.

The picture so far