When is a property expressed in infinitary logic also pseudo-elementary?

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This is joint work with Barbara Csima and Nancy Day.

It is also work in progress.

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G is connected iff
for all transitive $S \supseteq E$ and all a, b, $(a, b) \in S$.

G is not connected iff
there is a transitive $S \supseteq E$ and a, b such that $(a, b) \notin S$.

Let Δ be a set of sentences in the language of graphs.

Does

$$\Delta \quad \models \quad \text{“}G \text{ is connected”} \text{ ?}$$

Let Δ be a set of sentences in the language of graphs.

$$\Delta \quad \models \quad \text{"}G \text{ is connected"}$$

$$\Updownarrow$$

$$\Delta \cup \{R \supseteq E \text{ is transitive}\} \quad \models \quad \forall a, b \ \ (a, b) \in R$$

Definition

A class $\mathbb{K}$ of $\mathcal{L}$-structures is a PC-class if there is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and an elementary first-order sentence ϕ such that

$$\mathbb{K} = \{\mathcal{M} \mid \text{there is an } \mathcal{L}^*\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \vDash \phi\}.$$

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The class of disconnected graphs is a PC-class.

It is also defined by the infinitary sentence:

$$\exists x_1, x_2 \bigwedge_{n \in \mathbb{N}} \forall y_1, \ldots, y_n \quad \neg(x_1 E y_1 \wedge y_1 E y_2 \wedge \cdots \wedge y_n E x_2).$$

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Example

The class of non-well-founded linear orders is a PC-class.

It is not $\mathcal{L}_{\omega_1\omega}$-definable.

Example

Let ϕ be a first-order sentence. The class $\mathbb{K}$ of infinite models of ϕ is a PC-class.

$\mathcal{A} \models \phi$ is infinite if and only if there is a linear order $\preceq$ on $\mathcal{A}$ such that $(\forall x)(\exists y)[y \succ x]$.

$\mathbb{K}$ also defined by the infinitary sentence

$$\phi \wedge \bigwedge\!\!\!\bigwedge_{n \in \mathbb{N}} (\exists x_0, \ldots, x_n) \left[\bigwedge_{i \neq j} x_i \neq x_j \right].$$

Question

When is a pseudo-elementary class also definable by an infinitary sentence, and vice versa?

Background: Pseudo-elementary Classes

There are four variants of pseudo-elementary classes:

- PC
- PC′
- PC_Δ
- PC'_Δ

Definition

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Definition

A class $\mathbb{K}$ of $\mathcal{L}$-structures is a PC_Δ-class if there is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and an elementary first-order theory T such that

$$\mathbb{K} = \{\mathcal{M} \mid \text{there is an } \mathcal{L}^*\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \models T\}.$$

Definition

Let $\mathcal{L} \subseteq \mathcal{L}^*$ be a pair of languages, with a unary predicate $P \in \mathcal{L}^* \smallsetminus \mathcal{L}$. Given an $\mathcal{L}^*$-structure $\mathcal{A}$, we denote by $\mathcal{A}_P$ the substructure of $\mathcal{A} \mid \mathcal{L}$ whose domain is $P^{\mathcal{A}}$ (if this is an $\mathcal{L}$-structure; otherwise $\mathcal{A}_P$ is not defined).

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Definition

A class $\mathbb{K}$ of $\mathcal{L}$-structures is a PC′-class if there is a language $\mathcal{L}^* \supseteq \mathcal{L}$, with a unary relation $P \in \mathcal{L}^* \setminus \mathcal{L}$, and an $\mathcal{L}^*$-formula ϕ, such that

$$\mathbb{K} = \{\mathcal{A}_P \mid \mathcal{A} \vDash \phi \text{ and } \mathcal{A}_P \text{ is defined}\}.$$

Definition

Let $\mathcal{L} \subseteq \mathcal{L}^*$ be a pair of languages, with a unary predicate $P \in \mathcal{L}^* \setminus \mathcal{L}$. Given an $\mathcal{L}^*$-structure $\mathcal{A}$, we denote by $\mathcal{A}_P$ the substructure of $\mathcal{A} \mid \mathcal{L}$ whose domain is $P^{\mathcal{A}}$ (if this is an $\mathcal{L}$-structure; otherwise $\mathcal{A}_P$ is not defined).

Definition

A class $\mathbb{K}$ of $\mathcal{L}$-structures is a PC'_Δ-class if there is a language $\mathcal{L}^* \supseteq \mathcal{L}$, with a unary relation $P \in \mathcal{L}^* \setminus \mathcal{L}$, and an $\mathcal{L}^*$-theory T, such that

$$\mathbb{K} = \{\mathcal{A}_P \mid \mathcal{A} \vDash T \text{ and } \mathcal{A}_P \text{ is defined}\}.$$

PC	sentence	one sort
PC′	sentence	extra sorts
PC_Δ	theory	one sort
PC'_Δ	theory	extra sorts

There are some obvious relations:

- Every PC-class is a PC′-class and a PC_Δ-class.
- Every PC′-class or PC_Δ-class is a PC'_Δ-class.

Theorem (Makkai)

Let $\mathbb{K}$ be a class of structures.

- $\mathbb{K}$ *is a* PC_Δ*-class if and only if it is a* PC'_Δ*-class.*
- *If all the structures in* $\mathbb{K}$ *are infinite, then* $\mathbb{K}$ *is a* PC*-class if and only if it is a* PC'*-class.*

Example

There is a PC_{Δ}-class which is not a PC'-class.

Proof. Let $A \in \mathbb{N}$ be a set which is not computably enumerable.

Let $\mathbb{K}$ be the class of all graphs which have no cycles of length n for $n \in A$.

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If φ were a sentence in an expanded language defining $\mathbb{K}$ as a PC'-class, then

$$n \in A \iff \varphi \vdash \text{“there are no cycles of length } n\text{”}.$$

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If φ were a sentence in an expanded language defining $\mathbb{K}$ as a PC'-class, then

$$n \in A \iff \varphi \vdash \text{"there are no cycles of length } n\text{"}.$$

This would mean that A is computably enumerable.

So $\mathbb{K}$ is an elementary class but not a PC'-class.

In fact, we have:

$$\mathsf{PC} \subsetneq \mathsf{PC}' \subsetneq \mathsf{PC}_{\Delta} = \mathsf{PC}'_{\Delta}\ .$$

We will show later that the second containment is strict.

Background: Infinitary Logic

$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countably infinite conjunctions and disjunctions.

Definition

The $\mathcal{L}_{\omega_1\omega}$-formulas are built up inductively as follows:

- atomic formulas
- $\neg\varphi$, where φ is an $\mathcal{L}_{\omega_1\omega}$-formula
- $(\exists x)\varphi$, where φ is an $\mathcal{L}_{\omega_1\omega}$-formula
- $(\forall x)\varphi$, where φ is an $\mathcal{L}_{\omega_1\omega}$-formula
- if $(\varphi_i)_{i\in\omega}$ are $\mathcal{L}_{\omega_1\omega}$-formulas, then so is $\bigwedge\!\!\!\bigwedge_{i\in\omega}\varphi_i$
- if $(\varphi_i)_{i\in\omega}$ are $\mathcal{L}_{\omega_1\omega}$-formulas, then so is $\bigvee\!\!\!\bigvee_{i\in\omega}\varphi_i$

A formula is computable if the conjunctions and disjunctions are over computable sets of formulas.

Example

There is a computable infinitary sentence which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n \in \mathbb{N}} nx = 0.$$

Example

There is a computable infinitary formula which describes the dependence relation on triples x, y, z in a $\mathbb{Q}$-vector space:

$$\bigvee_{(a,b,c) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}} ax + by + cz = 0$$

Example

There is a computable infinitary sentence which says that a $\mathbb{Q}$-vector space has finite dimension:

$$\bigvee\!\!\!\!\bigvee_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n)(\forall y)\, y \in \operatorname{span}(x_1, \ldots, x_n).$$

Example

There is a computable infinitary sentence which says that a $\mathbb{Q}$-vector space has infinite dimension:

$$\bigwedge\!\!\!\!\bigwedge_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n)\, \operatorname{Indep}(x_1, \ldots, x_n).$$

Definition

An $\mathcal{L}_{\omega_1\omega}$-sentence φ is a $\bigwedge$-formula if it can be written in normal form without any infinite disjunctions.

More concretely, the $\bigwedge$-formulas are defined inductively as follows:

- every finitary quantifier-free sentence is a $\bigwedge$-formula
- if φ is a $\bigwedge$-formula, then so are $(\exists x)\varphi$ and $(\forall x)\varphi$
- if $(\varphi_i)_{i\in\omega}$ are $\bigwedge$-formulas, then so is $\bigwedge_{i\in\omega}\varphi_i$.

New Results

Theorem (Interpolation Theorem)

Suppose ϕ_1 *is a* $\bigwedge\!\!\!\bigwedge$*-sentence and* ϕ_2 *is an* $\mathcal{L}_{\omega_1\omega}$*-sentence with* $\phi_1 \models \phi_2$*.*

There is a $\bigwedge\!\!\!\bigwedge$*-sentence* θ *such that* $\phi_1 \models \theta$*,* $\theta \models \phi_2$*, and every relation, function and constant symbol occurring in* θ *occurs in both* ϕ_1 *and* ϕ_2*.*

Theorem (Interpolation Theorem)

Suppose ϕ_1 is a $\bigwedge\!\!\!\bigwedge$-sentence and ϕ_2 is an $\mathcal{L}_{\omega_1\omega}$-sentence with $\phi_1 \vDash \phi_2$.

There is a $\bigwedge\!\!\!\bigwedge$-sentence θ such that $\phi_1 \vDash \theta$, $\theta \vDash \phi_2$, and every relation, function and constant symbol occurring in θ occurs in both ϕ_1 and ϕ_2.

Corollary

Let $\mathbb{K}$ be a class of $\mathcal{L}$-structures closed under isomorphism. If $\mathbb{K}$ is both a PC_Δ-class and $\mathcal{L}_{\omega_1\omega}$-elementary, then it is defined by a $\bigwedge\!\!\!\bigwedge$-sentence.

Question

If $\mathbb{K}$ is both a PC-class and $\mathcal{L}_{\omega_1\omega}$-elementary, then is it defined by a computable $\bigwedge\!\!\!\bigwedge$-sentence?

Theorem

Let $\mathbb{K}$ be a class definable by a computable $\bigwedge\!\!\!\!\bigwedge$-sentence in a finite language.

Then $\mathbb{K}$ is a PC' class.

Corollary

Every computably axiomatizable class in a finite language is a PC' class.

Let G be a graph with edge relation E.

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Example

The class of graphs with no cycles of prime length is a PC'-class.

Example

The class of graphs with at least one cycle of length p for each prime p is a PC'-class.

Theorem

Let $\mathbb{K}$ be a class definable by a $\bigwedge\!\!\!\bigwedge$-sentence.

Then $\mathbb{K}$ is a $\mathsf{PC}_{\boldsymbol{\Delta}}$ *class.*

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Let $\mathbb{K}$ be a class definable by a $\bigwedge\!\!\!\bigwedge$-sentence.

Then $\mathbb{K}$ is a PC_{Δ} class.

Corollary

Let $\mathbb{K}$ be a class of structures. The following are equivalent:

- *$\mathbb{K}$ is both a PC_{Δ}-class and $\mathcal{L}_{\omega_1\omega}$-elementary.*
- *$\mathbb{K}$ is defined by a $\bigwedge\!\!\!\bigwedge$-sentence.*

Theorem (Mal'tsev, Tarski)

If $\mathbb{K}$ *is a* PC'_{Δ}*-class which is closed under substructures, then it axiomatized by a set of universal sentences.*

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Theorem

Let $\mathbb{K}$ be a class of structures. The following are equivalent:

- *$\mathbb{K}$ is a PC'-class which is closed under substructures,*
- *$\mathbb{K}$ is axiomatized by a computable universal theory.*

Example

Orderable groups are a PC-class.

They are also universally axiomatizable by saying that every finite subset can be ordered in a way that is compatible with the group operation.

Example

There is a c.e. universal theory T whose models are a PC'-class but not a PC-class.

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For every finite graph G, we can decide effectively whether there is an expansion of G to a model of ϕ_n.

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For every finite graph G, we can decide effectively whether there is an expansion of G to a model of ϕ_n.

For each n, let C_n be a cycle of length n.

Let T be the theory which says that there is no cycle of length n for exactly those n where C_n has an expansion to a model of ϕ_n.

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Let T be the theory which says that there is no cycle of length n for exactly those n where C_n has an expansion to a model of ϕ_n.

T is c.e., universal, and different from each PC-class.

Conjecture

Let $\mathbb{K}$ be a class of structures. The following are equivalent:

- $\mathbb{K}$ *is a* PC*-class closed under substructures,*
- $\mathbb{K}$ *is axiomatized by a universal theory T and we can decide in polynomial time for each universal formula φ whether $T \vdash \varphi$.*

Conjecture

Let $\mathbb{K}$ be a class of structures. The following are equivalent:

- $\mathbb{K}$ *is both a* PC$'$*-class and $\mathcal{L}_{\omega_1\omega}$-elementary.*
- $\mathbb{K}$ *is defined by a computable $\bigwedge\!\!\!\!\bigwedge$-sentence.*

Thanks!