Computable Structures of High Scott Rank

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$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countably infinite conjunctions and disjunctions.

There is a hierarchy of $\mathcal{L}_{\omega_1\omega}$-formulas based on their quantifier complexity after putting them in normal form. Formulas are classified as either Σ^0_α or Π^0_α, for $\alpha < \omega_1$.

- A formula is Σ^0_0 and Π^0_0 is it is finitary quantifier-free.
- A formula is Σ^0_α if it is a disjunction of formulas $(\exists \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is Π^0_β for $\beta < \alpha$.
- A formula is Π^0_α if it is a conjunction of formulas $(\forall \bar{y})\varphi(\bar{x}, \bar{y})$ where φ is Σ^0_β for $\beta < \alpha$.

A formula is computable if the conjunctions and disjunctions are over computable sets of formulas.

Example

There is a computable Π^0_2 sentence which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee\!\!\!\bigvee_{n \in \mathbb{N}} nx = 0.$$

Example

There is a computable Σ^0_1 formula which describes the dependence relation on triples x, y, z in a $\mathbb{Q}$-vector space:

$$\bigvee\!\!\!\bigvee_{(a,b,c) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}} ax + by + cz = 0$$

Example

There is a computable Σ^0_3 sentence which says that a $\mathbb{Q}$-vector space has finite dimension:

$$\bigvee_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n)(\forall y)\ y \in \operatorname{span}(x_1, \ldots, x_n).$$

Example

There is a computable Π^0_3 sentence which says that a $\mathbb{Q}$-vector space has infinite dimension:

$$\bigwedge_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n) \operatorname{Indep}(x_1, \ldots, x_n).$$

Let $\mathcal{A}$ be a countable structure.

Theorem (Scott)

There is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that:

$$\mathcal{B} \text{ countable}, \mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

φ is a *Scott sentence* of $\mathcal{A}$.

Example

$(\omega, <)$ has a computable Π^0_3 Scott sentence consisting of the Π^0_2 axioms for infinite linear orders together with:

$$\forall y_0 \bigvee_{n\in\omega} \exists y_n < \cdots < y_1 < y_0 \left[\forall z\ (z > y_0) \vee (z = y_0) \vee (z = y_1) \vee \cdots \vee (z = y_n)\right].$$

Example

$(\omega + \omega, <)$ has a computable Σ^0_4 Scott sentence consisting of the Π^0_2 axioms for infinite linear orders together with:

there are two elements *a* and *b* such that *a* is the least element and *b* is greater than *a*, and there are infinitely many elements between *a* and *b* and infinitely many elements greater than *b*, and every element is an *n*th successor (for some *n*) of either *a* or *b*.

Let $\mathcal{A}$ be a countable structure.

Definition (Montalbán)

$SR(\mathcal{A})$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi^0_{\alpha+1}$ Scott sentence.

Theorem (Montalbán)

Let α a countable ordinal. The following are equivalent:

- *$\mathcal{A}$ has a $\Pi^0_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is Σ^0_α-definable without parameters.*
- *$\mathcal{A}$ is uniformly (boldface) $\mathbf{\Delta}^0_\alpha$-categorical without parameters.*

For computable structures, we need to talk about computable ordinals.

Definition

- ω_1^{CK} is the least non-computable ordinal.
- ω_1^x is the least non-x-computable ordinal.

Theorem (Sacks)

The countable admissible ordinals greater than ω are exactly the ordinals of the form ω_1^x.

Let $\mathcal{A}$ be a computable structure.

Theorem (Nadel)

$\mathcal{A}$ has Scott rank $\leq \omega_1^{CK} + 1$.

Moreover:

- $SR(\mathcal{A}) < \omega_1^{CK}$ if $\mathcal{A}$ has a computable Scott sentence.
- $SR(\mathcal{A}) = \omega_1^{CK}$ if each automorphism orbit is definable by a computable formula, but $\mathcal{A}$ does not have a computable Scott sentence.
- $SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is an automorphism orbit which is not defined by a computable formula.

This all relativizes.

Let $\mathcal{A}$ be an x-computable structure.

Theorem (Nadel)

$\mathcal{A}$ has Scott rank $\leq \omega_1^x + 1$.

Moreover:

- $SR(\mathcal{A}) < \omega_1^x$ if $\mathcal{A}$ has a x-computable Scott sentence.
- $SR(\mathcal{A}) = \omega_1^x$ if each automorphism orbit is definable by an x-computable formula, but $\mathcal{A}$ does not have an x-computable Scott sentence.
- $SR(\mathcal{A}) = \omega_1^x + 1$ if there is an automorphism orbit which is not defined by an x-computable formula.

It is not too hard to build computable structures of each computable Scott rank $\alpha < \omega_1^{CK}$.

The more difficult cases are building computable structures of Scott rank ω_1^{CK} and $\omega_1^{CK} + 1$. We say that such structures have high Scott rank.

Theorem (Harrison)

There is a computable linear order of Scott rank $\omega_1^{CK} + 1$ with order type $\omega_1^{CK}(1 + \mathbb{Q})$.

Theorem (Chan, Montalbán, Harrison, Kleene)

There is a computable operator Φ so that for all $x \in 2^\omega$, $\Phi(x)$ is a linear order with Scott rank $\omega_1^x + 1$ and order type $\omega_1^x(1 + \mathbb{Q})$.

Note that the order type does not depend on x, but only on ω_1^x.

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Steps of the construction:

1. The "relation y is not hyperarithmetic in x" is $\Sigma^1_1(x)$ and so there is an x-computable tree T whose paths are pairs $\langle y, f \rangle$ where f witnesses that y is not hyperarithmetic in x. T is x-computable uniformly in x and has no x-hyperarithmetic path.

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2. Take the Kleene-Brouwer order on T: $s \leq_{KB} t$ if and only if
    - $t < s$ or
    - $s(n) < t(n)$ and $t \upharpoonright n = s \upharpoonright n$.

   We get an x-computable linear order L with no x-hyperarithmetic descending sequence.

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2. Take the Kleene-Brouwer order on T: $s \leq_{KB} t$ if and only if
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Steps of the construction:

1. The "relation y is not hyperarithmetic in x" is $\Sigma_1^1(x)$ and so there is an x-computable tree T whose paths are pairs $\langle y, f\rangle$ where f witnesses that y is not hyperarithmetic in x. T is x-computable uniformly in x and has no x-hyperarithmetic path.
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   We get an x-computable linear order L with no x-hyperarithmetic descending sequence.
3. L has order type $\omega_1^x(1+\mathbb{Q}) + \alpha$ for some $\alpha < \omega_1^x$.
4. $L \cdot \omega$ has order type $\omega_1^x(1+\mathbb{Q})$.

How do you build a computable structure of Scott rank ω_1^{CK}?

Theorem (Makkai)

There is a Δ_2^0 structure of Scott rank ω_1^{CK}.

Theorem (Knight, Millar)

There is a computable structure of Scott rank ω_1^{CK}.

I will talk about a later construction of Calvert, Knight, and Millar.

Theorem (Calvert, Knight, Millar)

There is a computable thin homogeneous tree with no bound on the ordinal tree ranks at all levels. It has Scott rank ω_1^{CK}.

Assign to each node in a tree its tree rank:

- $rk(x) = 0$ if x is a leaf.
- $rk(x)$ is otherwise the least ordinal (or possibly ∞) greater than the ranks of the children of x.

If $rk(x) = \infty$, then there is a path through x.

Definition

A tree T is *thin* if there is a computable ordinal bound on the ordinal tree ranks at each level of the tree.

Definition

A tree T is *homogenous* if:

- Whenever x has a successor of rank α, it has infinitely many successors of rank α.
- If some element at level n has a successor of rank α, every element at level n with rank $> \alpha$ has a successor of rank α.

We want to build a computable tree which is thin, homogeneous, and has no bound on the ordinal tree ranks.

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Definition

A tree T is *homogenous* if:

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The Harrison order is a "natural structure":

- it is easily described (in natural language)
- it does not require making any choices
- it can be built with a construction which relativizes

The structure of Scott rank ω_1^{CK} that we built is not natural in any of these senses.

Question

Is there a natural computable structure of Scott rank ω_1^{CK}?

We will rephrase this question as asking for a construction which relativizes.

Question

Is there a computable (or even Borel) operator Φ such that:

- for all $x, y \in 2^\omega$, if $\Phi(x) \cong \Phi(y) \iff \omega_1^x = \omega_1^y$
- for all $x \in 2^\omega$, $\Phi(x)$ is a computable structure of Scott rank ω_1^x

Let F_{ω_1} be the equivalence relation which makes

$$x \; F_{\omega_1} \; y \iff \omega_1^x = \omega_1^y.$$

Descriptive set theorists call such an operator Φ a classification of F_{ω_1} by structures.

Theorem (Chan)

Suppose that Φ *is a* Δ^1_1 *operator such that for all* $x, y \in 2^\omega$,

$$\Phi(x) \cong \Phi(y) \iff \omega_1^x = \omega_1^y.$$

Then for all x, $SR(\Phi(x)) \geq \omega_1^x$.

We show that there is no natural construction of a computable structure of Scott rank ω_1^{CK}:

Theorem (Chan, HT, Marks)

There is no Borel operator Φ such that:

- *for all $x, y \in 2^\omega$, $\Phi(x) \cong \Phi(y) \iff \omega_1^x = \omega_1^y$*
- *for all $x \in 2^\omega$, $\Phi(x)$ is a computable structure of Scott rank ω_1^x*

Equivalently: If Φ is a classification of F_{ω_1} by structures, then for some x, $\Phi(x)$ has Scott rank $\omega_1^x + 1$.

Question

Is there a Δ^1_1 operator Φ such that:

- for all $x, y \in 2^\omega$, $\Phi(x) \cong \Phi(y) \iff \omega_1^x = \omega_1^y$
- for some $x \in 2^\omega$, $\Phi(x)$ is a computable structure of Scott rank ω_1^x?

Until recently, the structures we have talked about and built were essentially all of the examples we had.

Because there are so few examples of computable structures of high Scott rank, there are many general questions about them that we don't know the answer to.

I'm going to talk about two other constructions of new models of high Scott rank:

- Structures of Scott rank ω_1^{CK} and $\omega_1^{CK}+1$ which are not computably approximable.
- A structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

The latter is joint work with Greg Igusa and Julia Knight.

Definition

Given a model $\mathcal{A}$, we define the computable infinitary theory of $\mathcal{A}$,

$$Th_\infty(\mathcal{A}) = \{\varphi \text{ a computable } \mathcal{L}_{\omega_1\omega} \text{ sentence} \mid \mathcal{A} \models \varphi\}.$$

The computable infinitary theory of the Makkai-Knight-Millar structure was $\aleph_0$-categorical.

Question (Millar-Sacks)

Is there a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical?

Any other models of the same theory would necessarily be non-computable and of Scott rank at least $\omega_1^{CK} + 1$.

Theorem (Millar-Sacks)

There is a structure $\mathcal{A}$ of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

$\mathcal{A}$ is not computable, but $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. ($\mathcal{A}$ lives in a fattening of $\mathcal{L}_{\omega_1^{CK}}$.)

Freer generalized this to arbitrary admissible ordinals.

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Freer generalized this to arbitrary admissible ordinals.

Theorem (HT-Igusa-Knight)

There is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

The Harrison linear order is approximated by the computable ordinals:

For every computable sentence φ true of the Harrison linear order, there is a computable ordinal α such that $(\alpha, <) \models \varphi$.

So the Harrison linear order is a "limit" of the computable ordinals.

Let T be the computable tree of Scott rank ω_1^{CK} from the previous slides.

Theorem (Calvert, Knight, Millar 2006)

There is a sequence T_α of computable trees such that $SR(T_\alpha) < \omega_1^{CK}$ and $T_\alpha \equiv_\alpha T$.

So T is a limit of computable structures of low Scott rank in the same way.

Definition

$\mathcal{A}$ is computably approximable if every computable infinitary sentence φ true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \not\cong \mathcal{A}$ with $SR(\mathcal{B}) < \omega_1^{CK}$.

The Harrison linear order is computably approximated by the computable ordinals.

Question (Goncharov, Calvert, Knight)

Is every computable model of high Scott rank computably approximable?

Theorem (HT)

There is a computable model $\mathcal{A}$ *of Scott rank* $\omega_1^{CK}+1$ *and a* Π_2^c *sentence* ψ *such that:*

- $\mathcal{A} \models \psi$
- $\mathcal{B} \models \psi \implies SR(\mathcal{B}) = \omega_1^{CK}+1$.

The same is true for Scott rank ω_1^{CK}.

Corollary

There are computable models of Scott rank ω_1^{CK} *and* $\omega_1^{CK}+1$ *which are not computably approximable.*

I was initially interested in a different question.

Let φ be a sentence of $\mathcal{L}_{\omega_1\omega}$.

Definition

The *Scott spectrum* of φ is the set

$$SS(T) = \{\alpha \in \omega_1 \mid \alpha \text{ is the Scott rank of a countable model of } T\}.$$

Question

Classify the Scott spectra.

Theorem (HT, in ZFC + PD)

The Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences are exactly the sets of the following forms, for some $\boldsymbol{\Sigma}^1_1$ class of linear orders $\mathcal{C}$:

1. *the well-founded parts of orderings in $\mathcal{C}$,*
2. *the orderings in $\mathcal{C}$ with the non-well-founded part collapsed to a single element, or*
3. *the union of (1) and (2).*

The construction, from $\mathcal{C}$, of an $\mathcal{L}_{\omega_1\omega}$-sentence does not use PD, and:

- We can get a $\Pi^{\mathtt{in}}_2$ sentence.
- If the class $\mathcal{C}$ is lightface, then we get a Π^c_2 sentence.
- The Harrison linear order, with each element named by a constant, forms a Σ^1_1 class with a single member. From (1) we get $\{\omega_1^{CK}\}$ as a Scott spectrum and from (2) we get $\{\omega_1^{CK} + 1\}$.

Definition

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$ is the least countable ordinal α such that, for all computable $\mathcal{L}_{\omega_1\omega}$-sentences T:

$$T \text{ has a model of Scott rank } \alpha$$
$$\Downarrow$$
$$T \text{ has models of arbitrarily high Scott ranks.}$$

Question (Sacks)

What is $\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$?

Definition

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$ is the least countable ordinal α such that, for all computable $\mathcal{L}_{\omega_1\omega}$-sentences T:

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$$\Downarrow$$
$$T \text{ has models of arbitrarily high Scott ranks.}$$

Question (Sacks)

What is $\mathrm{sh}(\mathcal{L}_{\omega_1,\omega})$?

Theorem (Sacks, Marker, HT)

$\mathrm{sh}(\mathcal{L}_{\omega_1,\omega}) = \delta^1_2$, *the least ordinal which has no* Δ^1_2 *presentation.*

Question

Classify the Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences in ZFC.

Question

Classify the Scott spectra of computable $\mathcal{L}_{\omega_1\omega}$-sentences.

Question

Classify the Scott spectra of first-order theories.